Fuzzy field theories and new random matrix ensembles

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Outline

I will talk about

- random matrix ensembles,
- fuzzy spaces,
- ensembles of random matrices related to physics on fuzzy spaces.

Take home message

- Random matrix ensembles are more than just a bunch of random numbers in a table.
- Fuzzy spaces are finite mode approximations to compact manifolds.
- Scalar field theories on such spaces are described in terms of a very specific (and very purely understood) hermitian random matrix models.

Random matrices

Random matrices

$$\begin{pmatrix} -1.97209 & -0.0976152 & 1.35614 & 0.223808 & 1.11521 \\ 0.0626392 & 0.0996544 & 1.24676 & 0.178807 & 0.890936 \\ -0.352318 & 1.04726 & -0.416029 & -3.24653 & 1.36851 \\ -0.150889 & 0.083049 & 1.05206 & 0.622012 & -0.266355 \\ 1.29318 & -0.260398 & -1.36629 & 0.311455 & -0.0599934 \end{pmatrix}$$

Random matrices

$$\begin{pmatrix} -0.467628 & -0.293526 + 0.259101i & 0.208354 - 0.510098i \\ -0.293526 - 0.259101i & -0.422052 & 0.752265 + 0.0954037i \\ 0.208354 + 0.510098i & 0.752265 - 0.0954037i & 0.0384826 \end{pmatrix}$$

Random matrices

- Matrices random entries.
  Ensemble of matrices M, measure on this set dM and probability distribution
  $$P(M_{11}, M_{12}, \ldots) = P(M)\ .$$
- Expected value of some function f of the matrix is
  $$\langle f\rangle = \frac{1}{Z}\int dM\, f(M)P(M)\ .$$
- E.g.
  $$f(M) = M_{11}\ ,\ f(M) = M^2\ ,\ f(M) = \mathrm{Tr}\left(M^{12}\right)\ ,\ f(M) = \frac{1}{N}\mathrm{Tr}\left(M^{12}\right)\ .$$
- Interesting cases are $N = 1, N = 2, N \to \infty$.

Random matrices

- An important example - ensemble of $N \times N$ hermitian matrices with independent, normally distributed entries

$$p(M_{ii}) = N(0,1)\ ,\ p(Re\, M_{ij}) = p(Im\, M_{ij}) = N(0,1/2)$$

and

$$P(M) = \left[\prod_{i=1}^{N} p(M_{ii})\right]\left[\prod_{i<j} p(Re\, M_{ij}) p(Im\, M_{ij})\right] = e^{-\frac{1}{2}\mathrm{Tr}\left(M^2\right)}$$

and

$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij} Im\, M_{ij}\right]\ .$$

- Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$.
- Requirement of such invariance is very restrictive.

Random matrices - applications

- Originally - spectra of large nuclei.
- Large color limits of QCD.
- Discretization of surfaces.
- Quantum chaos.
- Mesoscopic physics.
- Riemann conjecture.
- Integrable hierarchies.
- Free probability.

Random matrices - eigenvalue decomposition

- We have
$$dM\,P(M) = dM\,e^{-\left[\frac{1}{2}\mathrm{Tr}\left(M^2\right)+\dots\right]} = dM\,e^{-V(M)}\ .$$
- If we ask invariant questions, we can turn
$$\langle f\rangle = \frac{1}{Z}\int dM\,f(M)P(M)$$
into an eigenvalue problem by diagonalization $M = U\Lambda U^\dagger$ for some $U\in SU(N)$ and $\Lambda = diag(\lambda_1,\dots,\lambda_N)$, the integration measure becomes
$$dM = dU\left(\prod_{i=1}^N d\lambda_i\right)\times\prod_{i<j}(\lambda_i-\lambda_j)^2$$
- We are to compute integrals like
$$\langle f\rangle \sim \int\left(\prod_{i=1}^N d\lambda_i\right) f(\lambda_i)\,e^{-\left[\sum_i V(\lambda_i)-2\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}\times\int dU$$

Random matrices - eigenvalue decomposition

- Term
$$2\sum_{i<j}\log|\lambda_i-\lambda_j|$$
is of order N^2 if $\lambda_i \sim 1$. Potential term
$$\sum_i V(\lambda_i)$$
is of order N.
- We need to enhance the probability measure by a factor of N to
$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i)-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
- This makes the N^2 dependence explicit.

Random matrices - eigenvalue decomposition

- We introduce eigenvalue distribution

$$\rho(\lambda) = \frac{1}{N} \sum_j \delta(\lambda - \lambda_j)$$

which gives for the averages

$$\langle f \rangle = \sum_i \rho(\lambda_i) f(\lambda_i) \ .$$

- The question is, how does do probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]}$$

translate into eigenvalue distribution ρ.

Random matrices - large N

- For finite N - orthogonal polynomials method.
- For $N \to \infty$ the question simplifies due to the factor N^2

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i) - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]} .$$

- For large N only configurations with small exponent contribute significantly to the integral. In the limit $N \to \infty$ only the extremal configuration

$$V'(\lambda_i) - \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} = 0 \quad \forall i$$

- Like a gas of particles with logarithmic repulsion.

Random matrices - large N

- In the $N \to \infty$ limit sums become integrals. We introduce resolvent

$$\omega(z) = \int dy \frac{\rho(y)}{z-y}$$

and

$$V'(\lambda_i) - \frac{1}{N}\sum_{i\neq j}\frac{1}{\lambda_i - \lambda_j} = 0 \quad \forall i$$

becomes a Riemann-Hilbert problem

$$\omega(x+i0^+) + \omega(x-i0^+) = V'(x)\ ,\ x \in \operatorname{supp}\rho\ .$$

- The eigenvalue distribution is then given by

$$\rho(\lambda) = -\frac{1}{2\pi i}\left[\omega(x+i0^+) - \omega(x-i0^+)\right]\ .$$

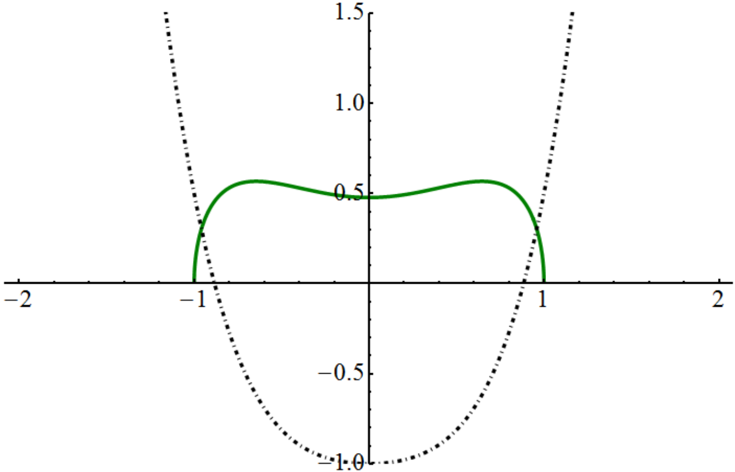
Random matrices - quartic potential

- The simplest case

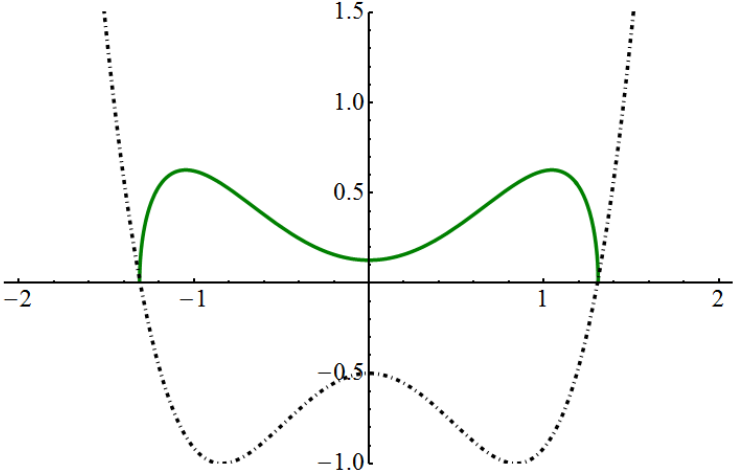
$$V(x) = \frac{1}{2}rx^2 + gx^4$$

- For $r > 0$ eigenvalues spread around the minimum at $x = 0$. For $r < 0$, peak can be high enough to spread the eigenvalues.
- This happens for $r = -4\sqrt{g}$.

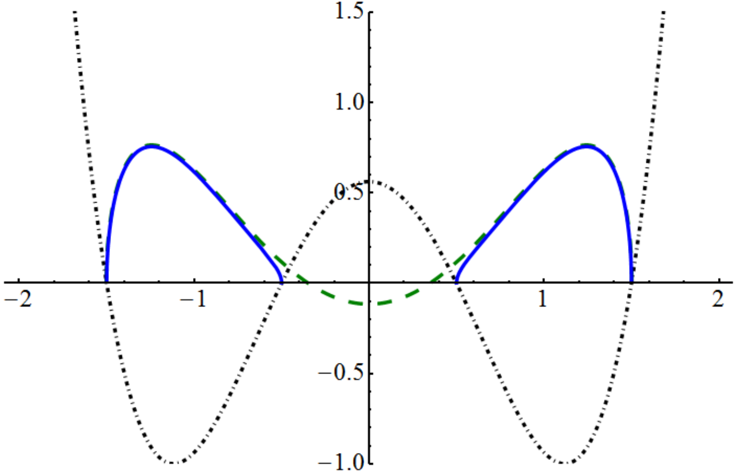
Random matrices - quartic potential



Random matrices - quartic potential



Random matrices - quartic potential



Fuzzy spaces

Fuzzy spaces - introduction

- Part of a more general concept of noncommutative spaces.
  Fuzzy spaces have finite volume.
- Promoting the phase space of QM to a true geometric space.
- The most important property is a "lattice" like short distance structure, but with continuous symmetries.

Fuzzy spaces - introduction

quantum theory + general relativity

||

?!?!?!

Fuzzy spaces - introduction

quantum theory + general relativity

$\Downarrow$

some nontrivial short distance structure of space

Doplicher, Fredenhagen, Roberststs '95

Fuzzy spaces - introduction

- To measure an event of spatial extent Δx, we need a particle with a similar wavelength. According to de Broglie, this particle has energy

$$E \sim \frac{1}{\Delta x} \ .$$

- As we lower Δx beyond a certain point, the concentration of energy will create a black hole. The result of the measurement will be hidden under the event horizon of this black hole and we can not obtain the information we were after.
- Rather unsurprisingly, this will happen at the Planck scale

$$R_S = \frac{2GM}{c^2} \ , \ E = Mc^2 \ , \ E = \frac{hc}{\lambda} \ \Rightarrow \ L = \sqrt{2} \underbrace{\sqrt{\frac{hG}{c^3}}}_{l_{pl}} \ .$$

Fuzzy spaces - introduction

quantum theory + general relativity

$\Downarrow$

some nontrivial short distance structure of space

$\Uparrow$

space noncommutatitvity

Fuzzy spaces - construction

Fuzzy spaces - construction

Fuzzy sphere Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder '90s; Steinacker '13

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- They form an (commutative) algebra and all the information about the sphere is encoded in this algebra (Gelfand, Naimark).
- To describe features at a length scale δx we need functions Y_m^l with

$$l\sim\frac{1}{\delta x}\ .$$

Fuzzy spaces - construction

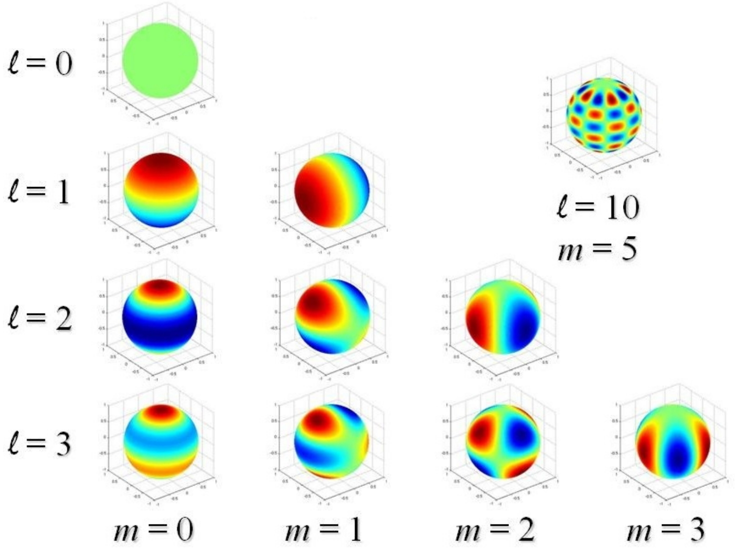


Image taken from http://principles.ou.edu/mag/earth.html

Fuzzy spaces - construction

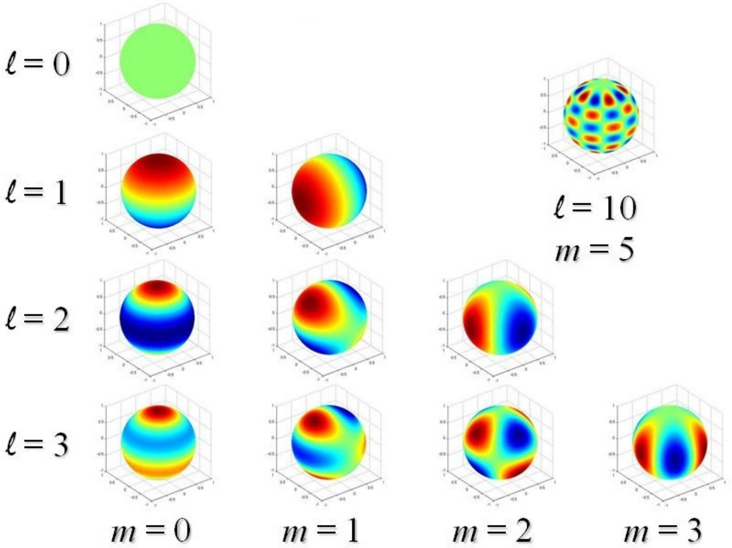
- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

we will not be able to see any features of functions at length scales under $\sim 1/L$.

- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces - construction



Fuzzy spaces - construction

- Number of independent functions with $l \leq L$ is

$$\sum_{l=0}^{L} \sum_{m=-l}^{l} 1 = (L+1)^2 = N^2 \ .$$

This is the same as the number of $N \times N$ hermitian matrices

$$N + 2\sum_{n=1}^{N}(n-1) = N^2 \ .$$

The idea is to map the former on the latter and borrow a closed product from there.

- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

Fuzzy spaces - construction

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product
$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$
- We have obtained a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces - an alternative construction

- The regualar sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \; ,$$

which generate the following algebra of functions

$$f = \left\{ \sum_{k \in \mathbb{N}^3} \left(a_{k_1 k_2 k_3} \prod_{i=1}^{3} x_i^{k_i} \right) \middle| x_i x_i = R^2 \right\} \; ,$$

which is by definition commutative.

- Information about the sphere is again hidden in this algebra.

Fuzzy spaces - an alternative construction

- For the fuzzy sphere S_F^2 we define

$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta\varepsilon_{ijk} x_k \ .$$

- Such x_i's generate a different, noncommutative algebra and S_F^2 is an object, which has this algebra as an algebra of functions.
- The conditions can be realized as an $N = 2j + 1$ dimensional representation of $SU(2)$

$$x_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} j(j+1) = r^2 \ .$$

- The group $SU(2)$ still acts on x_i's and this space enjoys a full rotational symmetry.
- And again, in the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces - an alternative construction

- x_i's are $N \times N$ matrices, functions on S_F^2 are combinations of all their possible products and thus hermitian matrices M.
- Such $N \times N$ matrix can be decomposed into

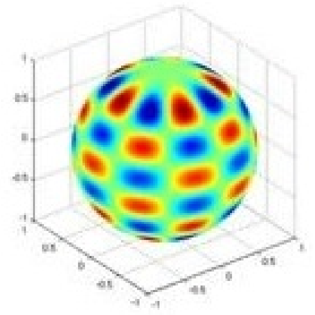
$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{m=l} c_{lm} T_{lm} \ .$$

where matrices T^l_m are called polarization tensors and

$$\begin{aligned} T^l_m &= \varphi(Y^l_m) \ , \\ \mathrm{Tr}\,(T_{lm} T_{l'm'}) &= \delta_{ll'} \delta_{mm'} \ , \\ [L_i, [L_i, T_{lm}]] &= l(l+1) T_{lm} \ . \end{aligned}$$

Fuzzy spaces - an alternative construction

- Either way, we have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.

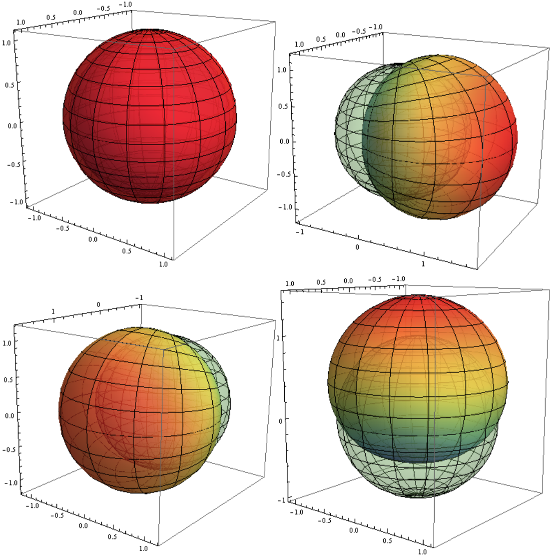


- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Fuzzy spaces - examples of function/matrix correspondence

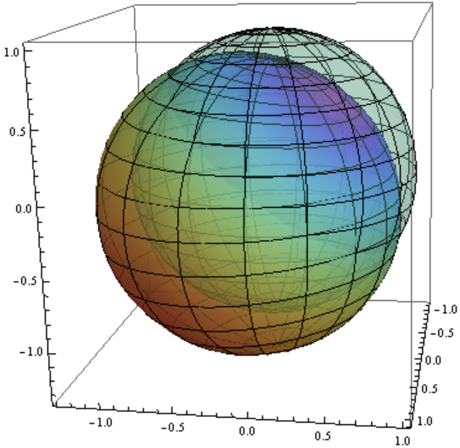
Examples of function/matrix correspondence

$N = 2$



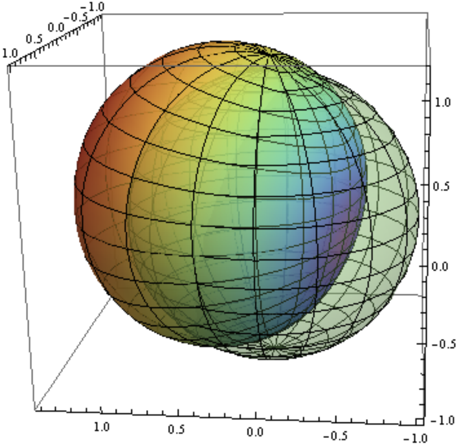
Examples of function/matrix correspondence

$$\begin{pmatrix} 0.153116 & -0.299646 + 0.0471715i \\ -0.299646 - 0.0471715i & 1.2326 \end{pmatrix}$$



Examples of function/matrix correspondence

$$\begin{pmatrix} 0.0948057 & -0.196209 - 0.572162i \\ -0.196209 + 0.572162i & -0.344324 \end{pmatrix}$$



Fuzzy physics - scalar field theory

Fuzzy physics - scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int dx \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \ .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

Fuzzy physics - scalar field theory

- **Commutative**

$$S(\Phi) = \int dx \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right]$$

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} \ .$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr}\left[\frac{1}{2} M \frac{1}{R^2}[L_i,[L_i,M]] + \frac{1}{2}m^2 M^2 + V(M)\right]$$

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \ .$$

Balachandran, Kürkçüoğlu, Vaidya ’05; Szabo ’03

Random matrix models of scalar field theories

Models of scalar fields

- Fuzzy field theory = Ensemble of hermitian $N \times N$ matrices with

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r \mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

- We are to compute integrals like

$$\langle F \rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[\frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum \lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]}$$

$$\times \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Models of scalar fields

- Fuzzy field theory = Ensemble of hermitian $N \times N$ matrices with

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

- We are to compute integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[S_{eff}(\lambda_i)+\frac{1}{2}r\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Steinacker '05

- How to compute S_{eff}?

Models of scalar fields - perturbative

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. O'Connor, Sämann '07; Sämann '10

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2 \varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- The most recent result is Sämann '15

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ - \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ - \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

where

$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

- Standard treatment of such multitrace matrix model yields a very unpleasant behaviour. Self interaction is way too strong in the important region.

Models of scalar fields - perturbative

- Interesting structure

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}t_2 - \varepsilon^2\frac{1}{24}t_2^2 + \varepsilon^4\frac{1}{2880}t_2^4\right] - \\ - \varepsilon^4\frac{1}{3456}\left[t_4 - 2t_2^2\right]^2 - \\ - \varepsilon^3\frac{1}{432}t_3^2$$

where

$$t_n = \frac{1}{N}\mathrm{Tr}\left[\left(M - \frac{1}{N}\mathrm{Tr}\left(M\right)\right)^n\right]$$

- First line - quadratic, second line vanishes for semicircle.

Models of scalar fields - nonperturbative

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues.
  Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2 - c_1^2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2 - c_1^2}{1 - e^{-(c_2 - c_1^2)}}\right) + \mathcal{R}$$

- We get a the following fuzzy-field-theory-like matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  Better behaved for S_F^2 but still not complete.
  Polychronakos '13; JT '15, JT '17
- In general function F given by the properties of the fuzzy space.

Models of scalar fields - different (?) approximation (?)

- It can be shown

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2 \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$
$$\approx \int dU\, e^{-N^2 \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [E,[E,U\Lambda U^\dagger]]\right)}$$

  for a single matrix E with given eigenvalues.
  Steinacker '16

- This is still an approximation, but a significantly simpler integral to compute.
- The Harish-Chandra-Itzykson-Zuber integral formula might be of use.

$$\int dU\, e^{\mathrm{Tr}\left(AUBU^\dagger\right)} = \left(\prod_{p=1}^{N-1} p!\right) \frac{\det e^{a_i b_j}}{\Delta(a)\Delta(b)}$$

  a_i - eigenvalues of A, b_j - eigenvalues of B, Δ - Vandermonde

Models of scalar fields - different approximation

- Recall the perturbative action

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2\right) - 3c_1^4 - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$
$$= \frac{1}{2}\frac{1}{2}c_2 - \frac{1}{4}c_1^2 - \frac{1}{24}c_2^2 - \frac{1}{432}c_3^2 - \frac{1}{3456}c_4^2 + \dots$$

- Function of the form

$$S_{eff} = \sum_{i,j} a\log\left|1 - b\,\lambda_i\lambda_j\right|$$

with $a = 3/2, b = 1/6$ correctly reproduces all four known coefficients.

Conclusion

Conclusion - the new random matrix ensembles

- Scalar field theories on fuzzy spaces give rise to certain specific hermitian random matrix ensembles.
- The new ingredient is in a term in the probability distribution coming from the kinetic term of the theory.
- The issue is computation of the following integral over the $U(N)$ group

$$\int dU\, e^{-N^2 \frac{1}{2} \mathrm{Tr}\left(U \Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

  L_i's are the generators of $SU(2)$ in the N dimensional representation for the case of S_F^2, more complicated for other spaces.

Thank you for your attention!

If time permits - Fuzzy spaces in physics

- Regularization of infinities in the standard QFT.
  Heisenberg ∼'30; Snyder '47, Yang '47
- Regularization of field theories for numerical simulations.
  Panero '16
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  Seiberg Witten '99; Douglas, Nekrasov '01
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  Steinacker '13
- Geometric unification of the particle physics and theory of gravity.
  van Suijlekom '15
- An effective description of various systems in a certain limit (eg. QHE).
  Karabali, Nair '06

If time permits - Symmetry breaking in NC field theories

- From now on ϕ^4 theory.
- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.

  Glimm, Jaffe, Spencer '75; Chang '76

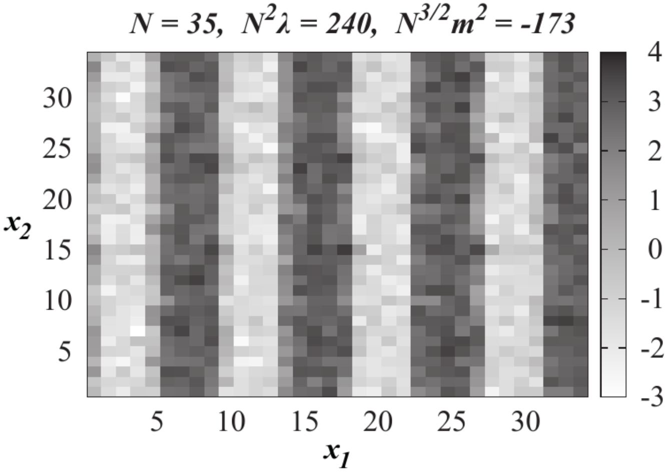
  Loinaz, Willey '98; Schaich, Loinaz '09
- In disorder phase the field oscillates around the value $\phi = 0$.
- In uniform order phase the field oscillates around a nonzero value which is a minimum of the potential.

If time permits - Symmetry breaking in NC field theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; G.-H. Chen and Y.-S. Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero '14; Mejía-Díaz, Bietenholz, Panero '14; Medina, Bietenholz, D. O'Connor '08; Bietenholz, Hofheinz, Nishimura '04; Lizzi, Spisso '12; Ydri, Ramda, Rouag '16
  Panero '15
- This phase is a result of some very interesting features/bugs of the theory.

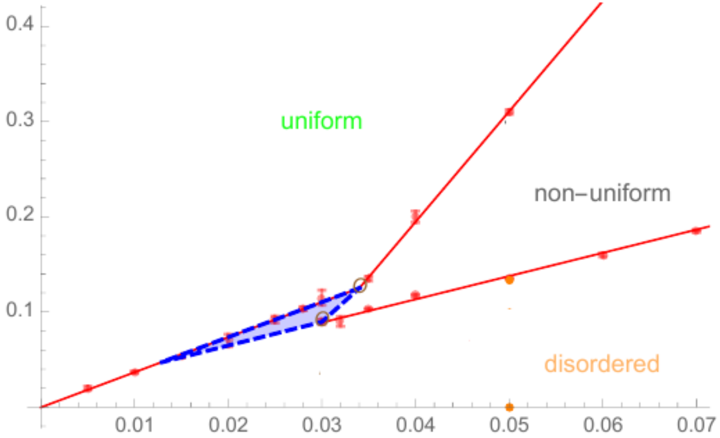
If time permits - Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero '14 for $\mathbb{R}^2_\theta$



If time permits - Symmetry breaking in NC field theories

O'Connor, Kováčik '18 for S_F^2



If time permits - Symmetry breaking in NC field theories

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$