Fuzzy field theories in the string modes formalism

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Take home message

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- There is an interesting (new) way to describe functions and operators on fuzzy spaces.
- The description works in position space and uses the coherent states also as a basis for the functions and operators.
- It leads to some new insights in the fuzzy scalar field theory.

Fuzzy field theories in the string modes formalism

Fuzzy sphere [Hoppe 1982; Madore 1992; Grosse, Klimčík, Prešnajder 1990s]

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i = 1,2,3 \ ,$$

which generate the following algebra of functions

$$f = \left\{ \sum_{k \in \mathbb{N}^3} \left(a_{k_1 k_2 k_3} \prod_{i=1}^{3} x_i^{k_i} \right) \middle| x_i x_i = R^2 \right\} \ ,$$

which is by definition commutative.

- Information about the sphere is hidden in this algebra.

Fuzzy spaces

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i\hat{x}_i = r^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

- Such $\hat{x}_i$'s generate a different, noncommutative, algebra and S_N^2 is an object, which has this algebra as an algebra of functions.
- The conditions can be realized as an $N = 2s+1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}}L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1}s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces

- $\hat{x}_i$'s are $N \times N$ matrices, functions on S_F^2 are combinations of all their possible products and thus hermitian matrices M.
- Such $N \times N$ matrix can be decomposed into

$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{m=l} c_{lm} T_{lm} \ .$$

where matrices T_{lm} are fuzzy analogues of spherical harmonics Y_{lm} (called polarization tensors)

$$\mathrm{Tr}\left(T_{lm} T_{l'm'}\right) = \delta_{ll'} \delta_{mm'} \ ,$$
$$[L_i, [L_i, T_{lm}]] = l(l+1) T_{lm} \ .$$

- The space of matrices M is N^2 dimensional.

Fuzzy field theories in the string modes formalism

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right]$$

$$\langle F\rangle = \frac{\int d\Phi\, F(\Phi)e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}}\ .$$

- **Noncommutative** (for S^2_F)

$$S(M) = \frac{4\pi r^2}{N}\mathrm{Tr}\left[\frac{1}{2}M\frac{1}{r^2}[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + V(M)\right]$$

$$\langle F\rangle = \frac{\int dM\, F(M)e^{-S(M)}}{\int dM\, e^{-S(M)}}\ .$$

[Balachandran, Kürkçüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]

- $N \to \infty$ limit of the effective action is different from the standard S^2 effective action [Chu, Madore, Steinacker 2001]

$$S_{\text{one loop}} = S_0 + \frac{1}{2}\int d^2x\,\phi(x)^2\delta m^2 - \frac{g}{12\pi}\int d^2x\,\phi(x)h(\tilde{\Delta})\phi(x) + \dots$$

$$\tilde{\Delta} Y_{lm} = l\, Y_{lm}\ ,\ \ h(n) = \sum_{k=1}^{n}\frac{1}{k}\ .$$

There is an extra, mildly nonlocal, term.

Fuzzy field theories in **the string modes** formalism

String modes - coherent states

- Natural basis in the auxiliary hilbert space $\mathcal{H}$ is the "spin" basis

$$|n\rangle = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \;,\; n = -s, \ldots, s \;,$$

  derived from the highest weight state $|s\rangle$.
- For any $x \in S^2$ with radius 1, choose some $g_x \in SO(3)$ such that $x = g_x \cdot p$, where p is the north pole on S^2. We define [Perelomov 1986]

$$|x\rangle = g_x \cdot |s\rangle \;,\; g_x \in SU(2)$$

  and call the set of all $|x\rangle$ the coherent states.
- $|x\rangle$ is located around x, but is an element of $\mathcal{H}$, and is a noncommutative analogue of the point x. [Steinacker 2020]

String modes - coherent states

- They form an over-complete set in $\mathcal{H}$ and

$$\mathbb{1} = \frac{N}{4\pi}\int d^2x\,|x\rangle\langle x| \quad , \quad \mathbb{1} = \sum_n |n\rangle\langle n| \ .$$

- They are orthogonal only in the large N limit

$$|\langle x|y\rangle|^2 = \left(\frac{1 + x\cdot y}{2}\right)^{N-1} \ .$$

String modes - coherent states

- Coherent states can be used to map (quantize) functions on S^2 on matrices

$$\phi(x) \to M = \int d^2x\, \phi(x)|x\rangle\langle x| \ .$$

  and matrices on functions (de-quantize)

$$M \to \phi(x) = \langle x| M |x\rangle \ .$$

- This maps T_{lm} on Y_{lm} up to normalization

$$T_{lm} \to \langle x| T_{lm} |x\rangle = \frac{1}{c_l} Y_{lm}(x) \ , \ c_l^2 = \frac{1}{4\pi} \frac{(N-1-l)!(N+l)!}{((N-1)!)^2} \sim \frac{N}{4\pi} e^{\frac{l^2}{N}} \ .$$

- For $l < \sqrt{N}$ coefficients c_l are approximately constant, quantization and de-quantization are inverse of each other.
  For $l > \sqrt{N}$ coefficient c_l grows extremely fast and the-quantized matrices are misleading.

String modes - representation of functions on S_F^2

- Functions on the fuzzy sphere are matrices acting on $\mathcal{H}$

$$M = \sum_{m,n=-s}^{s} M_{mn} \left| m \right\rangle \left\langle n \right| \ .$$

- We can express the matrix M in a similar fashion using the coherent states

$$M = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y\, \phi(x,y) \left| x \right\rangle \left\langle y \right| \ .$$

- Objects [Iso, Kawai, Kitazawa 2000; Steinacker 2016]

$$\left| x \right\rangle \left\langle y \right| =: \left| {x \atop y} \right)$$

form a basis of functions on the fuzzy sphere and we will call them the **string modes**.

String modes - representation of functions on S_F^2

- Such representation of matrix M by function $\phi(x, y)$ seems to be not unique (way more functions than matrices).
  But one can show that derivatives of $\phi(x, y)$ are bounded by $\sqrt{N}$, which means that the Fourier modes of ϕ to be restricted by $l_x, l_y \leq \sqrt{N}$.
- Functions $\phi(x, y)$ that represent functions on the fuzzy sphere have rather mild behavior. The coherent states are spread out over an area $\sim 4\pi/N$ and average out any larger oscillations.
- Large momentum UV wavelengths are smoothed out on the fuzzy sphere. But the price we pay is non-local string modes.

$$\left| \begin{matrix} x \\ y \end{matrix} \right)$$

- **Short modes** for $|x-y| < 1/\sqrt{N}$ can be shown to represent localized wave-packets with momentum $\sim N|x-y|$.
  This is the classical regime.
- Particularly string mode $\left| \begin{matrix} x \\ x \end{matrix} \right)$ represents a maximal localized function around point x, i.e. a fuzzy version of δ-function.
  Functions with $\phi(x,y) = \phi(x)\delta(x,y)$ are local and become the standard functions on S^2 in the commutative limit.
- **Long modes** for $|x-y| > 1/\sqrt{N}$ are non-local and have no classical analogue.
  This is the non-commutative regime.

String modes - representation of operators on functions

- When working with functions we encounter operators

$$\mathcal{O} : M \to \mathcal{O}(M) \ .$$

- For example the kinetic term of the field theory or the propagator of the theory

$$[L_i, [L_i, M]] =: \Box M \quad , \quad \frac{1}{\Box + m^2} \ .$$

- String modes are eigenfunctions of $\Box$

$$\Box \left|{x \atop y}\right) = \left(\frac{N^2}{4} |x - y|^2 + N \right) \left|{x \atop y}\right) \ .$$

String modes - representation of operators on functions

[Steinacker, T work in progress]

- A general representation of such operators in terms of the string modes is straightforward

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2x'\, d^2y\, d^2y' \left|{x \atop y}\right)\mathcal{O}(x,y;x',y')\left({x' \atop y'}\right| .$$

- There are two special cases
  - Local

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop x}\right)\mathcal{O}^L(x,y)\left({y \atop y}\right| .$$

  - Non-local, but diagonal,

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop y}\right)\mathcal{O}^D(x,y)\left({x \atop y}\right| .$$

- Functions $\mathcal{O}_L$ and $\mathcal{O}_D$ may have very different behavior for different operators (oscillation, singularity). Local representations are typically highly oscillatory, non-local representations are better behaved.

- Operator traces

$$\mathrm{Tr}\,\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle|\mathcal{O}\middle|\begin{smallmatrix} x \\ y \end{smallmatrix}\right) .$$

[used in the "I don't have time to show you details" part of Harold's talk @ Humboldt Kolleg]

- For the propagator

$$\frac{1}{\Box + m^2} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop y}\right) \mathcal{O}_P^D(x,y) \left({x \atop y}\right|$$

where

$$\mathcal{O}_P^D(x,y) = \left({x \atop y}\right| \frac{1}{\Box + m^2} \left|{x \atop y}\right) \ .$$

- For any function of the $\Box$ operator $f(\Box)$ we have

$$\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\middle| f(\Box) \middle| \begin{smallmatrix} y \\ y \end{smallmatrix}\right) = \frac{1}{N}\sum_l (2l+1) f\big(k(k+1)\big) e^{-l^2/N} P_l(\cos\vartheta)$$

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle| f(\Box) \middle| \begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{1}{N}\sum_{k,l} (2k+1)(2l+1)(-1)^{l+k+2s} f\big(k(k+1)\big) \left\{ \begin{matrix} l & s & s \\ k & s & s \end{matrix} \right\} e^{-l^2/N} P_l(\cos\vartheta)$$

where the curly bracket is the $6j$-symbol and $\cos\vartheta = x \cdot y$.

- For the propagator we obtain

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle| \frac{1}{\Box + m^2} \middle| \begin{smallmatrix} x \\ y \end{smallmatrix}\right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \; .$$

- Trace of propagator

$$\mathrm{Tr}\,\frac{1}{\Box+m^2} = \frac{N^2}{(4\pi)^2}\int d^2x\, d^2y \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right|\frac{1}{\Box+m^2}\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{N^2}{(4\pi)^2}\int \frac{1}{\frac{N^2}{4}|x-y|^2+m^2} =$$

$$= \frac{N^2}{2}\int_{-1}^{1} du\, \frac{1}{\frac{N^2}{2}(1-u)+m^2} \sim 2\log(N)\ .$$

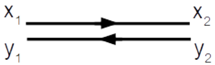
- This is consistent with

$$\mathrm{Tr}\,\frac{1}{\Box+m^2} = \sum_{l=0}^{N-1}\frac{2l+1}{l(l+1)+m^2} \sim N\int_0^1 \frac{2Nx}{N^2x^2+m^2} \sim 2\log(N)\ .$$

Fuzzy field theories in the string modes formalism

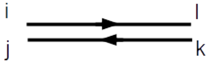
[Steinacker 2016; Steinacker, T work in progress]

- Feynman rules in string modes formalism - propagator



$$= \left({}^{x_2}_{y_2} \middle| \frac{1}{\Box + m^2} \middle| {}^{x_1}_{y_1} \right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \delta(x_1, x_2)\delta(y_1, y_2)$$

- Compare with the pure matrix models propagator

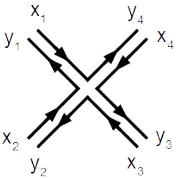


$$\sim \frac{1}{m^2}\delta_{il}\delta_{jk} \; .$$

and field theory action

$$S(M) = \frac{4\pi}{N}\mathrm{Tr}\left[\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + g\,M^4\right]$$

- Feynman rules in string modes formalism - vertex

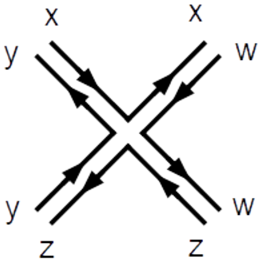


$$= g \langle y_1 | x_2 \rangle \langle y_2 | x_3 \rangle \langle y_3 | x_4 \rangle \langle y_4 | x_1 \rangle \approx g\, \delta(y_1, x_2)\delta(y_2, x_3)\delta(y_3, x_4)\delta(y_4, x_1) \ .$$

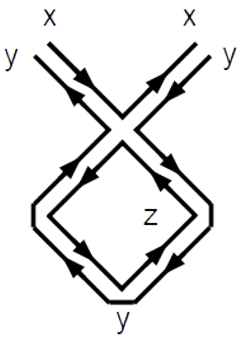
Loop computations and (non)locality in fuzzy QFT

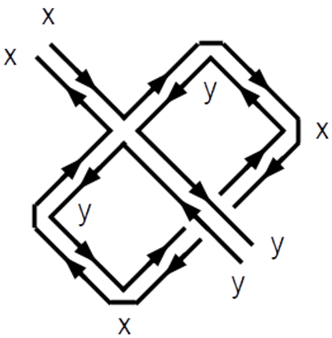
- String modes bring, in the large N limit, the best from the two worlds. They diagonalize the kinetic term and keep a simple structure of the vertices.
- Similar to the standard QFT calculations, but regular thanks to the effective noncommutative cutoff. No singularities and no issues when computing loop diagrams in position space.

One-loop two-point function and effective action



One-loop two-point function and effective action

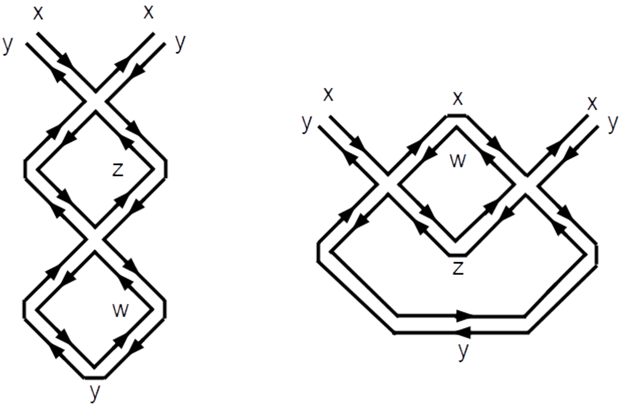




- We obtain the one-loop effective action for the classical fields $\phi(x,y) = \phi(x)\delta(x,y)$

$$S_{\text{eff}} = \int dx \phi(x) \frac{1}{2}(\Box + \mu^2)\phi(x) + \frac{g}{3}\frac{1}{4\pi}\int dx\, \phi(x)^2 \mu_N^2 +$$
$$+ \frac{g}{6}\left(\frac{N}{4\pi}\right)^2 \int dx\, dy\, \phi(x)\phi(y) \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \ .$$

- It can be shown that this is equivalent to the previous formula with $-\frac{g}{12\pi}\int d^2x\, \phi h(\tilde{\Delta})\phi$ but with a different interpretation. [Steinacker 2016]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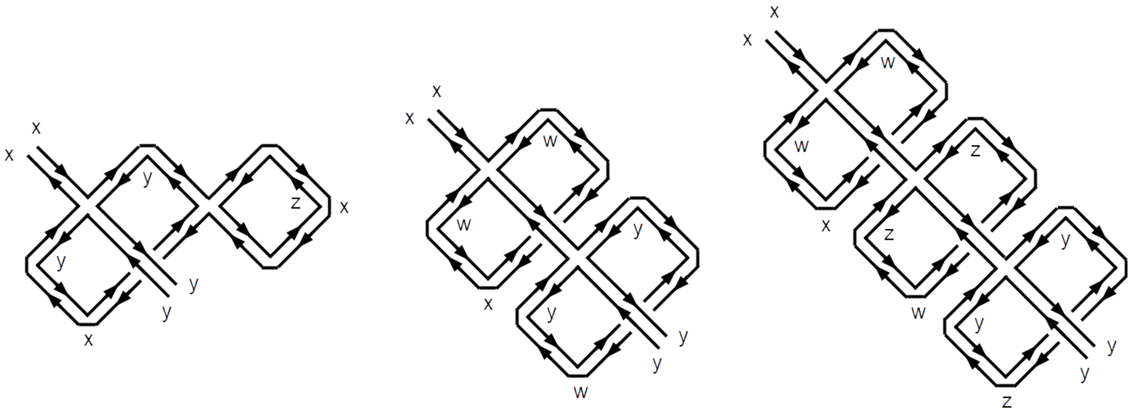
Two-loop two-point function and effective action



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