Fuzzy field theories in the string modes formalism

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Take home message

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- Fuzzy spaces are (among other things) toy models of spaces with quantum structure.
- Fuzzy scalar field theories are very different from their standard counterparts.
- There is an interesting **(new)** way to see and understand this difference in position space.

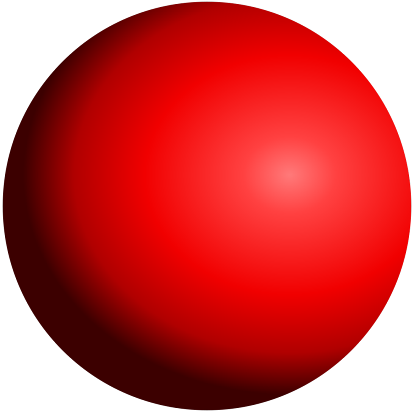
Fuzzy field theories in the string modes formalism

Fuzzy field theories in the string modes formalism

Quantum structure of spacetime

- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. Hossenfelder 1203.6191
- Reasons:
  - gravitational Heisenberg microscope,
  - instability of quantum gravitational vacuum, Doplicher, Fredenhagen, Roberts '95
  - emergent spacetime.

Fuzzy spaces



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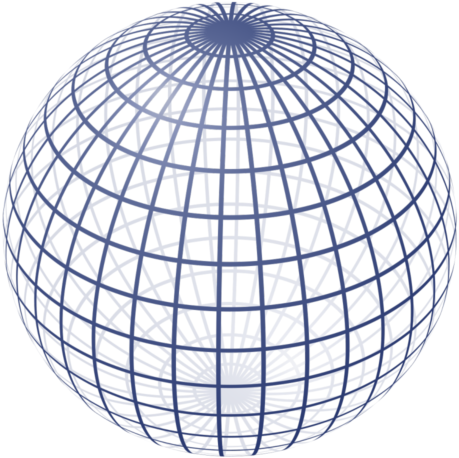
Fuzzy spaces



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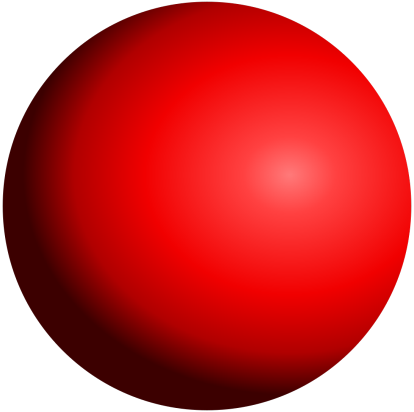
Fuzzy spaces



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Fuzzy sphere Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder '90s

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

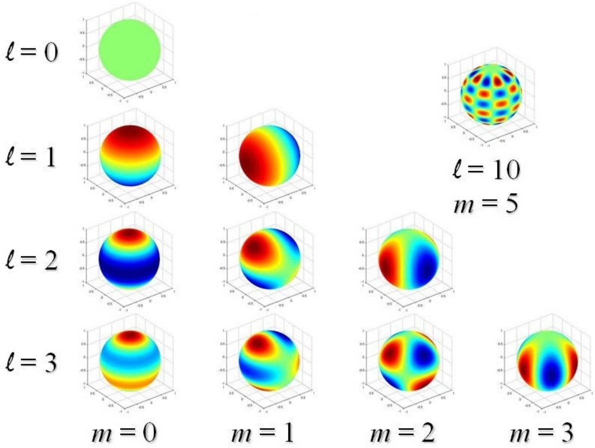


Image taken from http://principles.ou.edu/mag/earth.html

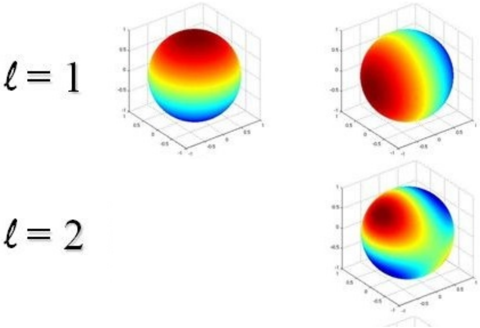
Fuzzy spaces

- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

  we will not be able to see any features of functions under certain length scales.
- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces



- Number of independent functions with $l \leq L$ is

$$\sum_{l=0}^{L} \sum_{m=-l}^{l} 1 = (L+1)^2 = N^2 \ .$$

This is the same as the number of $N \times N$ hermitian matrices

$$N + 2\sum_{n=1}^{N}(n-1) = N^2 \ .$$

Fuzzy spaces

- The idea is to map the functions on the matrices and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} * Y_{l'm'} := \varphi^{-1} \left(\varphi \left(Y_{lm} \right) \varphi \left(Y_{l'm'} \right) \right) .$$

Fuzzy spaces

- We have a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions.
  The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- In the limit N or $L \to \infty$ we recover the original sphere.

- The regualar sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i = 1,2,3 \ ,$$

which generate the following algebra of functions

$$f = \left\{ \sum_{k \in \mathbb{N}^3} \left(a_{k_1 k_2 k_3} \prod_{i=1}^{3} x_i^{k_i} \right) \middle| x_i x_i = R^2 \right\} \ ,$$

which is by definition commutative.

- Information about the sphere is again hidden in this algebra.

- For the fuzzy sphere S^2_F we define

$$\hat{x}_i \hat{x}_i = \rho^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta \varepsilon_{ijk} \hat{x}_k \ , \ i = 1,2,3 \ .$$

- Such $\hat{x}_i$'s generate a different, noncommutative algebra and S^2_F is an object, which has this algebra as an algebra of functions.
- The conditions can be realized as an $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- And again, in the limit $N \to \infty$ we recover the original sphere.

- Most importantly nonzero commutators

$$\hat{x}_i\hat{x}_i = \rho^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \ .$$

- Configuration space is analogous to phase space of quantum mechanics.
- In a similar fashion it is possible to construct an analogous deformation of the plane

$$\hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ij} \ , \ i = 1,2 \ .$$

Fuzzy spaces - an alternative construction

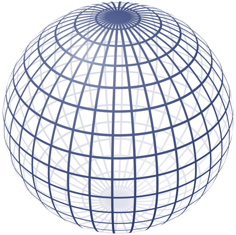
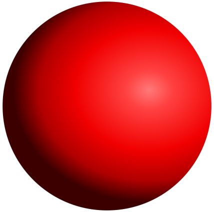
- $\hat{x}_i$'s are $N \times N$ matrices, functions on S_F^2 are combinations of all their possible products and thus hermitian matrices M.
- Such $N \times N$ matrix can be decomposed into

$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{m=l} c_{lm} T_{lm} \ .$$

where matrices T_m^l are called polarization tensors and

$$T_m^l = \varphi(Y_m^l) \ ,$$
$$\mathrm{Tr}\left(T_{lm} T_{l'm'}\right) = \delta_{ll'} \delta_{mm'} \ ,$$
$$[L_i, [L_i, T_{lm}]] = l(l+1) T_{lm} \ .$$

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.

An auxiliary Hilbert space has been used in both constructions.

- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Fuzzy field theories in the string modes formalism

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \; .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} \,.$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi r^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{r^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right]$$

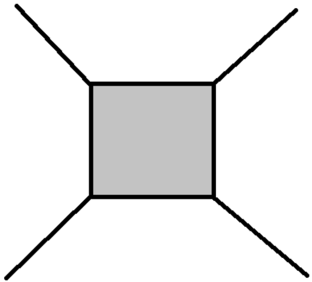
$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \,.$$

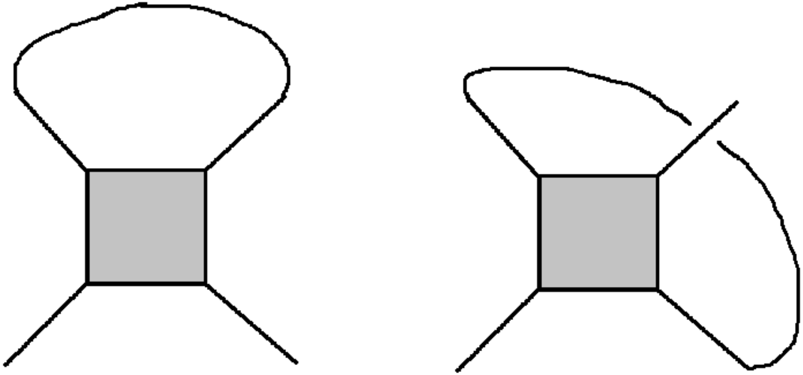
Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03; Ydri '16

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- In terms of diagrams different properties of planar and non-planar ones.
  The (matrix) vertex is not invariant under permutation of incoming momenta.

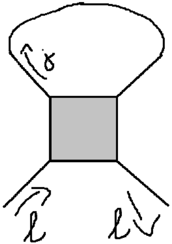
$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm} T_{lm} \ , \ S(M) = \frac{4\pi}{N} \mathrm{Tr} \left[\frac{1}{2} M[L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + g M^4 \right]$$

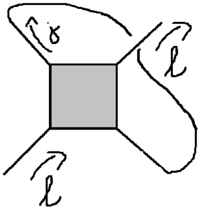
$$\mathrm{Tr}\left(M^4\right) = \sum_{l_{1\ldots4}} \sum_{m_{1\ldots4}} c_{l_1 m_1} c_{l_2 m_2} c_{l_3 m_3} c_{l_4 m_4} \mathrm{Tr}\left(T_{l_1 m_1} T_{l_2 m_2} T_{l_3 m_3} T_{l_4 m_4}\right)$$





Chu, Madore, Steinacker '01





$$I^P = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}$$

$$I^{NP} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} N(-1)^{l+j+N-1} \begin{Bmatrix} l & s & s \\ j & s & s \end{Bmatrix} \,, \quad s = \frac{N-1}{2}$$

Fuzzy scalar field theory - UV/IR mixing

$$I^{NP}-I^{P}=\sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+m^2}\left[N(-1)^{l+j+N-1}\begin{Bmatrix} l & s & s \\ j & s & s\end{Bmatrix}-1\right]$$

- This difference is finite in $N\to\infty$ limit.
- $N\to\infty$ limit of the effective action is different from the standard S^2 effective action.

$$S_{\text{one loop}}=S_0+\frac{1}{2}\int d^2x\,\phi^2\delta m^2-\frac{g}{12\pi}\int d^2x\,\phi h(\tilde{\Delta})\phi+\dots$$

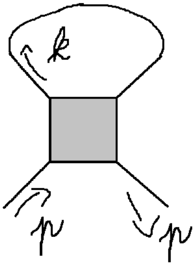
$$\tilde{\Delta}Y_{lm}=l\,Y_{lm}\ ,\ \ h(n)=\sum_{k=1}^{n}\frac{1}{k}$$

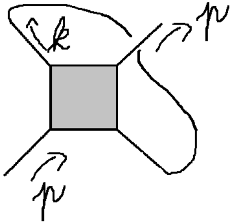
There is an extra, mildly nonlocal, term.

- Regularization of the field theory by NC space is anomalous.

Fuzzy scalar field theory - UV/IR mixing

- In the planar limit $S^2 \to \mathbb{R}^2$ one recovers singularities and the standard UV/IR-mixing.





- Nonplanar diagrams are divergent in the limit $p \to 0$.
- This leads to issues when such nonplanar loops appear on other loops. Technically we obtain effective action which diverges as $p \to 0$.

- The space (geometry) forgets where it came from, but the field theory (physics) remembers its fuzzy origin.
- Commutative limit of noncomutative theory is very different from commutative theory.

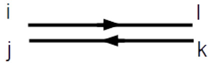
- Expressing M in terms of T_{lm} diagonalizes the kinetic term and leaves us to struggle with the interaction term.
- There is a different treatment that is more favorable to the interaction term but the kinetic term is the problematic one now.

Fuzzy scalar field theory - matrix formalism

- If we forget about the kinetic term for a moment

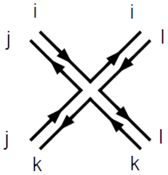
$$S(M) = \frac{4\pi}{N}\left[\frac{1}{2}m^2 M_{ij}M_{ji} + g\, M_{ij}M_{jk}M_{kl}M_{li}\right]$$

we can treat this model as a field theory with propagator



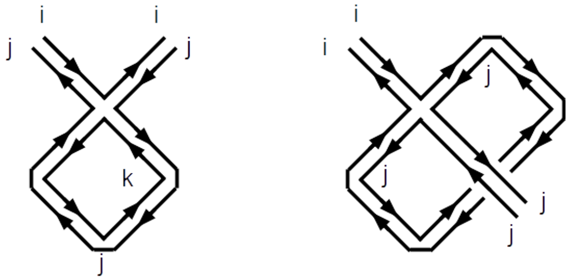
$$= \langle M_{ij}M_{kl}\rangle \sim \frac{1}{m^2}\delta_{il}\delta_{jk}$$

and vertex



$$= g$$

- Such graphs are called fat graphs or ribbon graphs and are well known in matrix models.



- Matrix indexes run in the loops, not the momenta!
- The problem is that the kinetic term leads to a nondiagonal propagator.

Fuzzy field theories in the string modes formalism

String modes - coherent states

- Natural basis in the auxiliary hilbert space $\mathcal{H}$ is the "spin" basis

$$|n\rangle = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}$$

derived from the highest weight state $|1\rangle$.

- For any $x \in S^2$ with radius 1, choose some $g_x \in SO(3)$ such that $x = g_x \cdot p$, where p is the north pole on S^2. We define

$$|x\rangle = g_x \cdot |1\rangle \ , \ g_x \in SU(2)$$

and call the set of all $|x\rangle$ the coherent states.

- Coherent states are optimally localized in the sense

$$\langle x|\, \hat{x}_i \hat{x}_i \,|x\rangle - \langle x|\, \hat{x}_i \,|x\rangle \langle x|\, \hat{x}_i \,|x\rangle \approx \frac{2}{N} \ .$$

- They form an over-complete basis in $\mathcal{H}$ and

$$\mathbb{1} = \frac{N}{4\pi} \int d^2x \,|x\rangle \langle x| \ .$$

- They are orthogonal only in the large N limit

$$|\langle x\,|y\rangle|^2 = \left(\frac{1 + x\cdot y}{2}\right)^{N-1} \ .$$

String modes - coherent states

- Coherent states can be used to map (quantize) functions on S^2 on matrices

$$\phi(x) \to M = \int dx \phi(x) |x\rangle\langle x| \ .$$

and matrices on functions (de-quantize)

$$M \to \phi(x) = \langle x| M |x\rangle \ .$$

- This maps Y_{lm} on T_{lm} up to normalization.

String modes - coherent states and representations of functions on fuzzy sphere

- Functions on the fuzzy sphere are matrices acting on $\mathcal{H}$

$$M = \sum_{m,n} M_{mn} \left| m \right\rangle \left\langle n \right| \ .$$

- We can express the matrix M in a similar fashion using the coherent states

$$M = \left(\frac{N}{4\pi} \right)^2 \int d^2x \, d^2y \, \phi(x,y) \left| x \right\rangle \left\langle y \right| \ .$$

- Objects

$$\left| x \right\rangle \left\langle y \right| =: \left| \begin{smallmatrix} x \\ y \end{smallmatrix} \right)$$

form a basis of functions on the fuzzy sphere and we will call them the **string modes**.

String modes - coherent states and representations of functions on fuzzy sphere

- Such representation of matrix M by function $\phi(x, y)$ seems to be not unique (way more functions than matrices). But one can show that derivatives of $\phi(x, y)$ are bounded by $\sqrt{N}$, which means that the Fourier modes of ϕ to be restricted by $l_x, l_y \leq \sqrt{N}$.
- Functions $\phi(x, y)$ that represent functions on the fuzzy sphere have rather mild behavior. The coherent states are spread out over an area $\sim 2/N$ and average out any larger oscillations.
- Large momentum UV wavelengths are smoothed out on the fuzzy sphere. But the price we pay are non-local string modes.

String modes - coherent states and representations of functions on fuzzy sphere

- Short modes $\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for $|x-y| < 1/\sqrt{N}$ can be show to represent localized wave-packets with momentum $\sim |x-y|$.
- Particularly $\left|\begin{smallmatrix} x \\ x \end{smallmatrix}\right)$ represents maximal localization around point x, i.e. a fuzzy version of δ-function.
- Long modes $\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for $|x-y| > 1/\sqrt{N}$ are non-local and have no classical analogue.

- When working with functions we encounter operators

$$\mathcal{O} : M \to \mathcal{O}(M) \ .$$

For example the kinetic term of the field theory $[L_i,[L_i,M]] =: \Box M$ or the propagator of the theory

$$\frac{1}{\Box + m^2} \ .$$

- For example

$$\left({x \atop y} \middle| \Box \middle| {x \atop y} \right) \sim |x-y|^2 \ .$$

String modes - representation of operators on functions

- A general representation of such operators in terms of the string modes is straightforward

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2x'\, d^2y\, d^2y' \left|{x \atop y}\right)\mathcal{O}(x,y;x',y')\left({x' \atop y'}\right| .$$

- There are two special cases
  - Local

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2y \left|{x \atop x}\right)\mathcal{O}_L(x,y)\left({y \atop y}\right| .$$

  - Non-local

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2y \left|{x \atop y}\right)\mathcal{O}_D(x,y)\left({x \atop y}\right| .$$

- Functions $\mathcal{O}_L$ and $\mathcal{O}_D$ may have very different behavior for different operators (oscillation, singularity). Local representations are typically highly oscillatory, non-local representations are better behaved.

- For any function of the $\Box$ operator $f(\Box)$ we have

$$\left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle|f(\Box)\middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\sum_{k,l}(2k+1)(2l+1)(-1)^{l+k+2s}f\big(k(k+1)\big)\frac{1}{N}\begin{Bmatrix}l & s & s\\ k & s & s\end{Bmatrix}e^{-l^2/N}P_l(\cos\vartheta)$$

where the curly bracket is the $6j$-symbol and $\cos\vartheta=x\cdot y$.

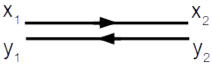
- Especially for propagator we obtain

$$\left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle|f(\Box)\middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right)\approx\frac{1}{\frac{N^2}{4}|x-y|^2+m^2}\ .$$

Fuzzy field theories in the string modes formalism

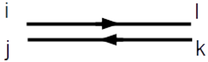
Loop computations and (non)locality in fuzzy QFT

- Feynman rules in string modes formalism - propagator



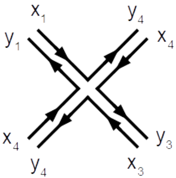
$$= \left({x_2 \atop y_2} \middle| \frac{1}{\Box + m^2} \middle| {x_1 \atop y_1} \right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2+\mu^2}\delta(x_1,x_2)\delta(y_1,y_2)$$

Compare with the matrix models propagator



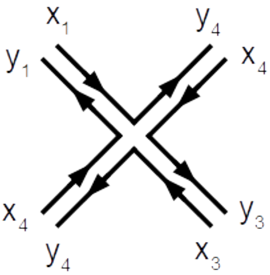
$$\sim \frac{1}{m^2}\delta_{il}\delta_{jk} \; .$$

- Feynman rules in string modes formalism - vertex

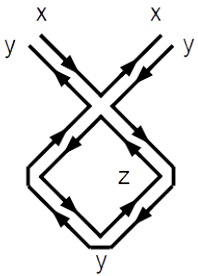


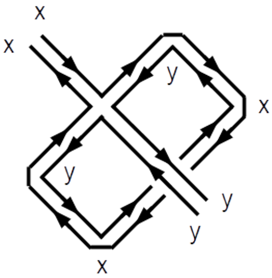
$$= \frac{g}{4!} \langle y_1 | x_2 \rangle \langle y_2 | x_3 \rangle \langle y_3 | x_4 \rangle \langle y_4 | x_1 \rangle \approx \frac{g}{4!} \delta(y_1, x_2) \delta(y_2, x_3) \delta(y_3, x_4) \delta(y_4, x_1) \ .$$

One-loop two-point function and effective action



One-loop two-point function and effective action





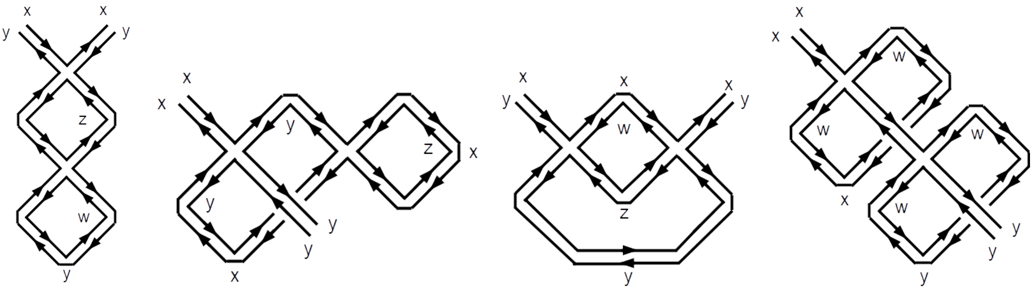
One-loop two-point function and effective action

- We obtain the one-loop effective action for the classical fields $\phi(x,y)=\delta(x,y)\phi(x)$

$$S_{\text{eff}}=\int dx\phi(x)\frac{1}{2}(\Box+\mu^2)\phi(x)+\frac{g}{3}\frac{1}{4\pi}\int dx\,\phi(x)^2\mu_N^2+\frac{g}{6}\left(\frac{N}{4\pi}\right)^2\int dx\,dy\,\phi(x)\phi(y)\frac{1}{\frac{N^2}{4}|x-y|^2+m^2}\ .$$

- It can be shown that this is equivalent to the previous formula with $-\frac{g}{12\pi}\int d^2x\,\phi h(\tilde{\Delta})\phi$ but with a different interpretation.

Two-loop two-point function and effective action



Take home message

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- Fuzzy spaces are (among other things) toy models of spaces with quantum structure.
- Fuzzy scalar field theories are very different from their standard counterparts.
- There is an interesting **(new)** way to see and understand this difference in position space.

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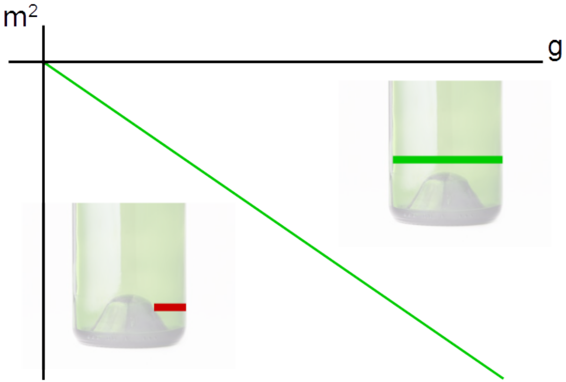
Thank you for your attention!

If time permits - Fuzzy spaces

- Regularization of infinities in the standard QFT.
  Heisenberg ∼'30; Snyder '47, Yang '47
- Regularization of field theories for numerical simulations.
  Panero '16
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  Seiberg Witten '99; Douglas, Nekrasov '01
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  Steinacker '13
- Geometric unification of the particle physics and theory of gravity.
  van Suijlekom '15
- An effective description of various systems in a certain limit (eg. QHE).
  Karabali, Nair '06

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + g\phi^4 \right)$$



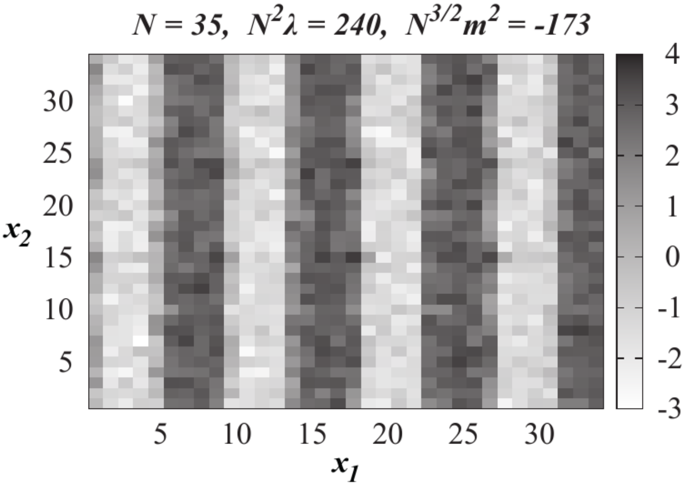
Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75; Chang '76
Loinaz, Willey '98; Schaich, Loinaz '09

If time permits - Symmetry breaking in NC field theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; Chen, Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero '14; Mejía-Díaz, Bietenholz, Panero '14; Medina, Bietenholz, D. O'Connor '08; Bietenholz, Hofheinz, Nishimura '04; Lizzi, Spisso '12; Ydri, Ramda, Rouag '16; Kováčik, O'Connor '18
  Panero '15

If time permits - Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero '14 for $\mathbb{R}^2_\theta$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$