Matrix models of fuzzy field theories

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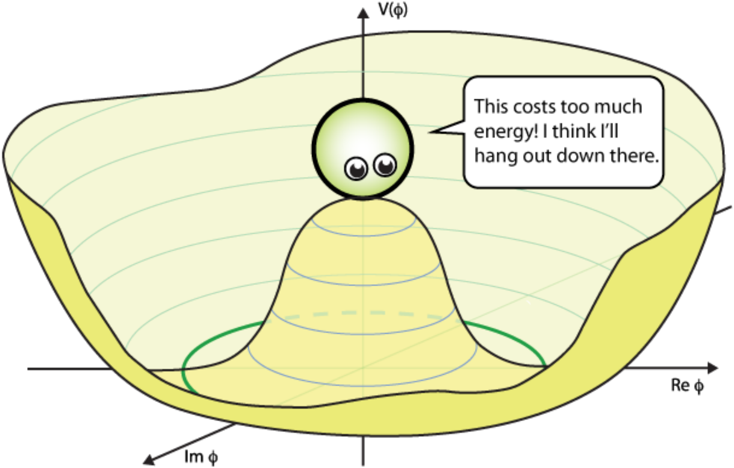
Action MP 1405
Quantum Structure of Spacetime

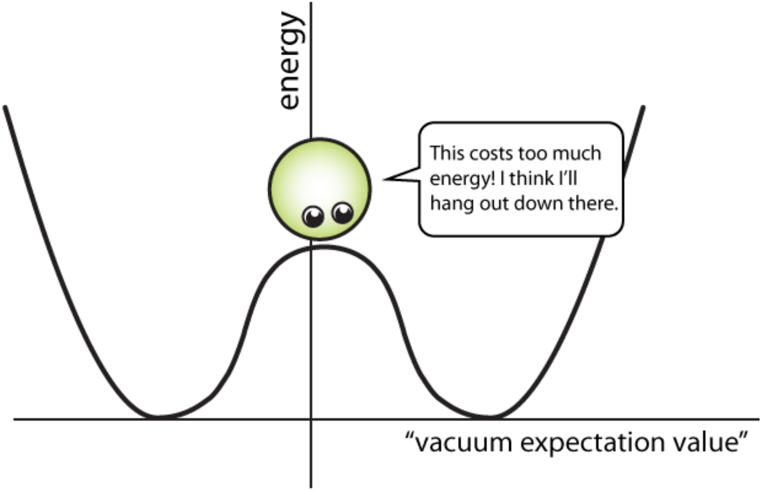
Quantum Spacetime '18, 21.2.2018, Sofia

[1512.00689 [hep-th]],[1711.02008 [hep-th]],[1802.05188 [hep-th]], work in progress

The ABEGHHK'tH mechanism



Real scalar ϕ^4 field on plane



Introduction and outline

In this talk, we will

- very briefly introduce fuzzy spaces and some aspects of fuzzy field theories,
- describe these theories in terms of a random matrix model,
- investigate the properties of this model.

Introduction and outline

Take home message.

- Symmetry breaking in noncommutative field theory is (very) different than in the commutative case.
- Matrix models are a great tool to analyze the(se) properties of scalar field theories on fuzzy spaces, and beyond.

Fuzzy spaces

Fuzzy sphere

- Noncommutative spaces introduce a shortest possible distance.
- Fuzzy spaces (= a finite dimensional algebra) have finite number of the "Planck cells" N.
- The hallmark example is the fuzzy sphere S_F^2.
  Hoppe '82; Madore '92; Grosse, Klimcik, Presnajder '90s
- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

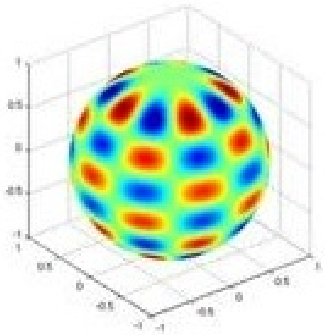


Image from

http://principles.ou.edu/mag/earth.html

Fuzzy sphere

Technically, this is done by

- truncating the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

- deforming the coordinate(function)s

$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta \varepsilon_{ijk} x_k \ .$$

Real functions on the fuzzy sphere are $N \times N$ hermitian and the eigenvalues of M represent the values of the function on the cells.

Fuzzy scalar field theory

Scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int dx \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \, .$$

- We construct the noncommutative theory as an analogue with
  - field $\to$ matrix,
  - functional integral $\to$ matrix integral,
  - spacetime integral $\to$ trace,
  - derivative $\to$ L_i commutator.

Scalar field theory

- **Commutative**

$$S(\Phi) = \int dx \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \, .$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right]$$

$$\langle F \rangle = \frac{\int dM \, F(M) e^{-S(M)}}{\int dM \, e^{-S(M)}} \, .$$

Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03

Spontaneous symmetry breaking

Symmetry breaking in NC field theories

- From now on ϕ^4 theory.
- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.

  Glimm, Jaffe, Spencer '75; Chang '76

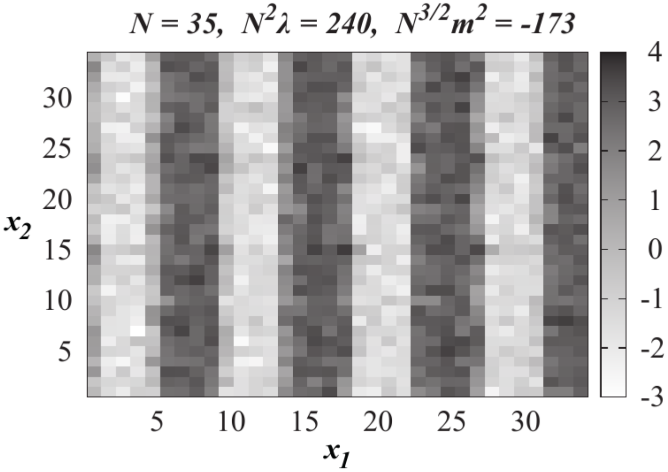
  Loinaz, Willey '98; Schaich, Loinaz '09
- In disorder phase the field oscillates around the value $\phi = 0$.
- In uniform order phase the field oscillates around a nonzero value which is a minimum of the potential.

Symmetry breaking in NC field theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; G.-H. Chen and Y.-S. Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero '14; Mejía-Díaz, Bietenholz, Panero '14; Medina, Bietenholz, D. O'Connor '08; Bietenholz, Hofheinz, Nishimura '04; Lizzi, Spisso '12; Ydri, Ramda, Rouag '16
  Panero '15

Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero '14 for $\mathbb{R}^2_\theta$

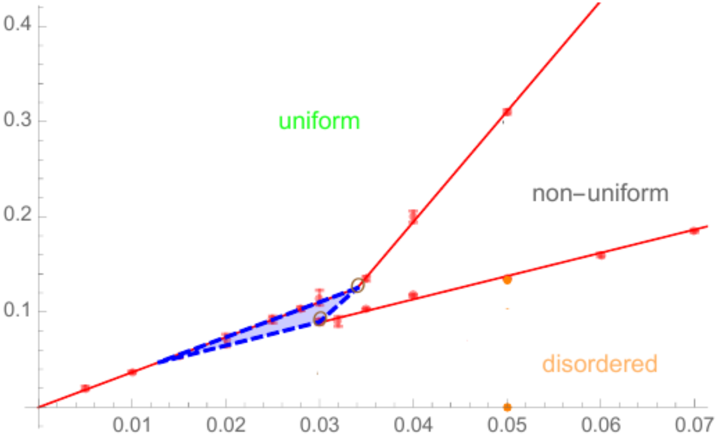


Symmetry breaking in NC field theories

- This phase is a result of the nonlocality of the theory.
- This phase survives the commutative limit of the noncommutative theory! Result of the UV/IR mixing.
- The commutative limit of such noncommutative theory is (even more) different than the commutative theory we started with.

Symmetry breaking in NC field theories

O'Connor, Kováčik '18 for S_F^2



Matrix model description of fuzzy field theories

Matrix models

- Ensemble of hermitian $N \times N$ matrices with a probability measure $S(M)$ and expectation values

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \ .$$

- This is the very same expression as for the real scalar field.
- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2} \mathrm{Tr}\left(M[L_i,[L_i,M]] \right) + \frac{1}{2} r\, \mathrm{Tr}\left(M^2 \right) + g\, \mathrm{Tr}\left(M^4 \right)$$

(minus the red Brezin, Itzykson, Parisi, Zuber '78)

Matrix models of fuzzy field theories

- The large N limit of the model **without** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

is **well** understood.

- The key is diagonalization and the saddle point approximation.

Matrix models of fuzzy field theories

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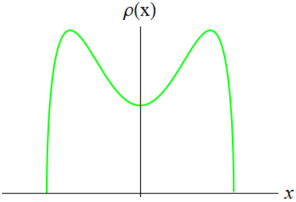
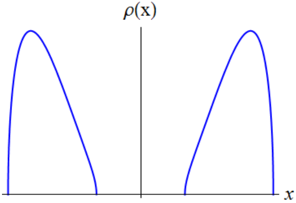
is **well** understood.

- The key results is that for $r < -4\sqrt{g}$ we get two cut eigenvalue density.





Matrix models of fuzzy field theories

- The model **with** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  is **not well** understood.

  Steinacker '05; JT Acta Physica Slovaca '15

- The key issue being that diagonalization no longer straightforward.

Matrix models of fuzzy field theories

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- The key issue being that diagonalization no longer straightforward.
- We are to compute integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[\frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Matrix models of fuzzy field theories

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- The key issue being that diagonalization no longer straightforward.
- We are to compute integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[S_{eff}(\lambda_i)+\frac{1}{2}r\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- How to compute S_{eff}?

Matrix models of fuzzy field theories

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. O'Connor, Sämann '07; Sämann '10

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU \, e^{-N^2 \varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- The most recent result is Sämann '15

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon \frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2 \frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4 \frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ - \varepsilon^4 \frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ - \varepsilon^3 \frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

where

$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

- The standard treatment of such multitrace matrix model yields a very unpleasant behaviour. Self interaction is way too strong in the important region.

Hermitian matrix model of fuzzy field theories

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

- Recall the perturbative action

$$\begin{aligned} S_{eff}(M) =& \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \end{aligned}$$

The first line is the first terms of the small c_2 expansion with $c_2 \to c_2 - c_1^2$.

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- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

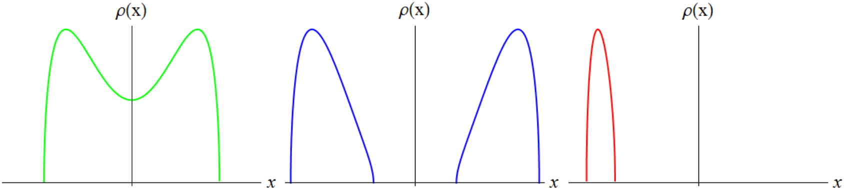
- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)$$

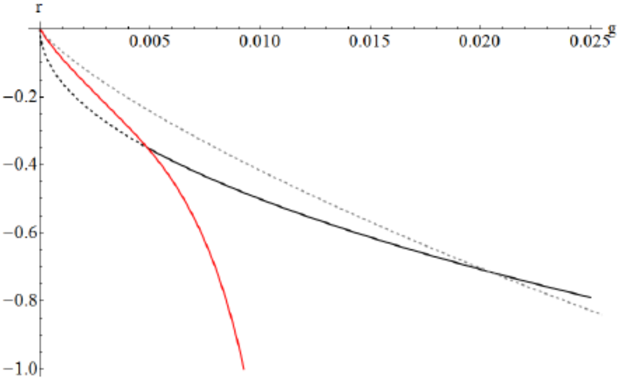
Polychronakos '13; JT '15, JT '17

Matrix models of fuzzy field theories

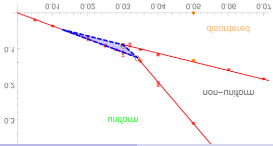
- Such F introduces a (not too strong) interaction among the eigenvalues. For some values of r, g an asymmetric configuration can become stable.
- It corresponds to the "standard" symmetry broken phase.



Hermitian matrix model



JT '17



- A very good qualitative agreement. A very good quantitative agreement in the critical coupling.
- Different value for the critical mass parameter and different behaviour of the asymmetric transition line for large $-r$.
- We need to include $\mathcal{R}$ in a nonperturbative way.
  work in progress with M. Šubjaková

- Recall the perturbative action

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\underbrace{\left(c_2 - c_1^2\right)}_{t_2} - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$

$$- \varepsilon^4\frac{1}{3456}\left[\underbrace{\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2}_{t_4 - 2t_2^2}\right]^2 -$$

$$- \varepsilon^3\frac{1}{432}\left[\underbrace{c_3 - 3c_1c_2 + 2c_1^3}_{t_3}\right]^2 \approx$$

$$\approx \frac{1}{2}F_2[t_2] + F_3[t_3] + F_4[t_4 - 2t_2^2]$$

- Find a function which gives a correct perturbative expansion and behaves well close to the triple point. E.g.

$$n\log\left(1+A\frac{t^2}{n}\right)\ ,\ \frac{1}{\left(1+A\frac{t^2}{n}\right)^2}-1\ ,$$

$$-An\log\left(1+\frac{t^2}{n}\right)\ ,\ A\left(\frac{1}{\left(1+A\frac{t^2}{n}\right)^2}-1\right)\ .$$

- So far it either does barely anything or completely ruins the model.

Conclusions

- Symmetry breaking in noncommutative field theory is (very) different than in the commutative case.
- Matrix models are a great tool to analyze the(se) properties of scalar field theories on fuzzy spaces, and beyond.

Outlook

To do list.

- Find (a more) complete understanding of the matrix model.
- Investigate matrix models corresponding to spaces beyond the fuzzy sphere.
- Investigate matrix models corresponding to theories without the UV/IR mixing.

Thank you for your attention!

If time permits I

- Recall the perturbative action

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$

$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2\right) - 3c_1^4 - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$

$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

$$= \frac{1}{2}\frac{1}{2}c_2 - \frac{1}{4}c_1^2 - \frac{1}{24}c_2^2 - \frac{1}{432}c_3^2 - \frac{1}{3456}c_4^2 + \dots$$

- This part can be interpreted as an additional two-particle interaction.

If time permits I

- Recall the perturbative action

$$S_{eff}(M) = \frac{1}{2}\frac{1}{2}c_2 - \frac{1}{4}c_1^2 - \frac{1}{24}c_2^2 - \frac{1}{432}c_3^2 - \frac{1}{3456}c_4^2 + \dots$$

- Function of the form

$$S_{eff} = \sum_{i,j} a \log(1 - b\,\lambda_i\lambda_j)$$

with $a = 3/2, b = 1/6$ correctly reproduces all four known coefficients.

If time permits II

Investigate matrix models corresponding to theories without the UV/IR mixing.

- For a noncommutative theory with no UV/IR mixing, the extra phase should not be present in the commutative limit of the phase diagram.
  - B.P. Dolan, D. O'Connor and P. Prešnajder [arXiv:0109084],
  - H. Grosse and R. Wulkenhaar [arXiv:0401128],
  - R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa [arXiv:0802.0791].
- Understanding the phase diagram of such theories, especially mechanism of the removal of the striped phase could teach us a lot technically and conceptually.

If time permits III