Phase diagram of modified scalar field theory on fuzzy sphere

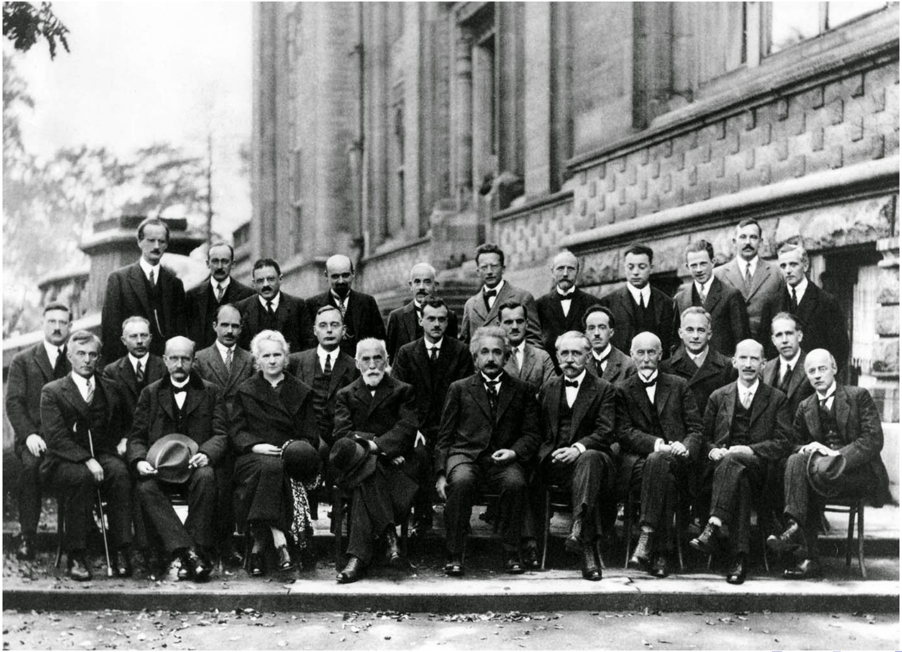
Juraj Tekel

Department of Theoretical Physics
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

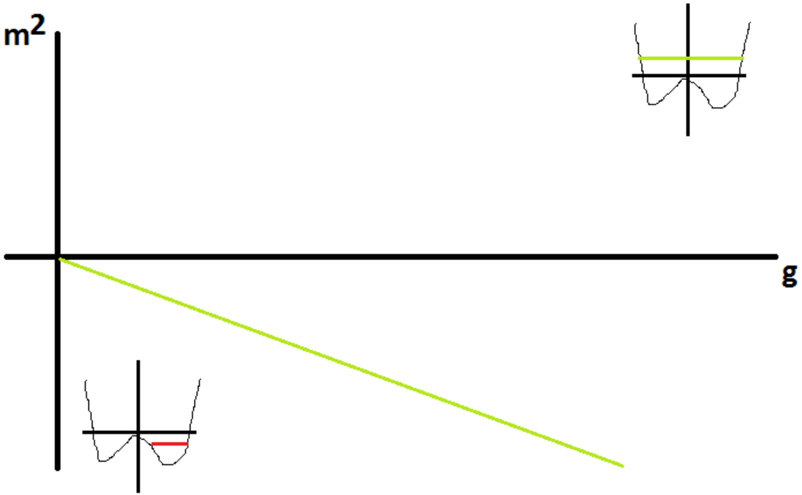
Action MP 1405
Quantum Structure of Spacetime

Physical applications of fuzzy spaces, 17.1.2019, COST QSpace Bruxelles Meeting

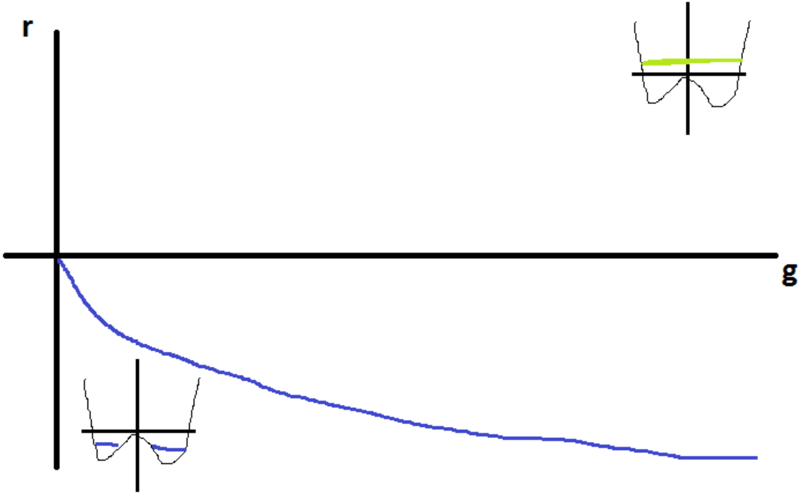
[1711.02008 [hep-th]],[1802.05188 [hep-th]], work in progress



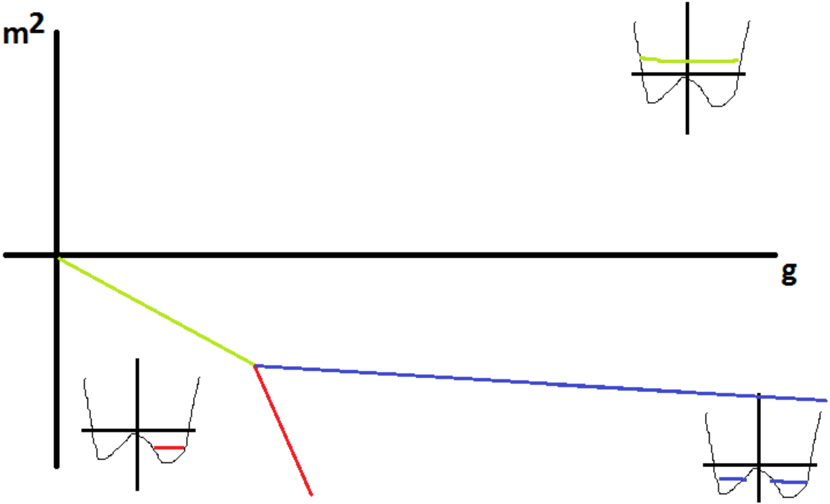
$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + g\phi^4\right)$$



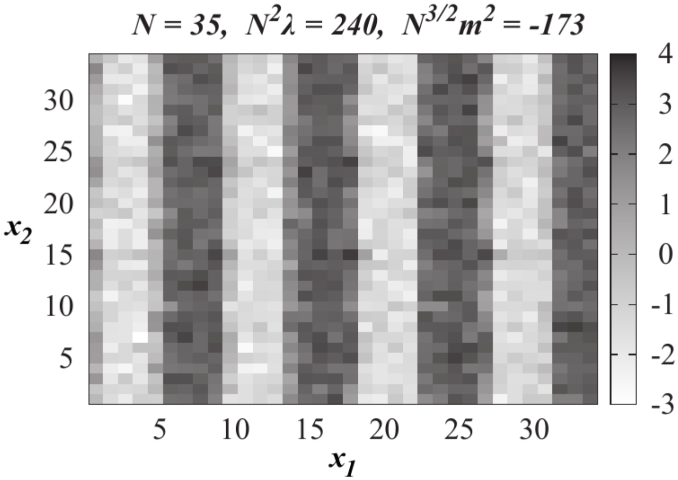
$$S[M] = \mathrm{Tr}\left(\frac{1}{2} r M^2 + g M^4\right)$$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + gM^4\right)$$



Mejía-Díaz, Bietenholz, Panero '14



Introduction and outline

In this seminar, I will

- briefly describe fuzzy field theories and the UV/IR mixing,
- describe fuzzy field theories in terms of a random matrix model,
- and investigate properties of models which should eventually describe a theory without the UV/IR mixing.

Fuzzy field theories

Scalar field theory on fuzzy sphere

- **Commutative**

$$S(\Phi) = \int dx \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

$$\langle F \rangle = \frac{\int D\Phi\, F(\Phi) e^{-S(\Phi)}}{\int D\Phi\, e^{-S(\Phi)}} \,.$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2}[L_i,[L_i,M]] + \frac{1}{2}m^2 M^2 + V(M) \right]$$

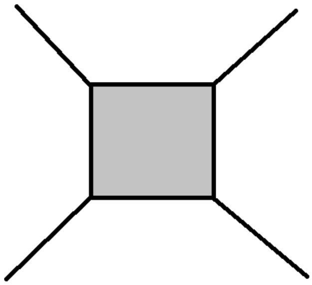
$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \,.$$

Grosse, Klimčík, Prešnajder '90s

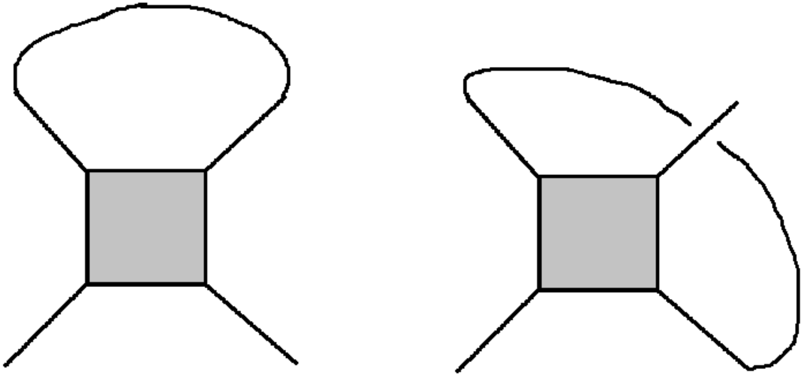
Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03

Scalar field theory on fuzzy sphere

$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm} T_{lm}$$

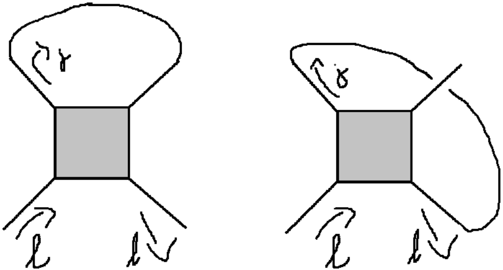


Scalar field theory on fuzzy sphere



UV/IR mixing

UV/IR on fuzzy sphere, Chu, Madore, Steinacker '01



$$I^P = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}$$

$$I^{NP} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} (-1)^{l+j+N-1} \left\{ \begin{array}{ccc} l & s & s \\ j & s & s \end{array} \right\} , \; s = \frac{N-1}{2}$$

UV/IR on fuzzy sphere, Chu, Madore, Steinacker '01

$$I^{NP} - I^{P} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} \left[(-1)^{l+j+N-1} \left\{ \begin{array}{ccc} l & s & s \\ j & s & s \end{array} \right\} - 1 \right]$$

- This difference is finite in $N \to \infty$ limit.
- One can get quite far for small l.
- $N \to \infty$ limit of the effective action is different from the standard S^2 effective action.
- In the planar limit $S^2 \to \mathbb{R}^2$ one recovers singularities and the standard UV/IR-mixing.

Removal of UV/IR mixing on the fuzzy sphere

Removal of UV/IR mixing on S_F^2, Dolan, O'Connor, Prešnajder '01

- These problems are genuine for the two point functions and there is no such anomaly in coupling renormalization.
- By properly modifying the kinetic term of the original naive theory one can subtract the problematic anomalous term

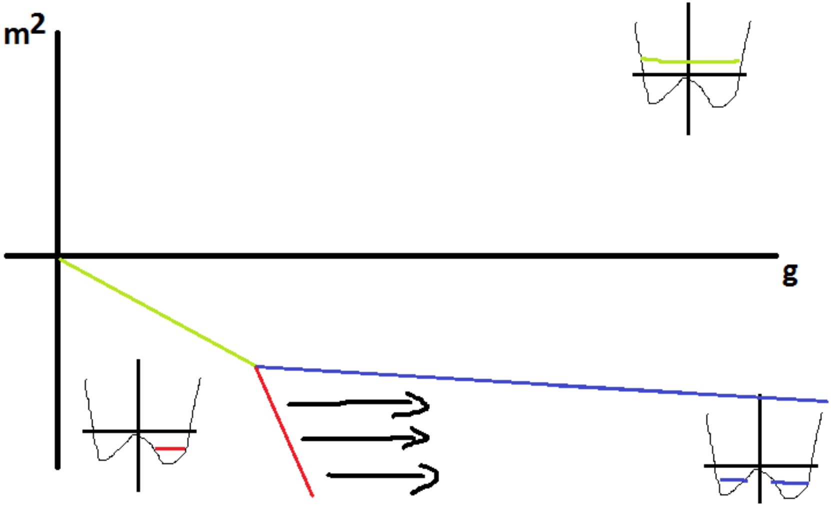
$$S = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] - 12gMQM + \frac{1}{2}m^2M + gM^4\right)$$

where

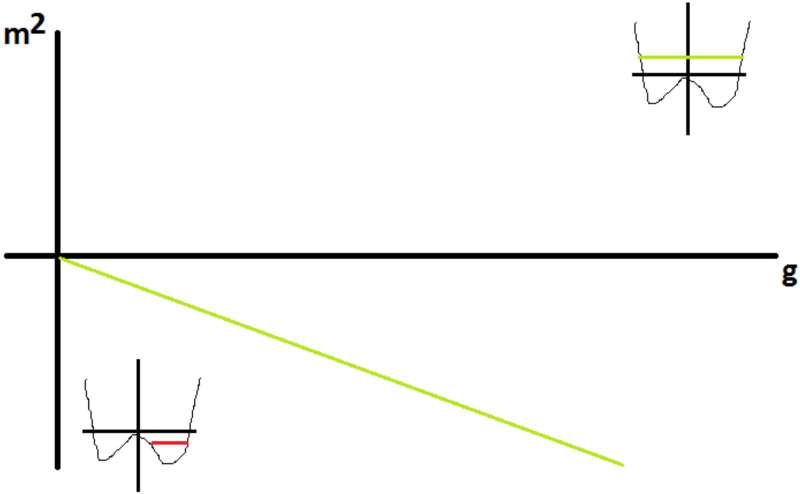
$$QT_{lm} = \underbrace{\left(\sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}\left[(-1)^{l+j+N-1}\begin{Bmatrix} l & s & s \\ j & s & s \end{Bmatrix} - 1\right]\right)}_{Q(l)} T_{lm} \ .$$

- How does the phase diagram of this theory look?

Removal of UV/IR mixing on S_F^2



Removal of UV/IR mixing on S_F^2



Second moment multitrace matrix model for fuzzy field theory

Matrix models

- Ensemble of hermitian $N \times N$ matrices with a probability measure $S(M)$ and expectation values

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \,.$$

- This is the very same expression as for the real scalar field.
- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2} \mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

(minus the red Brezin, Itzykson, Parisi, Zuber '78)

Matrix models of fuzzy field theories

- The large N limit of the model **without** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  is **well** understood.

- The key is diagonalization and the saddle point approximation.

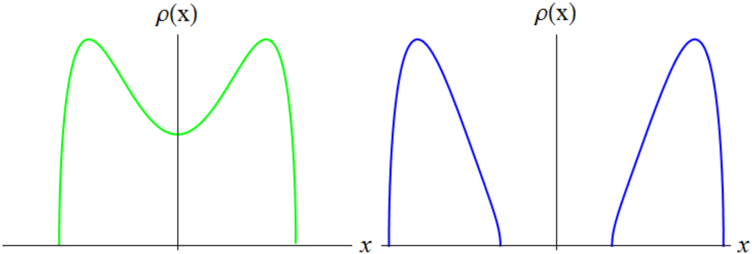
Matrix models of fuzzy field theories

- The large N limit of the model **without** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

is **well** understood.

- The key results is that for $r < -4\sqrt{g}$ we get two cut eigenvalue density.



Matrix models of fuzzy field theories

- The model **with** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  is **not well** understood.

  Steinacker '05; JT Acta Physica Slovaca '15

- The key issue being that diagonalization no longer straightforward.

Matrix models of fuzzy field theories

- The model **with** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  is **not well** understood.

  Steinacker '05; JT Acta Physica Slovaca '15

- The key issue being that diagonalization no longer straightforward.
- We are to compute integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[\frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Matrix models of fuzzy field theories

- The model **with** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  is **not well** understood.
  Steinacker '05; JT Acta Physica Slovaca '15
- The key issue being that diagonalization no longer straightforward.
- We are to compute integrals like

$$\langle F\rangle \sim \int\left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[S_{eff}(\lambda_i)+\frac{1}{2}r\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- How to compute S_{eff}?

Hermitian matrix model of fuzzy field theories

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}\ ,\ \ c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

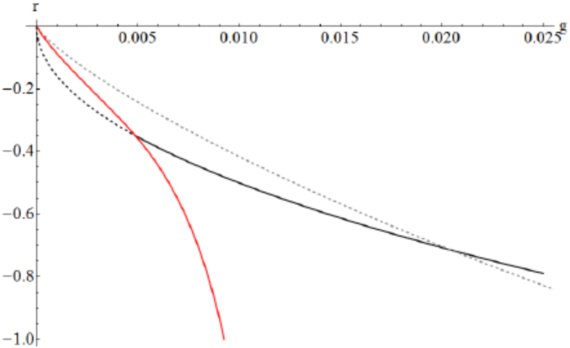
- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)\quad ,\quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)$$

Polychronakos '13; JT '15, JT '17

Matrix models of fuzzy field theories

- Such F introduces a (not too strong) interaction among the eigenvalues. For some values of r, g an asymmetric configuration can become stable.
- It corresponds to the "standard" symmetry broken phase.



JT '17; Šubjaková, JT '19

Matrix models of fuzzy field theories

- This is result of an analytic calculation.
- A very good qualitative agreement. A very good quantitative agreement in the critical coupling.
- Different value for the critical mass parameter and different behaviour of the asymmetric transition line for large $-r$.
- We need to include $\mathcal{R}$ in a nonperturbative way.
  work in progress with M. Šubjaková

Towards a matrix model of UV/IR free theory

Towards a matrix model of UV/IR free theory

- We would like to analyze the more complicated model

$$S = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] - a12gMQM + \frac{1}{2}m^2M + gM^4\right)$$

where

$$QT_{lm} = \underbrace{\left(\sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+m^2}\left[(-1)^{l+j+N-1}\begin{Bmatrix} l & s & s \\ j & s & s\end{Bmatrix} - 1\right]\right)}_{Q(l)} T_{lm} \ .$$

- The previous method works for any model with a kinetic term $\mathcal{K}$, which is diagonal in T_{lm} basis

$$\mathcal{K}T_{lm} = K(l)T_{lm} \ .$$

$$K(l) = l(l+1) - a12gQ(l) \ .$$

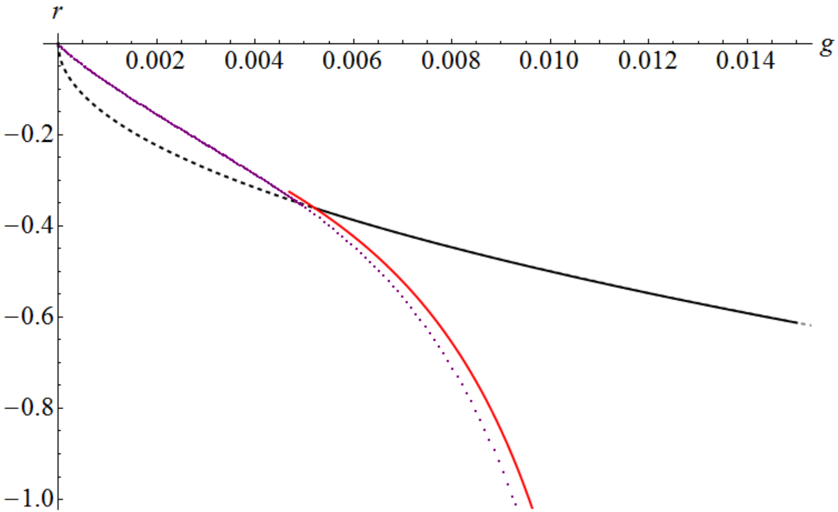
Towards a matrix model of UV/IR free theory

- Operator Q can be expressed as a power series in $C_2 = [L_i, [L_i, \cdot]]$

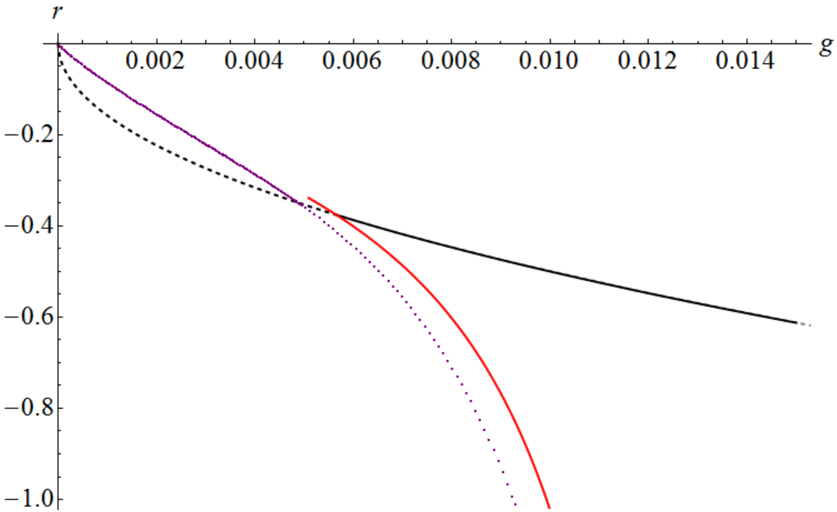
$$Q = q_1 C_2 + q_2 C_2^2 + \ldots$$

- As a starting point, it is interesting to see the phase structure of such simplified model.
  O'Connor, Säman '07

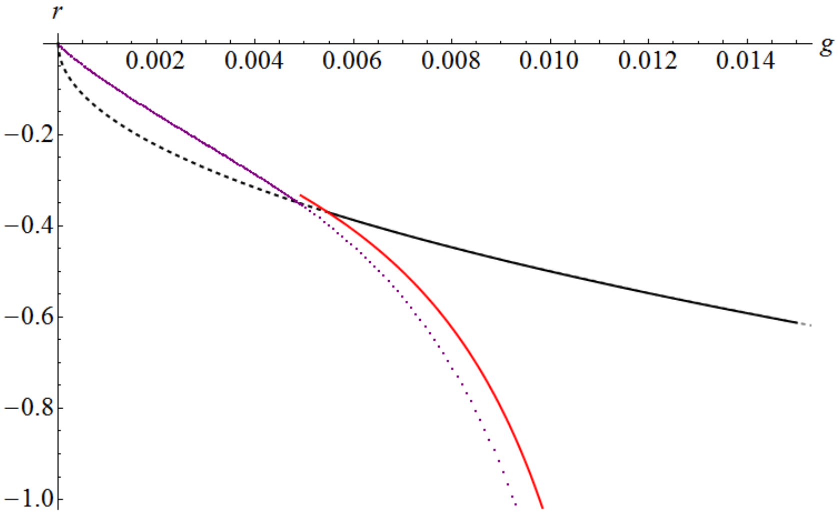
Towards a matrix model of UV/IR free theory



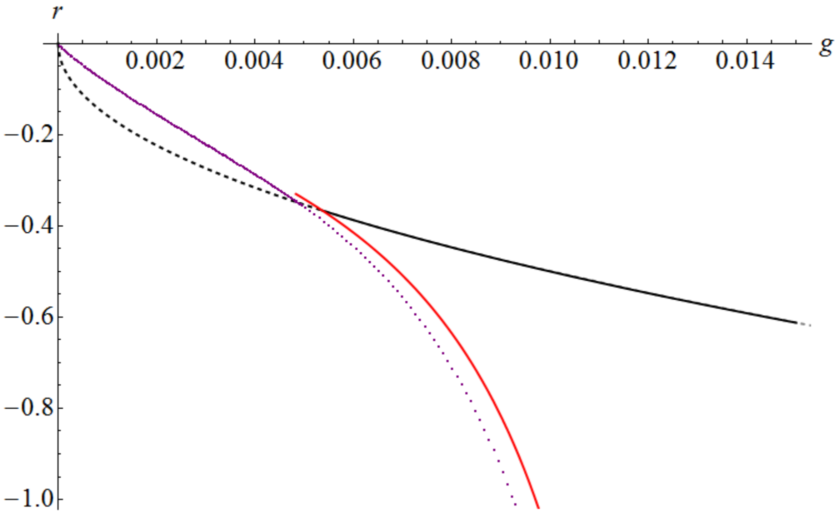
Towards a matrix model of UV/IR free theory



Towards a matrix model of UV/IR free theory



Towards a matrix model of UV/IR free theory



Conclusions and outlook

Conclusions

- We can achieve movement in the phase diagram by modifying the kinetic term of the theory.
- Making steps in the direction of the UV/IR free theory produces expected results.
- But there is plenty more.

Outlook

- Further analysis beyond $Q = q_1 C_2 + q_2 C_2^2$.
- Numerical analysis of the $\mathcal{K} = C_2 - 12gQ$ model.
- What about four dimensions. Especially $\mathbb{C}P^2$. second moment approximation.
- More complete analysis of the matrix model beyond the second moment approximation.

If time permits

Matrix models of fuzzy field theories

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M.
  O'Connor, Sämann '07; Sämann '10

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU \, e^{-N^2 \varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- The most recent result is Sämann '15

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

  where

$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

- The standard treatment of such multitrace matrix model yields a very unpleasant behaviour. Self interaction is way too strong in the important region.

Matrix models of fuzzy field theories

- Recall the perturbative action

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\underbrace{\left(c_2 - c_1^2\right)}_{t_2} - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$

$$- \varepsilon^4\frac{1}{3456}\left[\underbrace{\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2}_{t_4 - 2t_2^2}\right]^2 -$$

$$- \varepsilon^3\frac{1}{432}\left[\underbrace{c_3 - 3c_1c_2 + 2c_1^3}_{t_3}\right]^2 \approx$$

$$\approx \frac{1}{2}F_2[t_2] + F_3[t_3] + F_4[t_4 - 2t_2^2]$$

Matrix models of fuzzy field theories

- Find a function which gives a correct perturbative expansion and behaves well close to the triple point. E.g.

$$n \log\left(1 + A\frac{t^2}{n}\right) \;,\; \frac{1}{\left(1 + A\frac{t^2}{n}\right)^2} - 1 \;,$$
$$-An \log\left(1 + \frac{t^2}{n}\right) \;,\; A\left(\frac{1}{\left(1 + A\frac{t^2}{n}\right)^2} - 1\right) \;.$$

- So far it either does barely anything or completely ruins the model.