A LESS COMMUTATIVE VERSION OF QUARKONIUM MASSES

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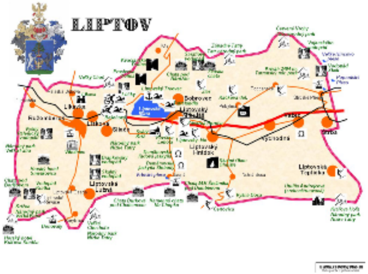
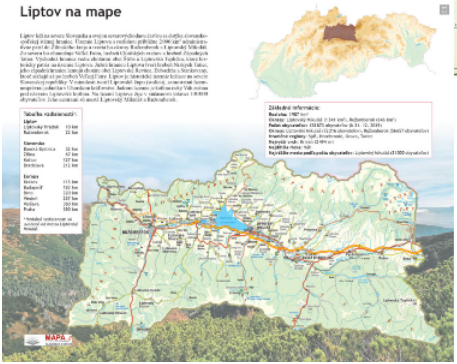
work with B. Bukor

arXiv: 2209.XXXXX [hep-th]


LIPTOV


LIPTÓ VÁRMEGYE
TÉRKÉPE


Liptov na mape

Take home message

Take home message

- Quantum mechanics and general relativity together lead to a quantum structure of spacetime.

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- Noncommutative spaces are a version of such a construction where some of the symmetries are preserved.

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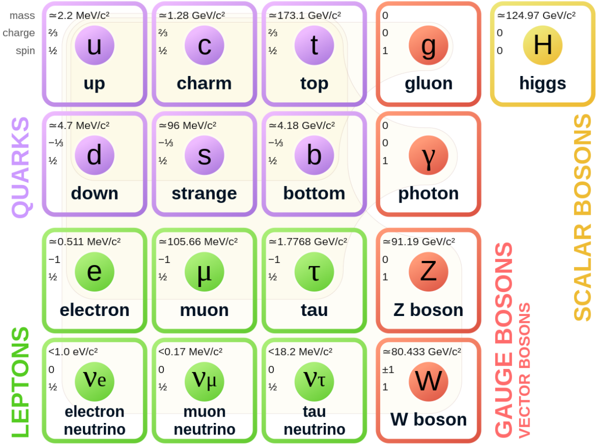
- Quantum mechanics and general relativity together lead to a quantum structure of spacetime.
- Noncommutative spaces are a version of such a construction where some of the symmetries are preserved.
- Quantum mechanics leads to some possibly observable corrections to things like energies of quantum systems – like masses of bound states of two heavy quarks.

Standard model and general relativity

Standard model



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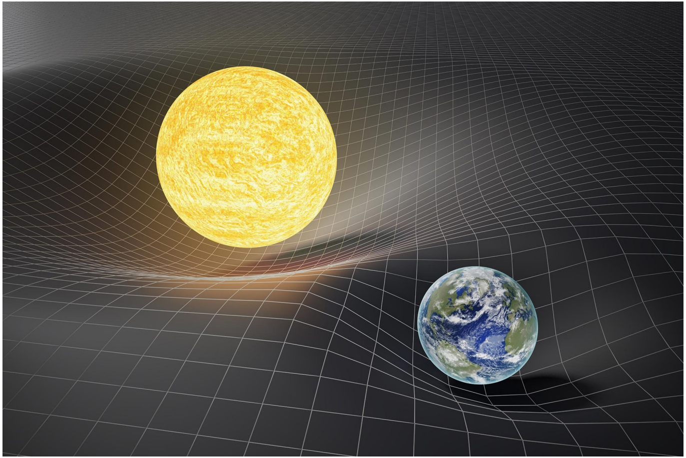
Standard model

$$SU(3) \times SU(2) \times U(1)$$

General relativity



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General relativity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

SM and GR

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} \hat{T}_{\mu\nu}$$

SM and GR

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} \;??\; \frac{8\pi G}{c^4} \hat{T}_{\mu\nu}$$

Quantum structure of spacetime

- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. [Hossenfelder 1203.6191]
- Reasons:
  - gravitational Heisenberg microscope,
  - emergent spacetime,
  - instability of quantum gravitational vacuum. [Doplicher, Fredenhagen, Roberts '95]

Quantum structure of spacetime

- Very energetic and localized quantum fluctuations can lead to black holes.
- A discrete structure solves this problem.
- Similar to the stabilization of the hydrogen atom in quantum mechanics.

Quantum structure of spacetime

$$\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$$

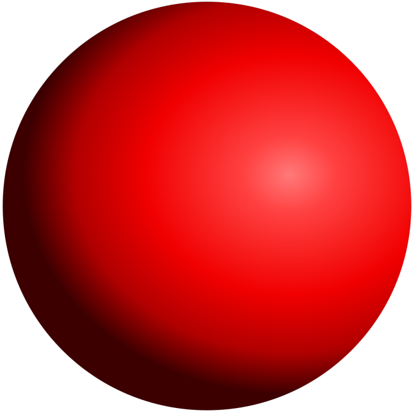
Quantum structure of spacetime

$$\Delta x \cdot \Delta y \geq \theta$$

- Natural scale for this is $\sqrt{\theta} \approx l_{\rm Pl} \approx 10^{-35}\ \mathrm{m}$.
- A fundamental volume, not length directly.
- Discrete, but preserves at least some of the continuous symmetries.

Fuzzy spaces

Fuzzy spaces



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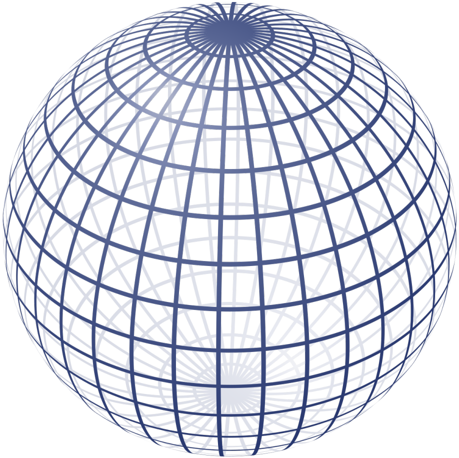
Fuzzy spaces



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Fuzzy sphere [Hoppe 1982; Madore 1992; Grosse, Klimčík, Prešnajder 1990s]

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i = 1,2,3 \ ,$$

which generate the following algebra of functions

$$f = \left\{ \sum_{k \in \mathbb{N}^3} \left(a_{k_1 k_2 k_3} \prod_{i=1}^{3} x_i^{k_i} \right) \middle| x_i x_i = R^2 \right\} \ ,$$

which is by definition commutative.

- Information about the sphere is hidden in this algebra.

Fuzzy spaces

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta \varepsilon_{ijk} \hat{x}_k \ , \ i = 1, 2, 3 \ .$$

- Such $\hat{x}_i$'s generate a different, non-commutative, algebra and S_N^2 is an object, which has this algebra as an algebra of functions.
- The conditions can be realized as an $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces

- Most importantly nonzero commutators

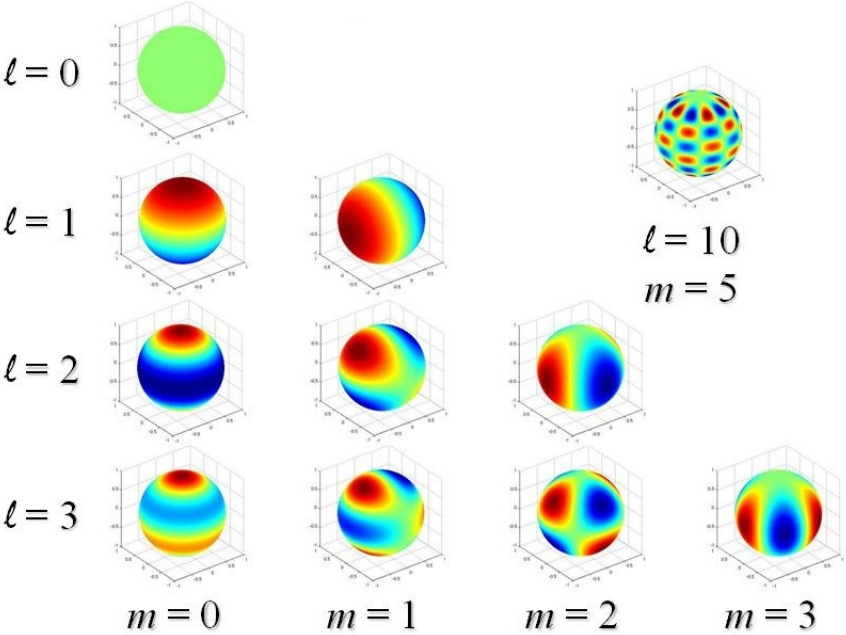
$$\hat{x}_i\hat{x}_i = \rho^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \ .$$

- Configuration space is analogous to phase space of quantum mechanics.
- In a similar fashion it is possible to construct an analogous deformation of the plane

$$\hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ij} \ , \ i = 1,2 \ .$$



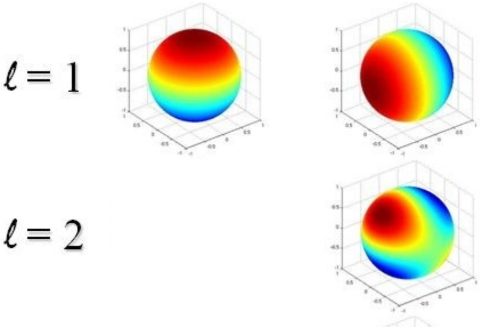
$\ell = 0$
$\ell = 1$
$\ell = 2$
$\ell = 3$
$\ell = 10$
$m = 5$
$m = 0$
$m = 1$
$m = 2$
$m = 3$

Fuzzy spaces

- If we truncate the possible values of l in the expansion of functions in terms of spherical harmonics

$$f=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ \to\ \sum_{l=0}^{L}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

  we will not be able to see any features of functions under certain length scales.
- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.


$\ell = 1$
$\ell = 2$

Fuzzy spaces

- Number of independent functions with $l \leq L$ is N^2, the same as the number of $N \times N$ hermitian matrices.
  The idea is to map the former on the latter and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{cccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

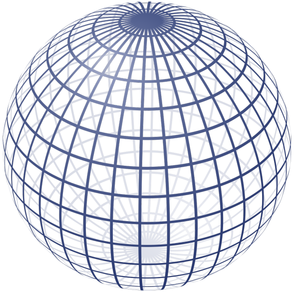
$$Y_{lm} * Y_{l'm'} := \varphi^{-1} \left(\varphi\left(Y_{lm}\right) \varphi\left(Y_{l'm'}\right) \right) \ .$$

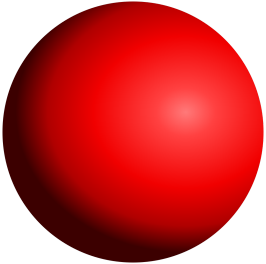
- We have a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

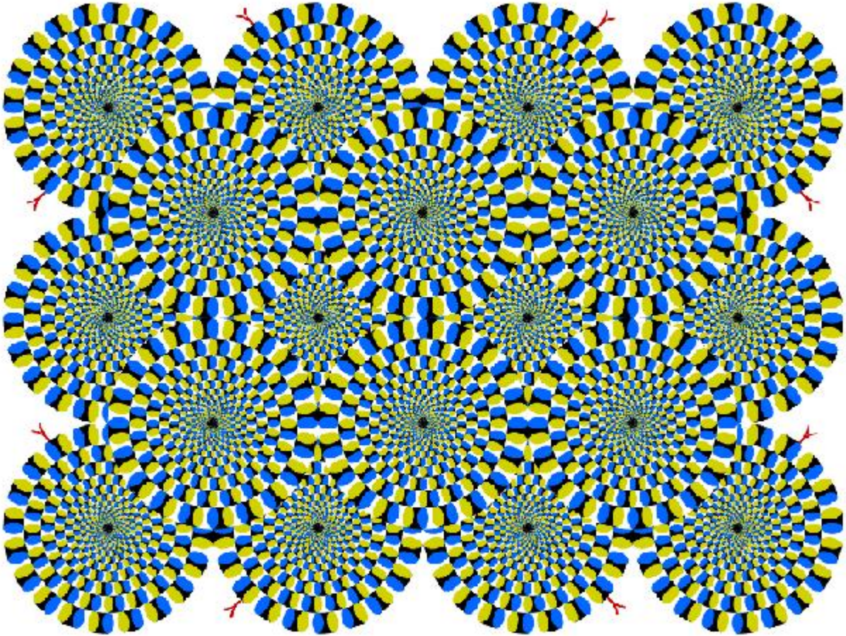
- In the limit N or $L \to \infty$ we recover the original sphere.

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.





- There are no sharp boundaries between the pieces and everything is blurred, or fuzzy.



Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼'30; Snyder '47, Yang '47]
- Regularization of field theories for numerical simulations.
  [Panero '16]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten '99; Douglas, Nekrasov '01]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  [Steinacker '13]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom '15]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair '06]

3D fuzzy spaces

3D fuzzy spaces

- The above construction does not work in odd dimension.
- For a 3D space one needs to be clever. [Galiková, Kováčik, Prešnajder '13, '15]

3D fuzzy spaces

Image from https://greatcharacters.miraheze.org/

Fuzzy onion

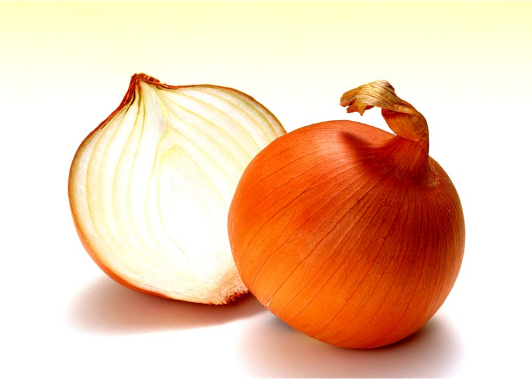


Image from https://www.rawpixel.com/

- The idea is to layer fuzzy spheres of increasing radius.

$$[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k \ , \ \ i, j, k = 1, 2, 3 \ .$$

- The most important point of the construction is that the radial step λ is compatible with the NC distance of the each fuzzy sphere.
- Some technicalities

$$x_j = \lambda\sigma^j_{\alpha\beta}a^\dagger_\alpha a_\beta \ , \ j \in \{1, 2, 3\} \ , \ \ r = \lambda\left(a^\dagger_\alpha a_\alpha + 1\right) \ , \ \tilde{r} = \lambda\left(a^\dagger_\alpha a_\alpha\right) \ .$$

- Fuzzy wave functions with separated variables

$$\Psi_{lm} = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \mathcal{K}_l(\tilde{r}) : \frac{a_1^{n_1}\,(-a_2)^{n_2}}{n_1!n_2!} \ .$$

- Action of r

$$\hat{r}\,\Psi_{lm} = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \ [\tilde{r}\mathcal{K} + (\lambda l + \lambda)\mathcal{K} + \lambda\tilde{r}\mathcal{K}'] : \frac{a_1^{n_1}\,(-a_2)^{n_2}}{n_1!n_2!} \ .$$

- The definition of the Laplace operator is

$$\hat{\Delta}_\lambda \Psi = -\frac{1}{\lambda r}\left[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi]\right] \ ,$$

where

$$\left[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi_{lm}]\right] = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} : \ [-\lambda\tilde{r}\mathcal{K}'' - 2(l+1)\lambda\mathcal{K}'] : \frac{a_1^{n_1}\,(-a_2)^{n_2}}{n_1!n_2!} \ .$$

Noncommutative quantum mechanics

3D fuzzy spaces

- Fuzzy Hamiltonian

$$\hat{H}_\lambda \Psi = \left[-\frac{\hbar^2}{2\mu} \hat{\Delta}_\lambda + V(\hat{r}) \right] \Psi$$

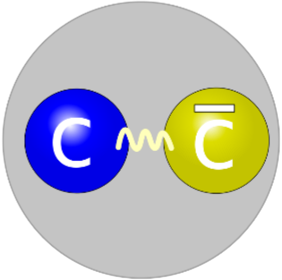
and NC version of Schrodinger equation

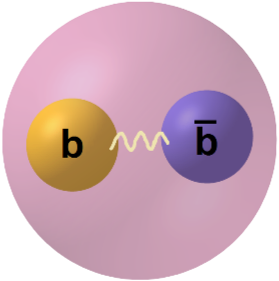
$$\frac{\hbar^2}{2\mu\lambda} \left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi] \right] + \hat{r} V(\hat{r}) \Psi = E \hat{r} \Psi \ .$$

- The equation for the radial wave function depends on the potential!

Quarkonium mesons

Quarkonium mesons





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Quarkonium mesons

Term symbol $n^{2S+1}L_J$	$I^G(J^{PC})$	Particle	mass (MeV/c^2)[4]
1^1S_0	$0^+(0^{-+})$	$\eta_c(1S)$	2 983.4 ± 0.5
1^3S_1	$0^-(1^{--})$	$J/\psi(1S)$	3 096.900 ± 0.006
1^1P_1	$0^-(1^{+-})$	$h_c(1P)$	3 525.38 ± 0.11
1^3P_0	$0^+(0^{++})$	$\chi_{c0}(1P)$	3 414.75 ± 0.31
1^3P_1	$0^+(1^{++})$	$\chi_{c1}(1P)$	3 510.66 ± 0.07
1^3P_2	$0^+(2^{++})$	$\chi_{c2}(1P)$	3 556.20 ± 0.09
2^1S_0	$0^+(0^{-+})$	$\eta_c(2S)$, or η'_c	3 639.2 ± 1.2
2^3S_1	$0^-(1^{--})$	$\psi(2S)$ or $\psi(3686)$	3 686.097 ± 0.025
1^1D_2	$0^+(2^{-+})$	$\eta_{c2}(1D)$	
1^3D_1	$0^-(1^{--})$	$\psi(3770)$	3 773.13 ± 0.35
1^3D_2	$0^-(2^{--})$	$\psi_2(1D)$	
1^3D_3	$0^-(3^{--})$	$\psi_3(1D)$[‡]	
2^1P_1	$0^-(1^{+-})$	$h_c(2P)$[‡]	
2^3P_0	$0^+(0^{++})$	$\chi_{c0}(2P)$[‡]	
2^3P_1	$0^+(1^{++})$	$\chi_{c1}(2P)$[‡]	
2^3P_2	$0^+(2^{++})$	$\chi_{c2}(2P)$[‡]	
$?^??_?$	$0^+(1^{++})$[*]	$X(3872)$	3 871.69 ± 0.17
$?^??_?$	$?^?(1^{--})$[†]	$Y(4260)$	$4263\,^{+8}_{-9}$

Term symbol $n^{2S+1}L_J$	$I^G(J^{PC})$	Particle	mass (MeV/c^2)[5]
1^1S_0	$0^+(0^{-+})$	$\eta_b(1S)$	9 390.9 ± 2.8
1^3S_1	$0^-(1^{--})$	$\Upsilon(1S)$	9 460.30 ± 0.26
1^1P_1	$0^-(1^{+-})$	$h_b(1P)$	9 899.3 ± 0.8
1^3P_0	$0^+(0^{++})$	$\chi_{b0}(1P)$	9 859.44 ± 0.52
1^3P_1	$0^+(1^{++})$	$\chi_{b1}(1P)$	9 892.76 ± 0.40
1^3P_2	$0^+(2^{++})$	$\chi_{b2}(1P)$	9 912.21 ± 0.40
2^1S_0	$0^+(0^{-+})$	$\eta_b(2S)$	
2^3S_1	$0^-(1^{--})$	$\Upsilon(2S)$	10 023.26 ± 0.31
1^1D_2	$0^+(2^{-+})$	$\eta_{b2}(1D)$	
1^3D_1	$0^-(1^{--})$	$\Upsilon(1D)$	
1^3D_2	$0^-(2^{--})$	$\Upsilon_2(1D)$	10 161.1 ± 1.7
1^3D_3	$0^-(3^{--})$	$\Upsilon_3(1D)$	
2^1P_1	$0^-(1^{+-})$	$h_b(2P)$	10 259.8 ± 1.2
2^3P_0	$0^+(0^{++})$	$\chi_{b0}(2P)$	10 232.5 ± 0.6
2^3P_1	$0^+(1^{++})$	$\chi_{b1}(2P)$	10 255.46 ± 0.55
2^3P_2	$0^+(2^{++})$	$\chi_{b2}(2P)$	10 268.65 ± 0.55
3^3S_1	$0^-(1^{--})$	$\Upsilon(3S)$	10 355.2 ± 0.5

Quarkonium mesons

- Phenomenologically described by Cornell (or Killingback) potential

$$V(r) = -\frac{C}{r} + B\,r + G\,r^2 \ .$$

Masses given by a two-body problem

$$M_{nl} = m_1 + m_2 + E_{nl} \ .$$

- Spectrum can not be solved for analytically, approximations needed.

Quarkonium mesons

- WKB approximation

$$\frac{1}{\hbar}\int_{r_1}^{r_2} \mathrm{d}r \sqrt{2\mu\left(E-\left(-\frac{C}{r}+Br+Gr^2\right)-\frac{(l+\frac{1}{2})^2\hbar^2}{2\mu r^2}\right)}=\left(n+\frac{1}{2}\right)\pi\ ,\ \ n\in\mathbb{Z}_0^+\ .$$

- Pekeris-type approximation – expand the integrand around the characteristic distance of the problem $r_Q = \sqrt{C/B}$.
- Treat B, C as free parameters of the model to be fixed by experimental data. The rest of the masses is a prediction of the model.

[Bukor, JT '22]

$c\bar{c}$ meson	$m_q = 1.27$ GeV	$B = 0.322$ GeV^2	$C = 0.891$
state	particle	present work M_{nl} [GeV]	experimental data M_{nl} [GeV]
1S	$J/\psi(1S)$	used for B, C	3.097
2S	$\psi(2S)$	used for B, C	3.686
3S	$\psi(4040)$	3.889	4.039
4S	–	3.982	no data
1P	$\chi_{C1}(1P)$	3.518	3.511
2P	$\chi_{C2}(3930)$	3.823	3.923
1D	$\psi(3770)$	3.787	3.774

Quarkonium mesons

[Bukor, JT '22]

$b\bar{b}$ meson	$m_q = 4.18$ GeV	$B = 1.266$ GeV^2	$C = 0.344$
state	particle	present work M_{nl} [GeV]	experimental data M_{nl} [GeV]
1S	$\Upsilon(1S)$	used for B, C	9.460
2S	$\Upsilon(2S)$	used for B, C	10.023
3S	$\Upsilon(3S)$	10.178	10.355
4S	$\Upsilon(4S)$	10.242	10.579
1P	$h_b(1P)$	9.942	9.899
2P	$h_b(2P)$	10.150	10.260
1D	$\Upsilon_2(1D)$	10.140	10.164

Quarkonium mesons

[Bukor, JT '22]

$c\bar{b}$ meson	$\mu = 0.97$ GeV	$B = 0.604$ GeV^2	$C = 0.603$
state	particle	present work M_{nl} [GeV]	experimental data M_{nl} [GeV]
1S	B_c^+	used for B, C	6.274
2S	$B_c^{\pm}(2S)$	used for B, C	6.871
3S	–	7.054	no data
4S	–	7.132	no data
1P	–	6.749	no data
2P	–	7.009	no data
1D	–	6.989	no data

Quarkonium mesons

[Bukor, JT '22]

quarkonium	μ [GeV c^{-2}]	B [GeV fm^{-1}]	C [GeV fm]	r_Q [10^{-16} m]
$c\bar{c}$	0.64	1.633	0.175	3.28
$b\bar{b}$	2.09	6.425	0.068	1.03
$c\bar{b}$	0.97	3.067	0.119	1.97

- We obtain reasonable sizes of the quarkonium states.

NC quarkonium

- Radial Schrodinger equation in dimensionless coordinates

$$R'' + \frac{2}{\zeta}R' - \frac{l(l+1)}{\zeta^2}R + \left(\frac{c}{\zeta} - b\zeta\right)R + \epsilon R + \sigma\left(\epsilon R' + \frac{\epsilon}{\zeta}R - 2b\zeta R' - 3bR\right) +$$
$$+ \sigma^2\left(-b\zeta R'' - 3bR' - \frac{b}{\zeta}R\right) = 0\ .$$

- Second order mass correction [Bukor, JT '22]

$$M^\sigma_{nl} = \left((m_1+m_2) - \frac{2\mu}{\hbar^2}\left[\frac{2C}{n+\frac{1}{2}+\sqrt{\frac{2\mu}{\hbar^2}C\sqrt{\frac{C}{B}}+\left(l+\frac{1}{2}\right)^2}}\right]^2 + 3\sqrt{BC}\right) +$$

$$+\sigma^2\frac{\hbar^2}{2\mu}\frac{B}{C}\left(\frac{b\left(105b^2+62bc+9c^2\right)}{8\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^2}+\frac{b(c+3b)l(l+1)}{2\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^2}+\right.$$

$$+\frac{b(c+3b)^4}{8\sqrt{b+(l+\frac{1}{2})^2}\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^5}-\frac{b}{4}(15b+4c)-$$

$$\left.-\frac{(45b-c)(c+3b)^3}{64\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^4}\right)+\dots\ ,\quad b=2\mu B r_Q^3/\hbar^2\ ,\quad c=2\mu C r_Q/\hbar^2$$

[Bukor, JT '22]

NC $c\bar{c}$ meson	$b = c = 1.883\ ,\quad \sigma^2 \approx 0.93\ \times\ 10^{-39}$
state	correction to the mass spectrum $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$0.522\,\sigma^2$
2S	$-1.422\,\sigma^2$
3S	$-2.613\,\sigma^2$
4S	$-3.301\,\sigma^2$
1P	$-0.456\,\sigma^2$
2P	$-1.936\,\sigma^2$
1D	$-1.062\,\sigma^2$

[Bukor, JT '22]

NC $b\bar{b}$ meson	$b = c = 0.750\ ,\quad \sigma^2 \approx 9.43 \times 10^{-39}$
state	correction to the mass spectrum $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$-0.261\,\sigma^2$
2S	$-1.289\,\sigma^2$
3S	$-1.753\,\sigma^2$
4S	$-1.973\,\sigma^2$
1P	$-0.738\,\sigma^2$
2P	$-1.480\,\sigma^2$
1D	$-1.042\,\sigma^2$

[Bukor, JT '22]

NC $c\bar{b}$ meson	$b = c = 1.174\ , \quad \sigma^2 \approx 2.58 \times 10^{-39}$
state	correction to the mass spectrum $\sigma^2 M_{nl}^{(2)}$ [GeV]
1S	$-0.001\,\sigma^2$
2S	$-1.442\,\sigma^2$
3S	$-2.210\,\sigma^2$
4S	$-2.609\,\sigma^2$
1P	$-0.672\,\sigma^2$
2P	$-1.777\,\sigma^2$
1D	$-1.147\,\sigma^2$

[Bukor, JT '22]

- The most precise known mass $J/\psi(1S)$ particle (a c$\bar{\text{c}}$ meson)

$$M_{00} = 3096.900 \pm 0.006\,\text{MeV} \ .$$

- This leads to the bound on NC length scale

$$\lambda \leq 1.11 \ \times \ 10^{-18}\,\text{m} \ .$$

Future prospect

- Find a system or an effect, for which the NC correction is first order in σ.

Take home message

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Thank you for your attention!