Second moment fuzzy-field-theory-like matrix models

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arXiv:2002.02317 [hep-th], work with M. Šubjaková

The law of instrument

"If the only tool you have is a hammer,
it is tempting to treat everything as if it were a nail."

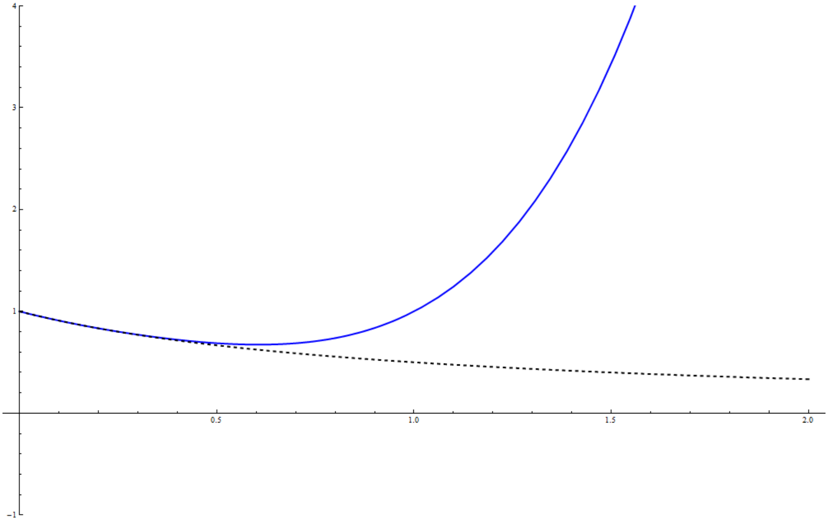
Abraham Harold Maslow (1908 – 1970)

Pade approximation

Pade approximation - Geometric series

$$1 - x + x^2 - x^3 + x^4$$

Pade approximation - Geometric series



Pade approximation - General

- Start with a series

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{N} a_n x^n$$

- Construct rational function

$$\frac{b_0 + b_1 x + \ldots + b_k x^k}{1 + c_1 x + \ldots + c_l x^l}$$

such that its expansion agrees with the first N terms of the series.

- Good, news. Mathematica can do it for you.

Pade approximation - General

- Plenty of interesting questions and uses.
- For us only a tool **to extend perturbative results**.

Take home message

Take home message

- Fuzzy spaces are (among other things) toy models of spaces with quantum structure.
- Fuzzy scalar field theories are described by matrix models.
- As such, they are a great laboratory to investigate consequences of quantum structure.

Second moment **fuzzy-field-theory**-like matrix models

Fuzzy spaces

Fuzzy sphere Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder '90s

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta,\phi) \ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi) \ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

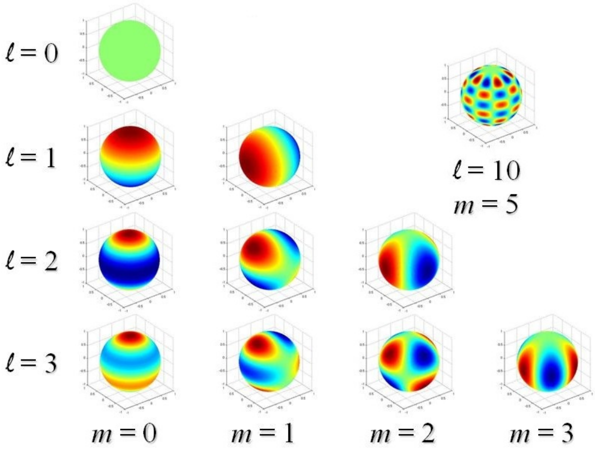


Image taken from http://principles.ou.edu/mag/earth.html

Fuzzy spaces

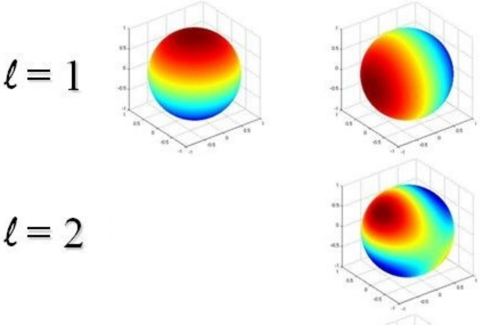
- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

we will not be able to see any features of functions under certain length scales.

- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces



- Number of independent functions with $l \leq L$ is N^2, the same as the number of $N \times N$ hermitian matrices.
  The idea is to map the former on the latter and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} * Y_{l'm'} := \varphi^{-1} \left(\varphi \left(Y_{lm} \right) \varphi \left(Y_{l'm'} \right) \right) \ .$$

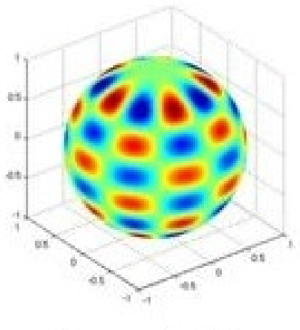
- We have a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- In the limit N or $L \to \infty$ we recover the original sphere.

Fuzzy spaces - an alternative construction

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.



- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Fuzzy spaces

- Regularization of infinities in the standard QFT.
  Heisenberg ∼'30; Snyder '47, Yang '47
- Regularization of field theories for numerical simulations.
  Panero '16
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  Seiberg Witten '99; Douglas, Nekrasov '01
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  Steinacker '13
- Geometric unification of the particle physics and theory of gravity.
  van Suijlekom '15
- An effective description of various systems in a certain limit (eg. QHE).
  Karabali, Nair '06

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} \ .$$

- We construct the noncommutative theory as an analogue with
  - field $\to$ matrix,
  - functional integral $\to$ matrix integral,
  - spacetime integral $\to$ trace,
  - derivative $\to$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \, .$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right]$$

$$\langle F \rangle = \frac{\int dM \, F(M) e^{-S(M)}}{\int dM \, e^{-S(M)}} \, .$$

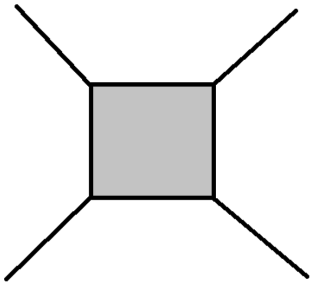
Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03; Ydri '16

Fuzzy scalar field theory - UV/IR mixing

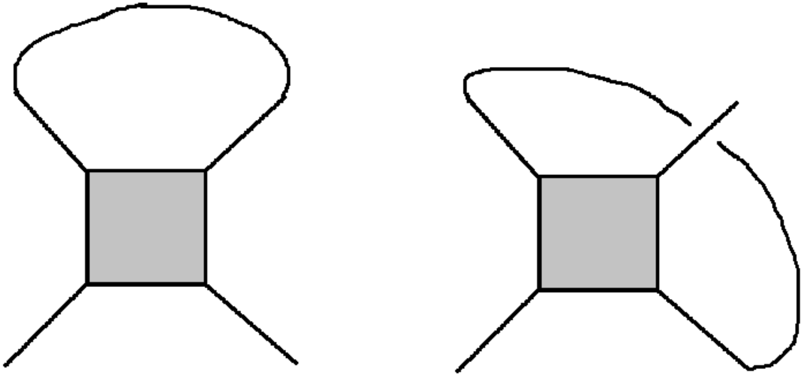
- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- In terms of diagrams different properties of planar and non-planar ones.
  The (matrix) vertex is not invariant under permutation of incoming momenta.

Fuzzy scalar field theory - UV/IR mixing

$$M=\sum_{l=0}^{N-1}\sum_{m=-l}^{l}c_{lm}T_{lm}\ ,\ \mathrm{Tr}\left(M^4\right)=\sum_{l_{1\ldots4}}\sum_{m_{1\ldots4}}c_{l_1,m_1}c_{l_2,m_2}c_{l_3,m_3}c_{l_4,m_4}\mathrm{Tr}\left(T_{l_1,m_1}T_{l_2,m_2}T_{l_3,m_3}T_{l_4,m_4}\right)$$

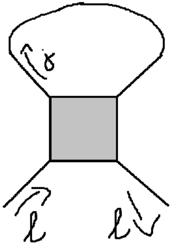


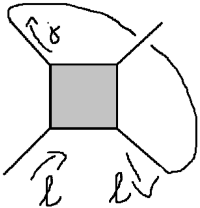
Fuzzy scalar field theory - UV/IR mixing



Fuzzy scalar field theory - UV/IR mixing

Chu, Madore, Steinacker '01





$$I^P = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}$$

$$I^{NP} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}(-1)^{l+j+N-1}\begin{Bmatrix} l & s & s \\ j & s & s \end{Bmatrix}, \quad s = \frac{N-1}{2}$$

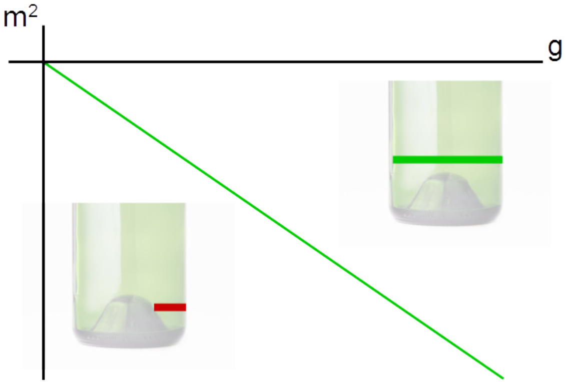
Fuzzy scalar field theory - UV/IR mixing

$$I^{NP} - I^{P} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} \left[(-1)^{l+j+N-1} \left\{ \begin{array}{ccc} l & s & s \\ j & s & s \end{array} \right\} - 1 \right]$$

- This difference is finite in $N \to \infty$ limit.
- $N \to \infty$ limit of the effective action is different from the standard S^2 effective action. Regularization of the field theory by NC space is anomalous.
- In the planar limit $S^2 \to \mathbb{R}^2$ one recovers singularities and the standard UV/IR-mixing.
- The space (geometry) forgets where it came from, but the field theory (physics) remembers its fuzzy origin.

Symmetry breaking in NC field theories

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + g\phi^4 \right)$$



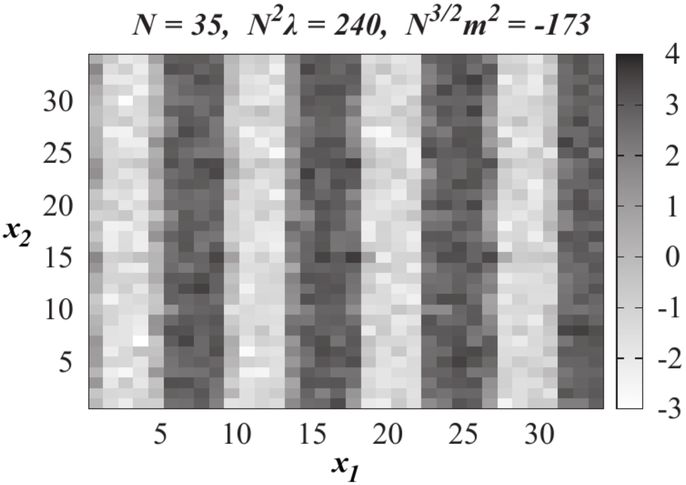
Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75; Chang '76
Loinaz, Willey '98; Schaich, Loinaz '09

Symmetry breaking in NC field theories

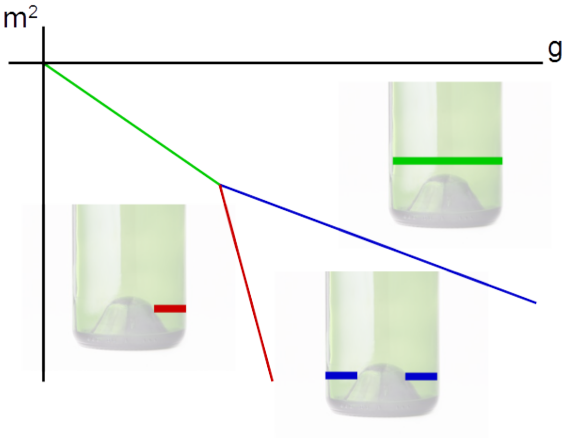
- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; Chen, Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero '14; Mejía-Díaz, Bietenholz, Panero '14; Medina, Bietenholz, D. O'Connor '08; Bietenholz, Hofheinz, Nishimura '04; Lizzi, Spisso '12; Ydri, Ramda, Rouag '16; Kováčik, O'Connor '18
  Panero '15

Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero '14 for $\mathbb{R}^2_\theta$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$



Second moment fuzzy-field-theory-like **matrix models**

Random matrices

$$\begin{pmatrix} -1.97209 & -0.0976152 & 1.35614 & 0.223808 & 1.11521 \\ 0.0626392 & 0.0996544 & 1.24676 & 0.178807 & 0.890936 \\ -0.352318 & 1.04726 & -0.416029 & -3.24653 & 1.36851 \\ -0.150889 & 0.083049 & 1.05206 & 0.622012 & -0.266355 \\ 1.29318 & -0.260398 & -1.36629 & 0.311455 & -0.0599934 \end{pmatrix}$$

Random matrices

$$\begin{pmatrix} -0.467628 & -0.293526 + 0.259101i & 0.208354 - 0.510098i \\ -0.293526 - 0.259101i & -0.422052 & 0.752265 + 0.0954037i \\ 0.208354 + 0.510098i & 0.752265 - 0.0954037i & 0.0384826 \end{pmatrix}$$

Random matrices

- Matrices random entries.
  Ensemble of matrices M, measure on this set dM and probability distribution

$$P(M_{11}, M_{12}, \ldots) = P(M) \ .$$

- Expected value of some function f of the matrix is

$$\langle f \rangle = \frac{1}{Z} \int dM\, f(M) P(M) \ .$$

- E.g.

$$f(M) = M_{11} \ , \ f(M) = M^2 \ , \ f(M) = \mathrm{Tr}\left(M^{12}\right) \ , \ f(M) = \frac{1}{N} \mathrm{Tr}\left(M^{12}\right) \ .$$

- Interesting cases are $N = 1, N = 2, N \to \infty$.

Random matrices

- An important example - ensemble of $N \times N$ hermitian matrices with

$$P(M) = e^{-\mathrm{Tr}(V(M))}\ ,\ \text{usually}\ V(x) = \frac{1}{2} r x^2 + g x^4$$

  and

$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij} Im\, M_{ij}\right] .$$

- Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$.
- Requirement of such invariance is very restrictive.

- If we ask invariant questions, we can turn

$$\langle f \rangle = \frac{1}{Z} \int dM\, f(M) P(M)$$

into an eigenvalue problem by diagonalization $M = U \Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, the integration measure becomes

$$dM = dU \left(\prod_{i=1}^{N} d\lambda_i \right) \times \prod_{i<j} (\lambda_i - \lambda_j)^2$$

- We are to compute integrals like

$$\langle f \rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i \right) f(\lambda_i)\, e^{-\left[\sum_i V(\lambda_i) - 2 \sum_{i<j} \log |\lambda_i - \lambda_j|\right]} \times \int dU$$

Random matrices - eigenvalue decomposition

- Term
$$2\sum_{i<j}\log|\lambda_i-\lambda_j|$$
is of order N^2 if $\lambda_i \sim 1$. Potential term
$$\sum_i V(\lambda_i)$$
is of order N.
- We need to enhance the probability measure by a factor of N to
$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i)-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
- This makes the N^2 dependence explicit.

Random matrices - eigenvalue decomposition

- We introduce eigenvalue distribution

$$\rho(\lambda) = \frac{1}{N} \sum_j \delta(\lambda - \lambda_j)$$

which gives for the averages

$$\langle f \rangle = \int d\lambda \, \rho(\lambda) f(\lambda) \ .$$

- The question is, how does do probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]}$$

translate into eigenvalue distribution ρ.

Random matrices - large N

- For finite N - orthogonal polynomials method.
- For $N \to \infty$ the question simplifies due to the factor N^2

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i) - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]} .$$

- For large N only configurations with small exponent contribute significantly to the integral. In the limit $N \to \infty$ only the extremal configuration

$$V'(\lambda_i) - \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} = 0 \quad \forall i$$

- Like a gas of particles with logarithmic repulsion. This gives us nice intuition.

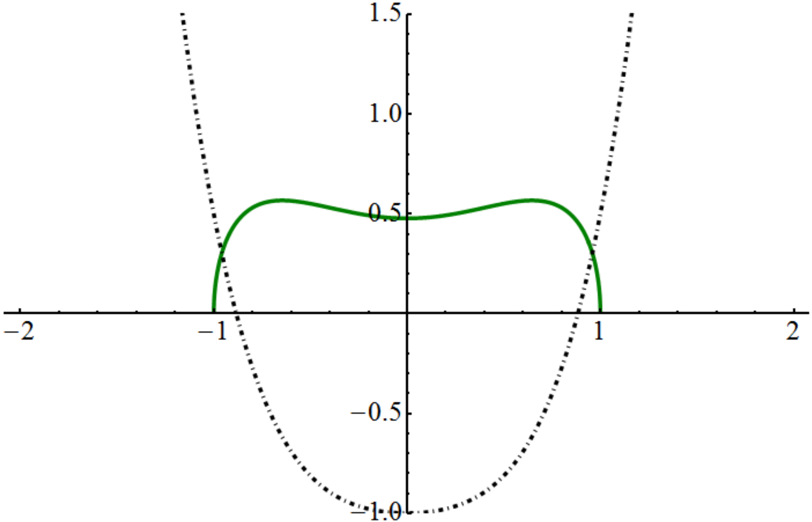
Random matrices - quartic potential

- The simplest case

$$V(x) = \frac{1}{2} r x^2 + g x^4$$

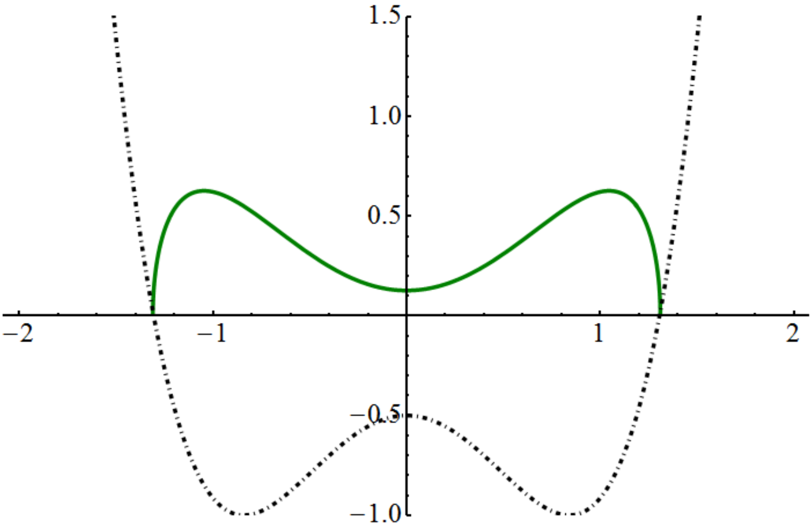
Random matrices - quartic potential

$V(x) = rx^2/2 + gx^4$ and $r > 0$



RANDOM MATRICES - QUARTIC POTENTIAL

$V(x) = rx^2/2 + gx^4$ and $r > 0$



Random matrices - quartic potential

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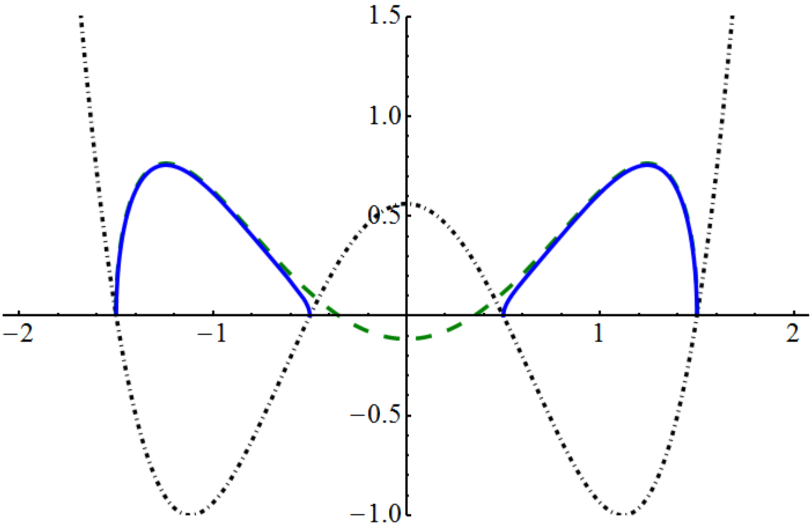

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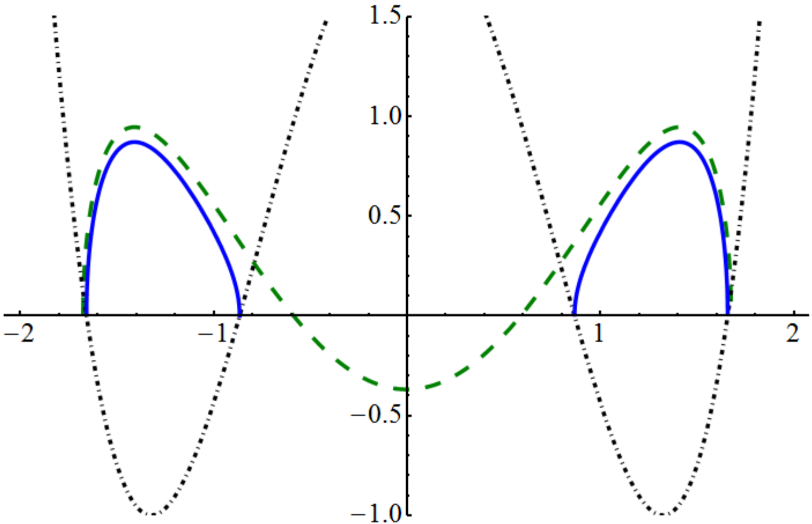

Random matrices - quartic potential

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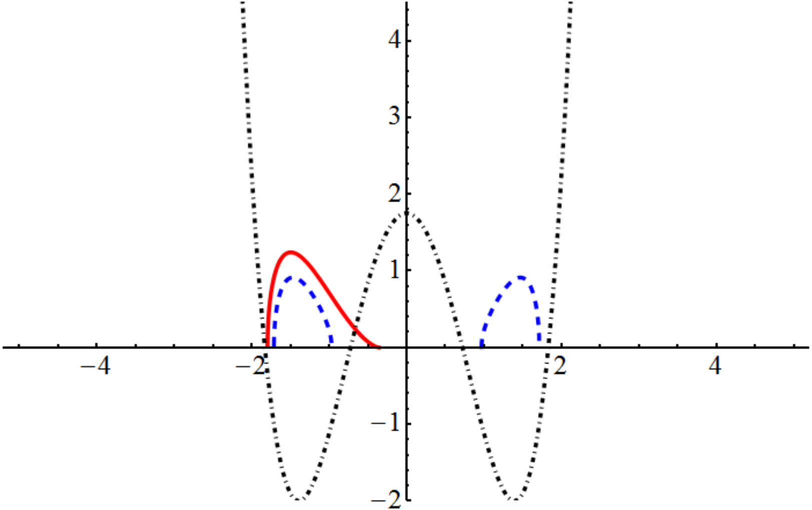


Random matrices - quartic potential

- If more than one solution is possible, the one with lower energy

$$\mathcal{F} = -N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j| \right]$$

  is the preferred one.
- The probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j| \right]}$$

  i.e. the more probable solution.

Second moment fuzzy-field-theory-like matrix models

Second moment approximation

- Recall the action of the fuzzy scalar field theory

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

  This is a particular case of a matrix model since we need

$$\int dM\, F(M) e^{-S(M)}$$

- The large N limit of the model with the kinetic term is not well understood.
  The key issue being that diagonalization $M = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ no longer straightforward.
- Integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) dU\ F(\lambda_i,U)\ e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Second moment approximation

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues.
  Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling.
  Polychronakos '13

$$S_{eff}(\lambda_i) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)$$

Polychronakos '13; JT

Second moment fuzzy-field-theory-**like** matrix models

Fuzzy-field-theory-like models

- We can consider model for a general function $F(x)$

$$S(M) = \frac{1}{2} F(c_2 - c_1^2) + \frac{1}{2} m^2 \,\mathrm{Tr}\left(M^2\right) + g \,\mathrm{Tr}\left(M^4\right)$$

- Such models appear within this approximation for generalized kinetic terms

$$M \frac{1}{2} [L_i, [L_i, M]] \;\rightarrow\; M K\Big([L_i, [L_i, \cdot]]\Big) M$$

for some $K(x)$ i.e. certain higher derivative actions.

Second moment fuzzy-field-theory-like matrix models

General solution

- **The symmetric one-cut solution.** Interval $(-\sqrt{\delta}, \sqrt{\delta})$ and

$$0 = \frac{4 - 3\delta^2 g}{\delta} - r - F'\left(\frac{4\delta + \delta^3 g}{16}\right) ,$$

$$\mathcal{F}_{S1C} = \frac{1}{4} + \frac{9}{128}\delta^4 g^2 + \frac{1}{8} r\delta + \frac{1}{32}\delta^3 g r + \frac{1}{2} F\left(\frac{4\delta + \delta^3 g}{16}\right) - \frac{1}{2}\log\left(\frac{\delta}{4}\right) .$$

- **The symmetric two-cut solution.** Intervals $(-\sqrt{D+\delta}, -\sqrt{D-\delta}) \cup (\sqrt{D-\delta}, \sqrt{D+\delta})$ and

$$0 = 4Dg + r + F'(D) \ , \ \delta^2 = \frac{1}{g} \ , \tag{1}$$

$$\mathcal{F}_{2C} = \frac{3}{8} + D^2 g + \frac{Dr}{2} + \frac{1}{2}F(D) + \frac{1}{4}\log(4g) \ . \tag{2}$$

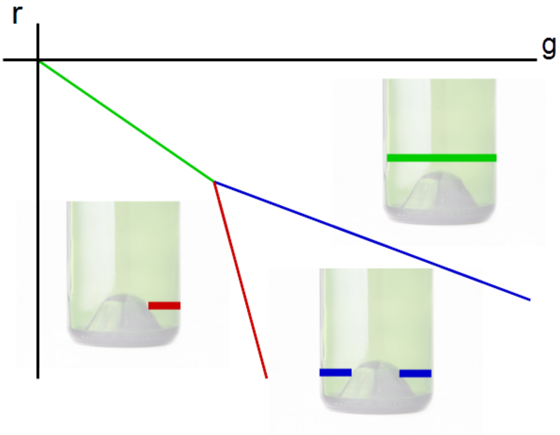
General solution

- **The asymmetric one-cut solution.** Interval $(D-\sqrt{\delta}, D+\sqrt{\delta})$ and

$$0 = 4\frac{4+15\delta^2 g+2r\delta}{\delta(4+9\delta^2 g)} - F'\left(\frac{\delta\left(64+160\delta^2 g+144\delta^4 g^2+81\delta^6 g^3+36\delta^3 gr+27\delta^5 g^2 r\right)}{64(4+9\delta^2 g)}\right),$$

$$\mathcal{F}_{AS1C} = -\frac{1}{128g\left(4+9\delta^2 g\right)^2}\Big[6075\delta^8 g^5+3240\delta^6 g^4(4+r\delta)+144\delta^4 g^3\left(29+40r\delta+3\delta^2 r^2\right)$$
$$+8\delta^2 g^2\left(-144+352r\delta+117\delta^2 r^2\right)+64g\left(-8+8r\delta+9\delta^2 r^2\right)+128r^2\Big]+$$
$$+\frac{1}{2}F\left(\frac{\delta\left(64+160\delta^2 g+144\delta^4 g^2+81\delta^6 g^3+36\delta^3 gr+27\delta^5 g^2 r\right)}{64(4+9\delta^2 g)}\right)-\frac{1}{2}\log\left(\frac{\delta}{4}\right).$$

$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + gM^4\right)$$



General solution

- The idea: for very negative r the terms F' do not play a role.

$$0 = 4Dg + r + F'(D)$$

- In this limit the solution is

$$D = -\frac{r}{4g} + \dots$$

  and corrections determined by the large t behaviour of $F(t)$.
- Similarly for asymmetric one-cut solution

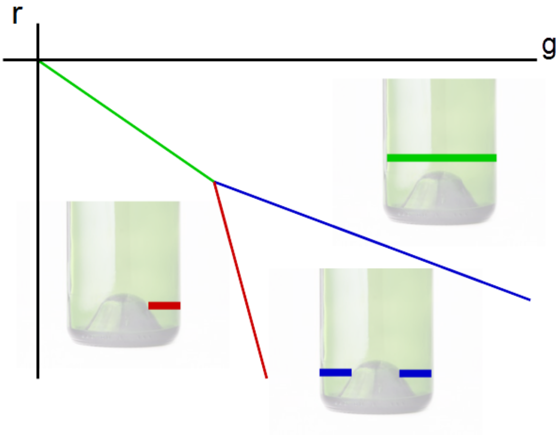
$$\delta = -\frac{2}{r} + \dots$$

  and corrections determined by the small t behaviour of $F(t)$.

General solution

- This leads to large $-r$ expansions of the free energies $\mathcal{F}$. Condition $\mathcal{F}_{AS1C} = \mathcal{F}_{2C}$ is then the phase transition condition.
- Solution for g as a series in powers of $1/r$.
- At the end, we treat it to a Pade approximation to obtain the phase transition line $g(r)$.

$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + gM^4\right)$$

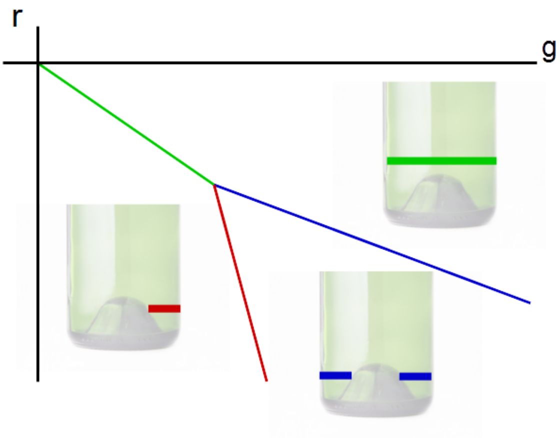


General solution

- The symmetric one-cut to symmetric two-cut phase transition can be solved analytically.

$$r(g) = -4\sqrt{g} - F'\left(\frac{1}{\sqrt{g}}\right)$$

$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + gM^4\right)$$



General solution

- The symmetric to asymmetric one-cut phase transition more tricky.
- Problem: the condition

$$0 = \frac{4 - 3\delta^2 g}{\delta} - r - F'\left(\frac{4\delta + \delta^3 g}{16}\right)$$

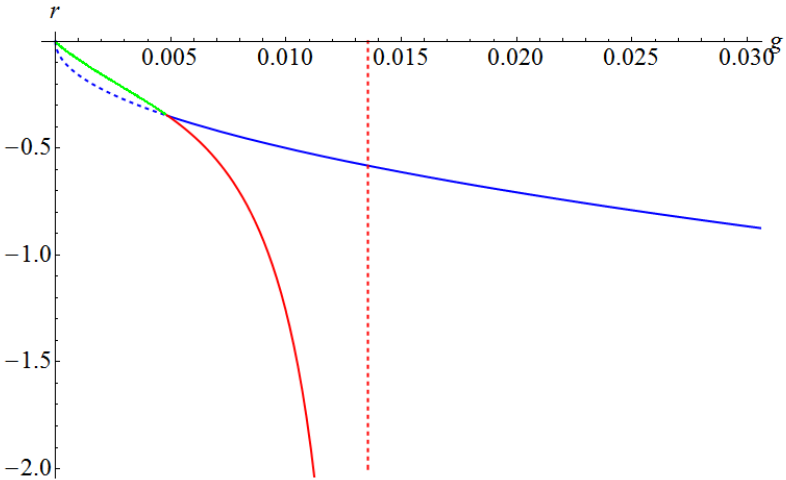
has no reasonable large $-r$ solution.

- Idea: we can solve this condition at the symmetric phase transition phase transition; then look to corrections around this point

$$\delta = \delta_0 + \delta_1(r - r_c) + \delta_2(r - r_c)^2 + \dots \ , \ r_c = -4\sqrt{g} - F'\left(\frac{1}{\sqrt{g}}\right)$$

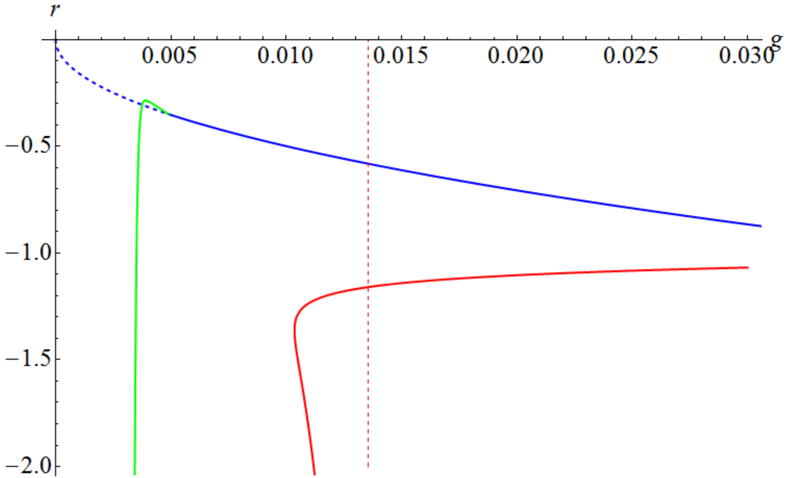
We obtain a usable expression for $\mathcal{F}_{S1C}$ and get $\mathcal{F}_{S1C} = \mathcal{F}_{AS1C}$ as the final phase transition condition.

Results - Fuzzy sphere



JT '18; Šubjaková, JT '20

Results - Fuzzy sphere



Results - Fuzzy sphere

- A very good qualitative agreement. A very good quantitative agreement in the critical coupling.
  Kováčik, O'Connor '18
- Different behaviour of the asymmetric transition line for large $-r$.
- We need to include $\mathcal{R}$, or the higher moments of the matrix, in a nonperturbative way.
  work in progress with M. Šubjaková

Towards a matrix model of UV/IR free theory

- We would like to analyse the more complicated model

$$S = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + 12gMQM + \frac{1}{2}m^2M + gM^4\right)$$

where

$$QT_{lm} = \underbrace{-\left(\sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}\left[(-1)^{l+j+N-1}\left\{\begin{array}{ccc} l & s & s \\ j & s & s \end{array}\right\} - 1\right]\right)}_{Q(l)} T_{lm} \ .$$

- This removes the UV/IR mixing in the theory. Dolan, O'Connor, Prešnajder '01

Towards a matrix model of UV/IR free theory

- Operator Q can be expressed as a power series in $C_2 = [L_i, [L_i, \cdot]]$

$$Q = q_1 C_2 + q_2 C_2^2 + \dots$$

- As a starting point, it is interesting to see the phase structure of such simplified model. O'Connor, Säman '07
- This is the case of

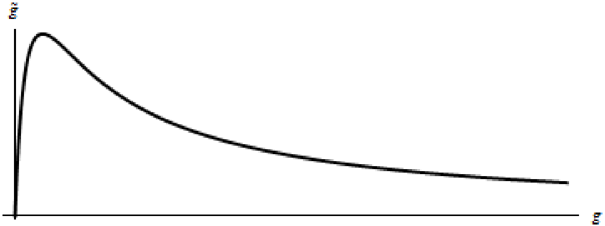
$$K(x) = (1 + ag)x \quad \text{or} \quad K(x) = (1 + ag)x + bg\, x^2 \ .$$

Results - Rescaled kinetic term

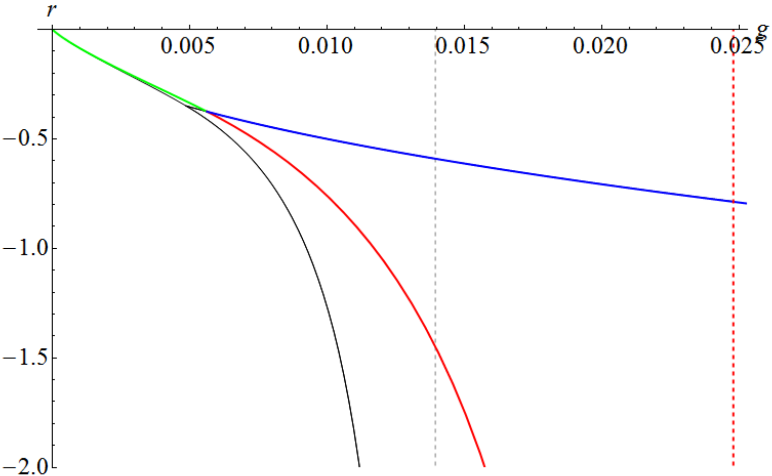
- Rescale the eigenvalues to obtain the original model with

$$\tilde{r} = \frac{r}{1+ag} \quad , \quad \tilde{g} = \frac{g}{(1+ag)^2}$$

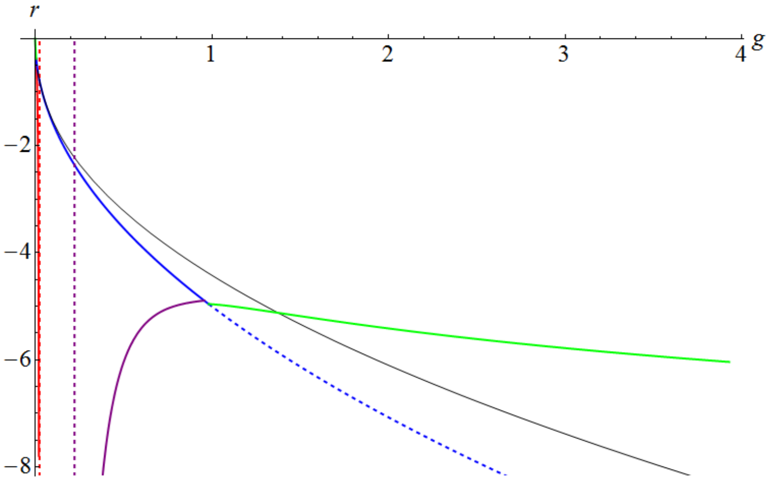
- A deformation of the original phase diagram.



Results - Rescaled kinetic term



Results - Rescaled kinetic term



- Rescale the eigenvalues to obtain the model $K(x) = x + \tilde{b}\,x^2$ and

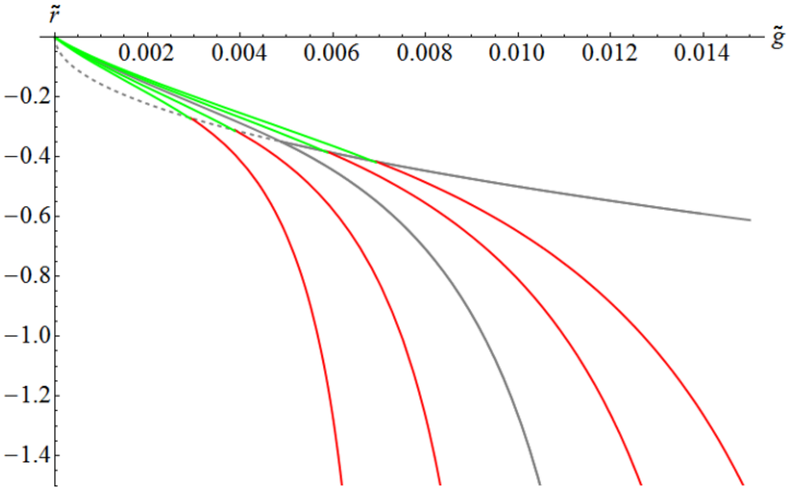
$$\tilde{b} = \frac{b\,g}{1+ag}\ ,\ \tilde{r} = \frac{r}{1+ag}\ ,\ \tilde{g} = \frac{g}{(1+ag)^2}\ .$$

- The same procedure with

$$F(t) \overset{t\to 0}{=} \frac{1}{6}(3+2\tilde{b})t - \frac{16\tilde{b}^2+30\tilde{b}+15}{360}t^2 + \frac{\tilde{b}(64\tilde{b}^2+126\tilde{b}+63)}{11340}t^3 + \dots\ ,$$

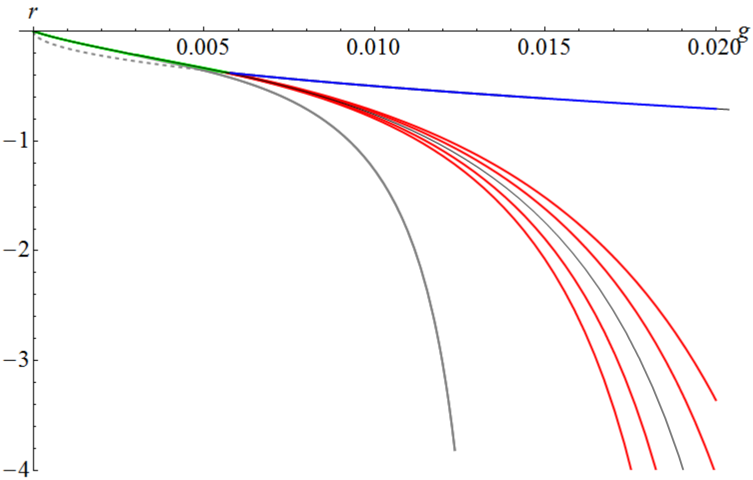
$$\overset{t\to\infty}{=} \log t + \left(1+\frac{1}{\tilde{b}}\right)\log\left(1+\tilde{b}\right) - 1 + \frac{1}{1+\tilde{b}}e^{-t} + \frac{1+\tilde{b}+2\tilde{b}t-\tilde{b}^2(1-2t)}{2(1+\tilde{b})^3}e^{-2t} + \dots\ ,$$

Results - Quadrakinetic term



$$\tilde{b} = -0.5, -0.25, 0.25, 0.5$$

Results - Quadrakinetic term



$a = 3e^{3/2}\ ,\ \ b = -4, -2, 2, 4$

Take home message

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- Fuzzy spaces are (among other things) toy models of spaces with quantum structure.
- Fuzzy scalar field theories are described by matrix models.
- As such, they are a great laboratory to investigate consequences of quantum structure.

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- Fuzzy spaces are (among other things) toy models of spaces with quantum structure.
- Fuzzy scalar field theories are described by matrix models.
- As such, they are a great laboratory to investigate consequences of quantum structure.

Thank you for your attention!

Correlation functions

- Quantity $\langle M(x)M(y)\rangle$ is U dependent, so we need to figure out what to do with

$$\int dU\ F(\Lambda, U)\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}\ .$$

Entanglement entropy

- We need to extended the model to $\mathbb{R}\times S^2_F$, i.e. $M(t)$

$$S(M)=\int dt\mathrm{Tr}\left(-\frac{1}{2}M\partial_t^2 M+\frac{1}{2}M[L_i,[L_i,M]]+\frac{1}{2}m^2M^2+gM^4\right)$$

Medina, Bietenholz, O'Connor '07; Ihl, Sachse, Sämann '10

Also the U dependence will play a role, but free theory where $\mathcal{R}=0$, is enough.

If time permits - GW solution

Grosse, Wulkenhaar '09 '14; Grosse, Sako, Wulkenhaar '16; Panzer, Wulkenhaar '18;
Grosse, Hock, Wulkenhaar '19 '19

- Model

$$S(M) = \mathrm{Tr}\left(EM^2 + gM^4\right)$$

  for a fixed external matrix E has been solved.
- An implicit formula for two point function and formulas for all higher correlation functions in terms of this two point function.
- Challenges: expressions are technically complicated to work with and work only for positive eigenvalues of E.