Correlation functions in fuzzy scalar field theories

Juraj Tekel

Department of theoretical physics
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

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30°C
Near record

Quick motivation

Quick motivation

quantum mechanics $+$ gravity $\Rightarrow$ short distance (quantum) structure of spacetime

Quick motivation

$$\text{quantum mechanics} \;+\; \text{gravity} \;\Rightarrow\; \text{short distance (quantum) structure of spacetime}$$

[many talks this week]

[Doplicher, Fredenhagen, Roberts 1995; Hosenfelder: 1203.6191 [gr-qc]]

Quick motivation – correlation functions

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Correlation functions and renormalization in a scalar field theory on the fuzzy sphere

Kohta Hatakeyama* and Asato Tsuchiya*
Department of Physics, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan
*E-mail: hatakeyama.kohta.15@shizuoka.ac.jp; tsuchiya.asato@shizuoka.ac.jp

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Renormalization on the fuzzy sphere

Kohta Hatakeyama[1,2,*], Asato Tsuchiya[1,2,*], and Kazushi Yamashiro[1,*]
[1]*Department of Physics, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan*
[2]*Graduate School of Science and Technology, Shizuoka University, 3-5-1 Johoku, Naka-ku, Hamamatsu 432-8011, Japan*
*E-mail: hatakeyama.kohta.15@shizuoka.ac.jp, tsuchiya.asato@shizuoka.ac.jp, yamashiro.kazushi.17@shizuoka.ac.jp

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PoS PROCEEDINGS OF SCIENCE

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[1]*Department of Physics, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan*
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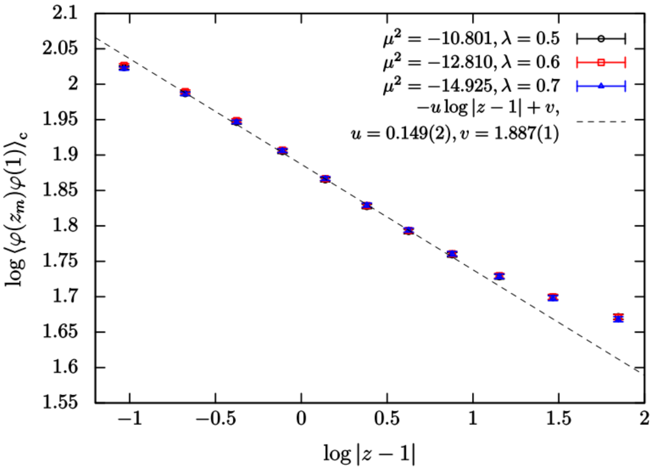
E-mail: hatakeyama.kohta.15@shizuoka.ac.jp, tsuchiya.asato@shizuoka.ac.jp, yamashiro.kazushi.17@shizuoka.ac.jp

Quick motivation – correlation functions



[Hatakeyama, Tsuchiya, Yamashiro 2018]

Moreover, it was observed that the behavior of the 2-point correlation functions is the same as that in a CFT at short distances and universally different from that at long distances. We consider the latter to be due to the UV/IR mixing.

The CFT observed at short distances seems to be different from the critical Ising model, because the value of $u/2$ in (3.5) disagrees with the scaling dimension of the spin operator, $\Delta_{\text{Ising}} = 1/8$. This indicates that the universality classes of the scalar field theory on the fuzzy sphere are totally different from those of an ordinary field theory[3].

[3] It should be noted that $\Delta_{\text{ours}} = u/2 \simeq 0.075 = 3/40$ coincides with the scaling dimension of the spin operator in the tricritical Ising model, which is the $(4,5)$ unitary minimal model.

[Hatakeyama, Tsuchiya, Yamashiro 2018]

Take home message

Take home message

- Plenty of interesting things happen on spaces with quantum structure.
- Among these are the properties of the correlation function.
- Doing field theory on fuzzy spaces is straightforward since they are essentially matrix models.

Fuzzy spaces

Fuzzy sphere [Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder 1990s]

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

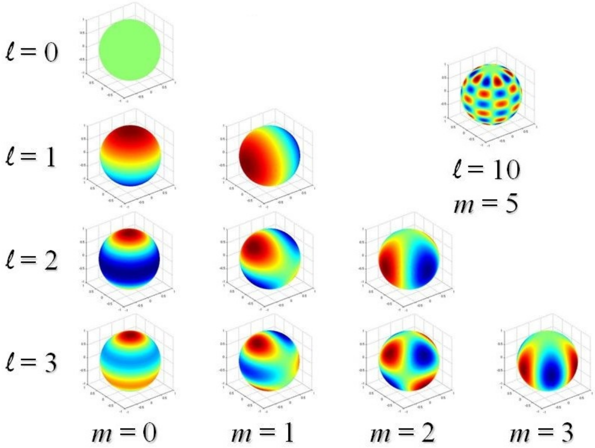


Image taken from http://principles.ou.edu/mag/earth.html

Fuzzy spaces

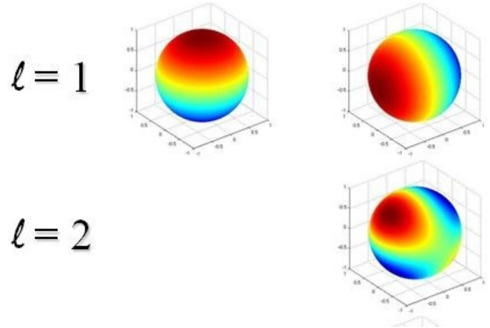
- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

we will not be able to see any features of functions under certain length scales.

- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces



- Number of independent functions with $l \leq L$ is N^2, the same as the number of $N \times N$ hermitian matrices.
  The idea is to map the former on the latter and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1} \left(\varphi \left(Y_{lm} \right) \varphi \left(Y_{l'm'} \right) \right) \ .$$

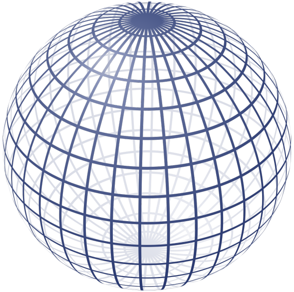
- We have a short distance structure, but the prize we had to pay was a noncommutative product $\star$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

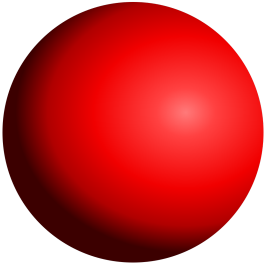
$$Y_{lm} \star Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- In the limit N or $L \to \infty$ we recover the original sphere.

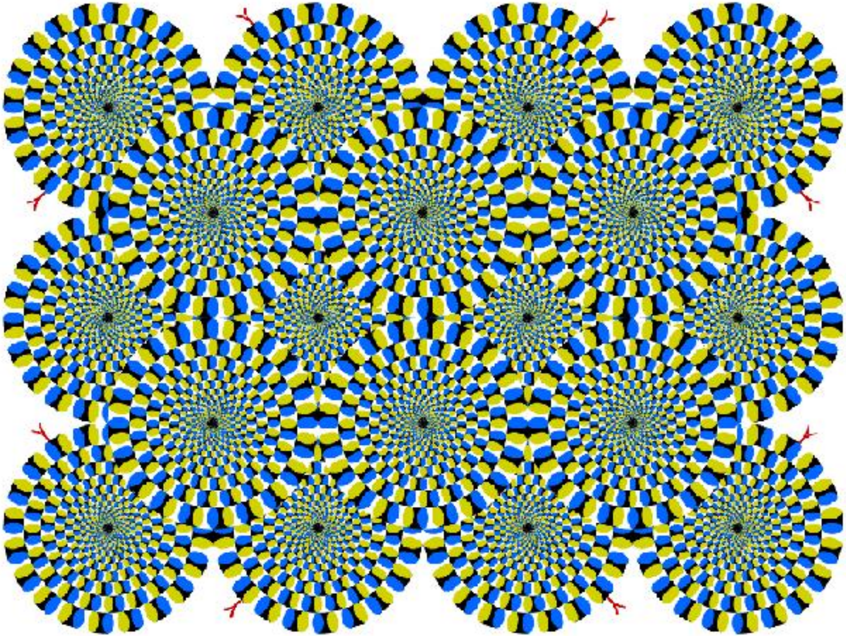
Fuzzy spaces - an alternative construction

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of ϕ represent the values of the function on these cells.





- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.



Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
  [Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  [Steinacker 2013]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair 2006]

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- An effective description of various systems in a certain limit (eg. QHE).
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- **Toy models of spaces with discrete quantum structure, which is expected to arise in quantum theory of gravity.**

Fuzzy field theories

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

  and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \; .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right] ,$$

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} .$$

- **Noncommutative** (for S_F^2)

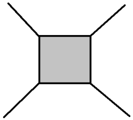
$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right] ,$$

$$\langle F \rangle = \frac{\int dM \, F(M) e^{-S(M)}}{\int dM \, e^{-S(M)}} .$$

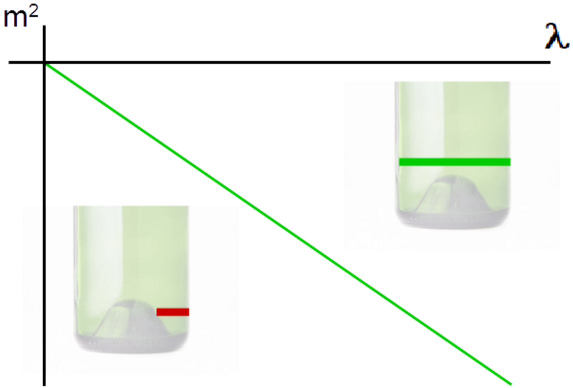
[Balachandran, Kürkçüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]

Fuzzy scalar field theory - UV/IR mixing

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  [Minwalla, Van Raamsdonk, Seiberg 2000; Vaidya 2001; Chu, Madore, Steinacker 2001]
- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- In terms of diagrams different properties of planar and non-planar ones.
  The (matrix) vertex is not invariant under permutation of incoming momenta.



$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\Phi\partial_i\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right)$$



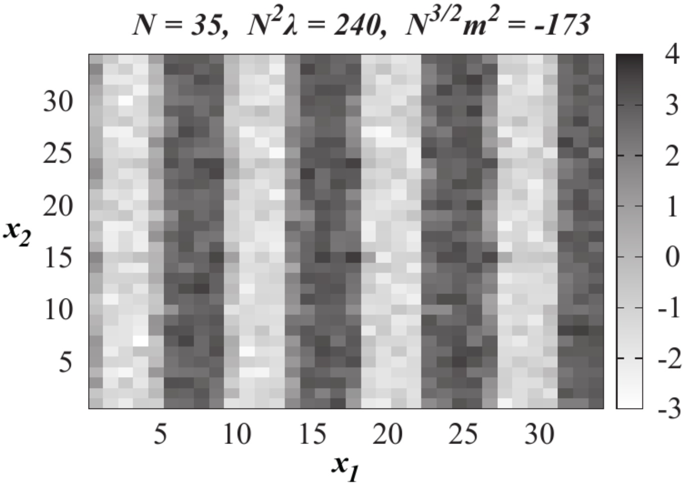
[Glimm, Jaffe 1974; Glimm, Jaffe, Spencer 1975; Chang 1976]
[Loinaz, Willey 1998; Schaich, Loinaz 2009]

Phases of fuzzy field theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  [Gubser, Sondhi 2001; Chen, Wu 2002]
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  [Martin 2004; García Flores, Martin, O'Connor 2006, 2009; Panero 2006, 2007; Ydri 2014; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero 2014; Mejía-Díaz, Bietenholz, Panero 2014; Medina, Bietenholz, D. O'Connor 2008; Bietenholz, Hofheinz, Nishimura 2004; Lizzi, Spisso 2012; Ydri, Ramda, Rouag 2016; Kováčik, O'Connor 2018]
  [Panero 2015]

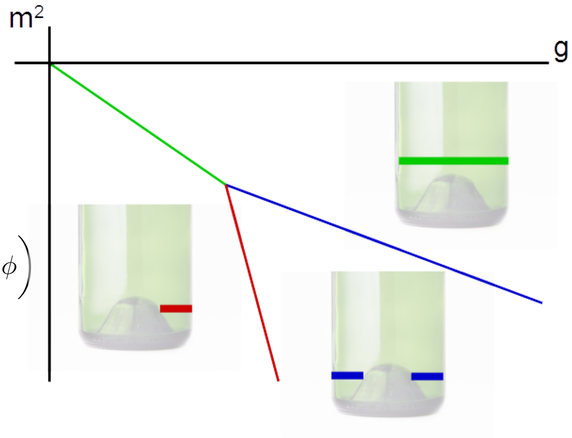
Phases of fuzzy field theories

[Mejía-Díaz, Bietenholz, Panero 2014] for $\mathbb{R}^2_\theta$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$

$$S = \int d^2x\left(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right)$$



Random matrices and fuzzy field theories

Random matrices

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Matrix model = ensemble of random matrices.
- An important example - ensemble of $N \times N$ hermitian matrices with

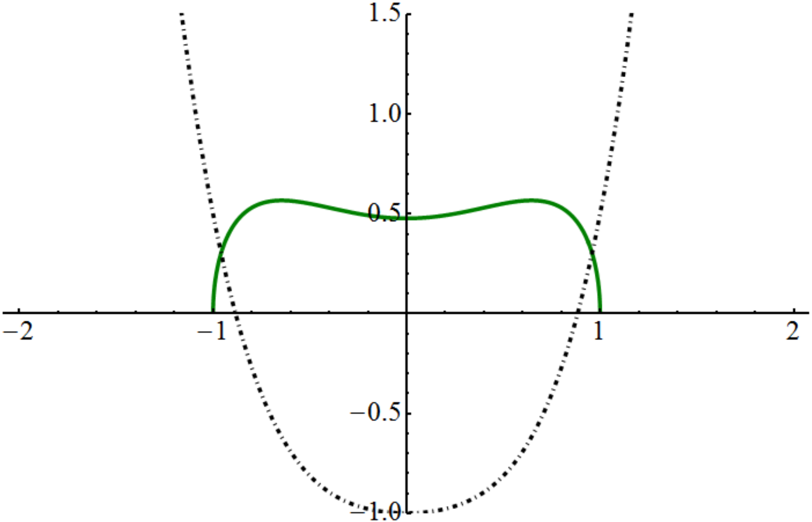
$$P(M) \sim e^{-N\mathrm{Tr}(V(M))} \ , \ \text{usually} \ V(x) = \frac{1}{2} r\, x^2 + g\, x^4$$

  and

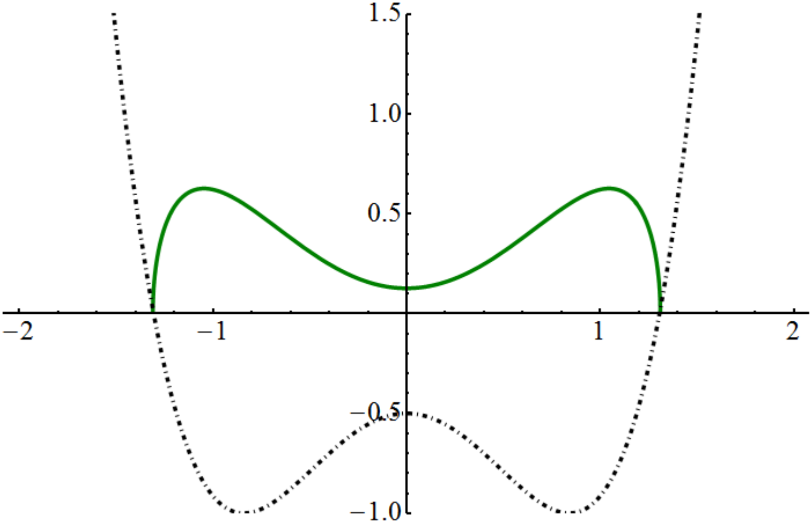
$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij}\, Im\, M_{ij}\right] .$$

- Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$.
- Requirement of such invariance is very restrictive. One is usually interested in the distribution of eigenvalues.

$V(x) = rx^2/2 + gx^4$ and $r > 0$

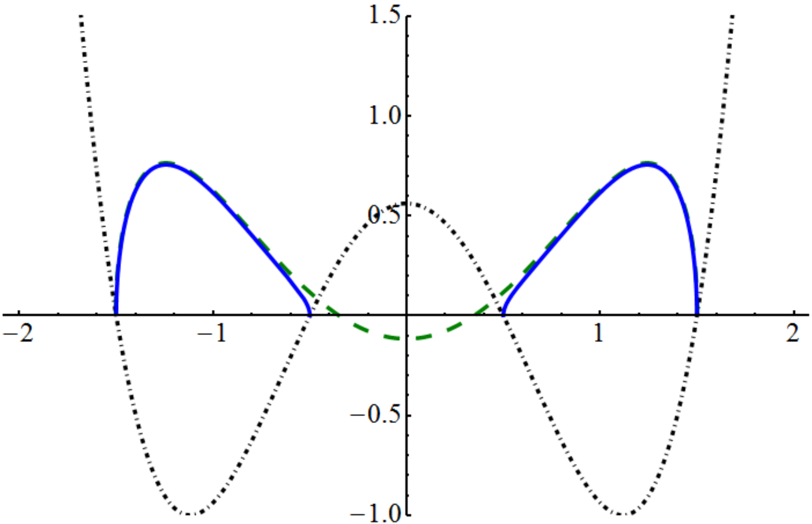


$V(x) = rx^2/2 + gx^4$ and $r < 0$



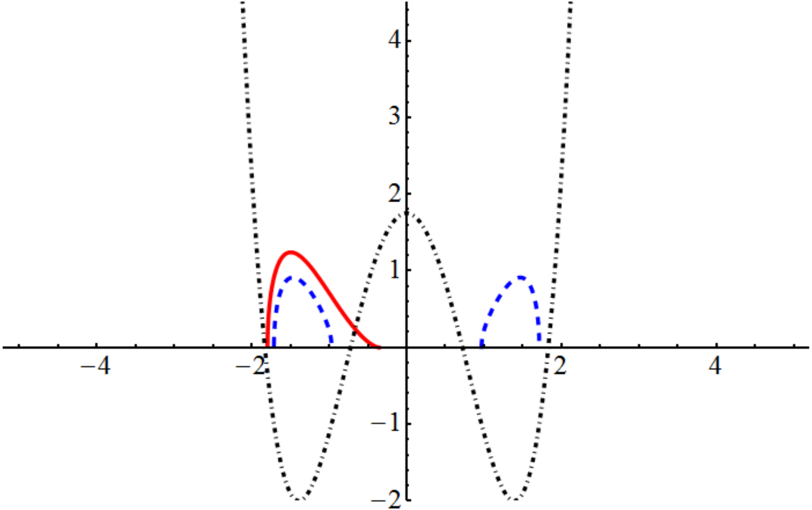
Random matrices - quartic potential

$V(x) = rx^2/2 + gx^4$ and $r < -4\sqrt{g}$



Random matrices - quartic potential

$V(x) = rx^2/2 + gx^4$ and $r << 0$



Kinetic term effective action

- Recall the action of the fuzzy scalar field theory

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \ .$$

This is a particular case of a matrix model since we need

$$\int dM\, F(M) e^{-S(M)} \ .$$

- The large N limit of the model with the kinetic term is not well understood. The key issue being that diagonalization $M = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ no longer straightforward.
- Integrals like

$$\langle F\rangle \sim \int d\Lambda \int dU \;\; F(\lambda_i, U)\; e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
$$\times\; e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)} \ .$$

$$e^{-N^2 S_{eff}(\Lambda)} = \int dU\, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. [O'Connor, Sämann 2007; Sämann 2010]
- The most recent result is [Sämann 2015]

$$S_{eff}(\Lambda) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n$$

- Standard treatment of such multitrace matrix model yields a very unpleasant behaviour close to the origin of the parameter space.

Perturbative calculation

$$e^{-N^2 S_{eff}(\Lambda)} = \int dU \, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

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$$\begin{aligned} S_{eff}(\Lambda) =& \frac{1}{2}\left[\varepsilon \frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2 \frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4 \frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4 \frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3 \frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n \end{aligned}$$

- More reasonable for large values of m^2, g.
  [Rea, Sämann 2015]

Second moment approximation

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. [Steinacker 2005]
- There is a unique parameter independent effective action that reconstructs this rescaling. [Polychronakos 2013]

$$S_{eff}(\Lambda) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}\ .$$

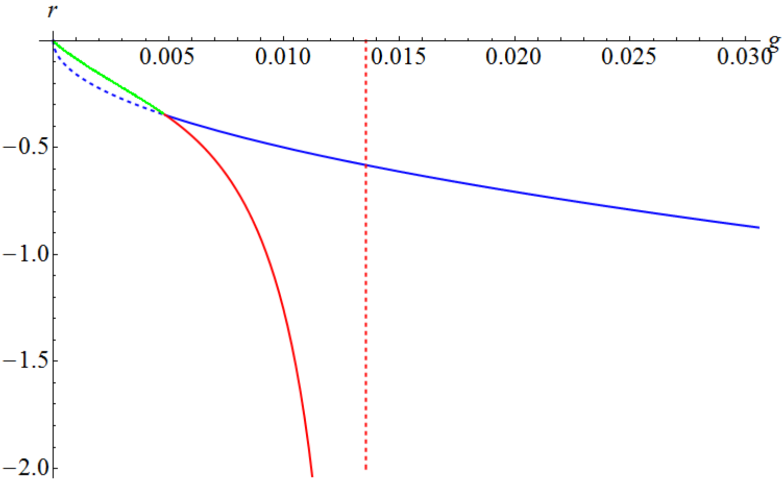
Can be generalized to more a more complicated kinetic term $\mathcal{K}$.

- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

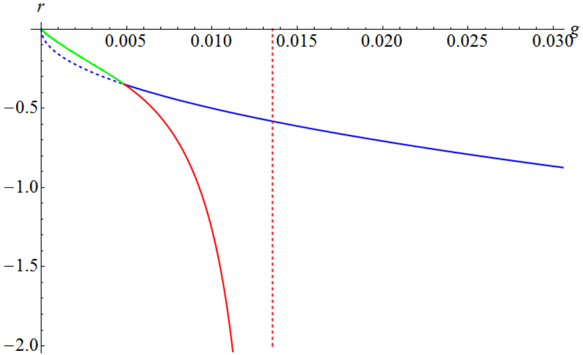
$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)\ .$$

[Šubjaková, JT PoS CORFU2019]

Second moment approximation



[JT '18; Šubjaková, JT 2020]


r
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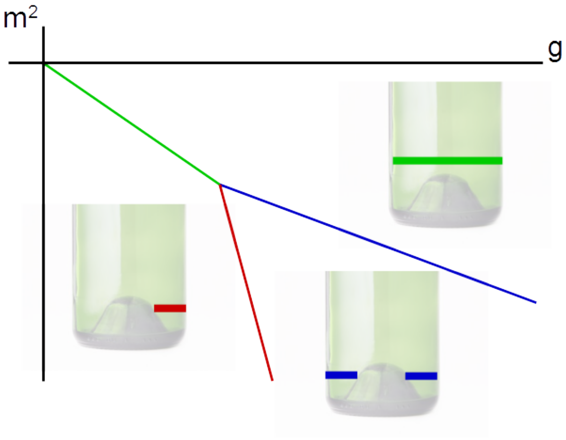

m2
g

Correlation functions

Coherent states

- We need to define value of the fuzzy function at a "point"

$$\phi(x)\ ,\ x \in S^2\ .$$

- One way to do that on a commutative space is

$$\phi(x) = \int dx\, \delta(x-y)\phi(y)\ .$$

Coherent states

- Natural basis in the auxiliary hilbert space $\mathcal{H}$ is the "spin" basis

$$|n\rangle = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} , \quad n = -s, \dots, s ,$$

derived from the highest weight state $|s\rangle$.

- For any $x \in S^2$ with radius 1, choose some $g_x \in SO(3)$ such that $x = g_x \cdot p$, where p is the north pole on S^2. We define [Perelomov 1986]

$$|x\rangle = g_x \cdot |s\rangle \, , \quad g_x \in SU(2)$$

and call the set of all $|x\rangle$ the coherent states.

- $|x\rangle$ is located around x, but is an element of $\mathcal{H}$, and is a non-commutative analogue of the point x. [Steinacker 2020]

Coherent states

- Fuzzy δ-function $|x\rangle\langle x|$ the most localized object at point x. We define the value $\phi(x)$ as

$$\phi(x) = \frac{4\pi}{N}\mathrm{Tr}\left(|x\rangle\langle x|\,\phi\right) = \langle x|\,\phi\,|x\rangle \ .$$

- Coherent states can be used to map (quantize) functions on S^2 on matrices

$$\phi(x) \to M = \int d^2x\,\phi(x)|x\rangle\langle x| \ .$$

and matrices on functions (de-quantize)

$$M \to \phi(x) = \langle x|\,M\,|x\rangle \ .$$

- They are orthogonal only in the large N limit

$$|\langle x|y\rangle|^2 = \left(\frac{1+x\cdot y}{2}\right)^{N-1} \quad .$$

- Some explicit formulas

$$|x\rangle \;=\; \sum_{s=-J}^{J} \sqrt{\binom{2J}{J+s}} \left(\cos\frac{\theta}{2}\right)^{J+s} \left(\sin\frac{\theta}{2}\right)^{J-s} e^{i(J-s)\varphi}\,|J,s\rangle \ ,$$

$$|z\rangle \;=\; \frac{1}{(1+|z|^2)^J} \sum_{s=-J}^{J} \sqrt{\binom{2J}{J+s}} z^{J-s}\,|J,s\rangle \ , \ z\in\mathbb{C} \ .$$

Correlation functions

- The two-point function of the model is

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{Z}\int d\phi\,\phi(x)\phi(y)e^{-S[\phi]}\ .$$

- We introduce the basis of polarization tensors

$$\phi_{ij} = \sum_{\mu} \mathrm{Tr}\,(\phi T_\mu)\,(T_\mu)_{ij}\ \ ,\ \ \mu = 0, 1, \ldots, N^2 - 1$$

and thus

$$\langle\phi(x)\phi(y)\rangle = x_i^* x_j y_k^* y_l\,\langle\phi_{ij}\phi_{kl}\rangle$$

with

$$\langle\phi_{ij}\phi_{kl}\rangle = (T_\mu)_{ij}\,(T_\nu)_{kl}\,\langle\mathrm{Tr}\,(\phi T_\mu)\,\mathrm{Tr}\,(\phi T_\nu)\rangle\ .$$

- We are after

$$\langle\mathrm{Tr}\,(\phi T_\mu)\,\mathrm{Tr}\,(\phi T_\nu)\rangle = \frac{1}{Z}\int d\Lambda e^{-S_{pot}-S_{vdm}}\int dU\mathrm{Tr}\left(U\Lambda U^\dagger T_\mu\right)\mathrm{Tr}\left(U\Lambda U^\dagger T_\nu\right)e^{-S_{kin}}\ .$$

Correlation functions

- Expand the kinetic part

$$\int dU\, \mathrm{Tr}\left(U\Lambda U^\dagger T_\mu\right) \mathrm{Tr}\left(U\Lambda U^\dagger T_\nu\right) \Big[1 - K_{\alpha\beta} \mathrm{Tr}\left(U\Lambda U^\dagger T_\alpha\right) \mathrm{Tr}\left(U\Lambda U^\dagger T_\beta\right) + \frac{1}{2}\left(K_{\alpha\beta}\mathrm{Tr}\left(U\Lambda U^\dagger T_\alpha\right) \mathrm{Tr}\left(U\Lambda U^\dagger T_\beta\right)\right)^2 + \ldots\Big]$$

with

$$K_{\mu\nu} := \frac{1}{2}\mathrm{Tr}\left(T_\mu[L_a,[L_a,T_\nu]]\right)$$

- We will need

$$I_{\mu_1\ldots\mu_n}(\Lambda) = \int dU \underbrace{\mathrm{Tr}\left(U\Lambda U^\dagger T_{\mu_1}\right)\ldots\mathrm{Tr}\left(U\Lambda U^\dagger T_{\mu_n}\right)}_{n} = \\ = \sum_\rho \frac{1}{\dim(\rho)}\chi_\rho(\Lambda)\,\mathrm{tr}_\rho\left(T_{\mu_1}\otimes\ldots\otimes T_{\mu_n}\right) \; ,$$

Correlation functions, $n = 2$

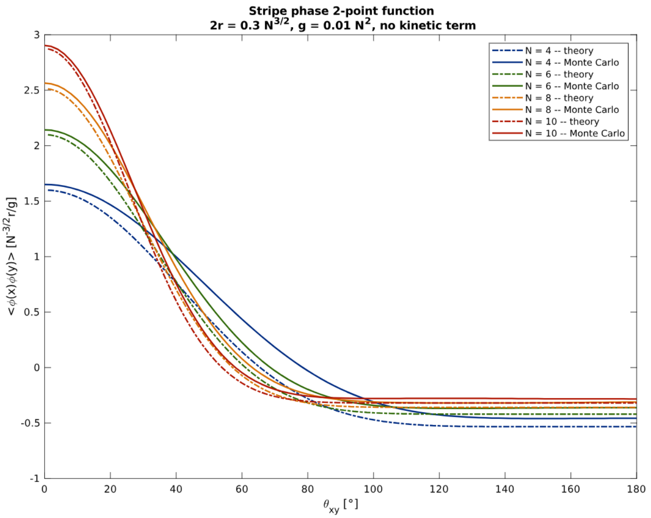
- For $n = 2$ and $c_k = \mathrm{Tr}\left(\phi^k\right)/N$ we get

$$
\begin{aligned}
I_{\mu_1\mu_2} &= \mathrm{Tr}\left(T_{\mu_1}\right)\mathrm{Tr}\left(T_{\mu_2}\right)\left(\frac{c_1^2 N^2}{N^2-1} - \frac{c_2}{N^2-1}\right) + \mathrm{Tr}\left(T_{\mu_1}T_{\mu_2}\right)\left(\frac{c_2 N}{N^2-1} - \frac{c_1^2 N}{N^2-1}\right) = \\
&= N\delta_{0\mu_1}\delta_{0\mu_2}\left(\frac{c_1^2 N^2}{N^2-1} - \frac{c_2}{N^2-1}\right) + \delta_{\mu_1\mu_2}\left(\frac{c_2 N}{N^2-1} - \frac{c_1^2 N}{N^2-1}\right) \; ,
\end{aligned}
$$

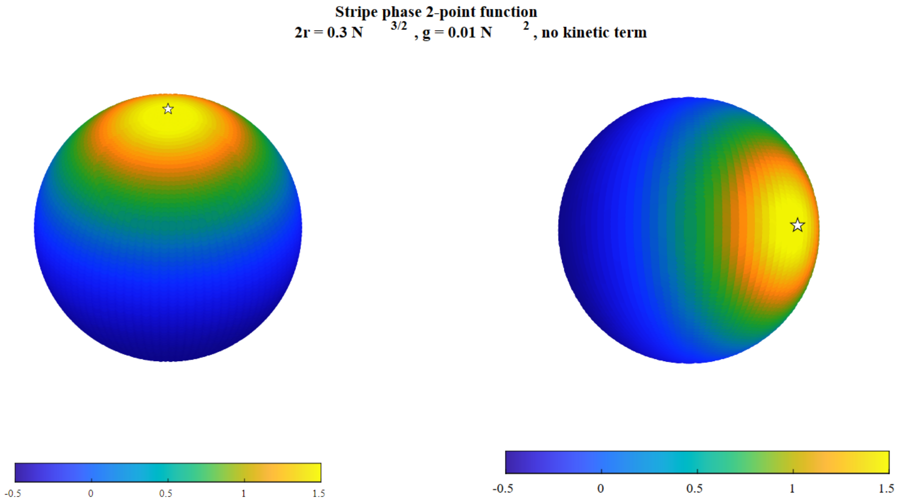
which leads to

$$
\begin{aligned}
\langle\phi_{ij}\phi_{kl}\rangle_{\text{pure potential}} &= \delta_{ij}\delta_{kl}\left(\frac{c_1^2 N^2}{N^2-1} - \frac{c_2}{N^2-1}\right) + \delta_{il}\delta_{jk}\left(\frac{c_2 N}{N^2-1} - \frac{c_1^2 N}{N^2-1}\right), \\
\langle\phi(x)(y)\rangle_{\text{pure potential}} &= \frac{1}{N^2-1}\left(N^2\left\langle c_1^2\right\rangle - \langle c_2\rangle\right) + |\langle x|y\rangle|^2 \frac{N}{N^2-1}\left(\langle c_2\rangle - \left\langle c_1^2\right\rangle\right) .
\end{aligned}
$$

Correlation functions, $n = 2$



Correlation functions, $n = 2$



Correlation functions, $n = 4$

- In the first order in the kinetic term, we get

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{Z}\int d\Lambda e^{-S_{pot}-S_{vdm}}\Big(x_i^* x_j y_k^* y_l \left(T_{\mu_1}\right)_{ij}\left(T_{\mu_2}\right)_{kl} I_{\mu_1\mu_2} - \varepsilon x_i^* x_j y_k^* y_l \left(T_{\mu_1}\right)_{ij}\left(T_{\mu_2}\right)_{kl} K_{\mu_3\mu_4} I_{\mu_1\mu_2\mu_3\mu_4}\Big) \ ,$$

  but now

$$Z = \int d\Lambda e^{-S_{pot}-S_{vdm}-S_{eff}} \ .$$

- This leads to

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{Z}\int d\Lambda e^{-S_{pot}-S_{vdm}-S_{eff}}\left(I_2 - \varepsilon I_4\right)\left(1 + \varepsilon S_1\right) = \langle I_2 - \varepsilon I_4 + \varepsilon S_1 I_2\rangle \ .$$

- I_4 includes combinations of moments $c_4, c_1 c_3, c_1^2 c_2, c_1^4$ which need to be rescaled when doing the saddle point approximation.

Correlation functions, $n = 4$

Correlation functions, $n = 4$

- A direct application of the saddle point approximation leads to essentially the same formula as for pure potential. The difference gets smaller as we increase N.
- An angular behavior different from $const + |\langle x|y\rangle|^2$ comes from

$$k_a (T_a)_{ij} (T_a)_{lm} x_i^* x_j y_k^* y_l$$

and

$$k_a d_{abc} d_{abd} (T_c)_{ij} (T_d)_{lm} x_i^* x_j y_k^* y_l \ .$$

Here

$$d_{abc} = \mathrm{Tr} \left(\{ T_a, T_b \} T_c \right) \ .$$

- Both of these are very small compared to the other contributions.

Truncated Heisenberg algebra and Grosse-Wulkenhaar model

- Grosse-Wulkenhaar model [2000's]

$$S_{GW} = \int d^2x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\Omega^2(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right) ,$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu \ .$$

- This model is renormalizable.
- Described by a matrix model in terms of truncated Heisenberg algebra. [Burić, Wohlgenannt 2010]

Truncated Heisenberg algebra

- The NC plane coordinates can be realized by

$$X=\frac{1}{\sqrt{2}}\begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix}\ ,\ Y=\frac{i}{\sqrt{2}}\begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix}\ ,$$

  then

$$[X,Y]=i\ .$$

- This algebra is then truncated to a finite dimension.

- Define finite matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \sqrt{N-1} \\ & & & \sqrt{N-1} & \end{pmatrix} , \quad Y = \dots \; ,$$

which gives

$$[X, Y] = i(1 - Z) \; , \; Z = \mathrm{diag}\,(0, \dots, N) \; .$$

- Original algebra is recovered in the $N \to \infty$ limit or under the $Z = 0$ condition.
- The kinetic term becomes

$$\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi \to [X, M][X, M] + [Y, M][Y, M] \; .$$

- The harmonic potential becomes

$$\frac{1}{2}\Omega^2(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)\to RM^2 \ ,$$

where R is a fixed external matrix

$$R=\frac{15}{2}-4Z^2-8\left(X^2+Y^2\right)=\frac{31}{2}-16\,\mathrm{diag}\,(1,2,\ldots,N-1,8N) \ .$$

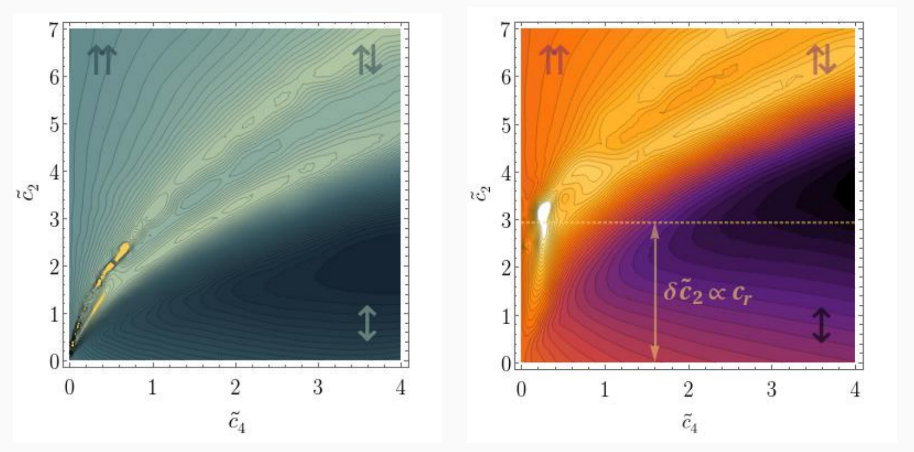
- Interpretation of coupling to the curvature of the space.
- We are thus left with a matrix model with action

$$S=\mathrm{Tr}\,(M[X,[X,M]]+M[Y,[Y,M]])-g_r\mathrm{Tr}\left(RM^2\right)-g_2\mathrm{Tr}\left(M^2\right)+g_4\mathrm{Tr}\left(M^4\right) \ .$$

Removal of stripes – GW model

[Prekrat, Todorović-Vasović, Ranković ’21; Prekrat ’21]

- Numerical investigation of this matrix model leads to



[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]

- The effect of the curvature term

$$S(M) = \mathrm{Tr}\left(M\mathcal{K}M\right) - \mathrm{Tr}\left(g_r R M^2\right) - g_2 \,\mathrm{Tr}\left(M^2\right) + g_4 \,\mathrm{Tr}\left(M^4\right) \; .$$

- Effective action up to g_r^4

$$S(\Lambda) = N\,\mathrm{Tr}\Big(- g_2\Lambda^2 + 8g_r\Lambda^2 + g_4\Lambda^4 - \frac{32}{3}g_r^2\Lambda^4\Big) + \frac{1024}{45}g_r^4\Lambda^8 + \\ + \frac{32}{3}g_r^2\left(\mathrm{Tr}\left(\Lambda^2\right)\right)^2 + \frac{1024}{15}g_r^4\left(\mathrm{Tr}\left(\Lambda^4\right)\right)^2 - \frac{4096}{45}g_r^4\,\mathrm{Tr}\left(\Lambda^6\right)\mathrm{Tr}\left(\Lambda^2\right) \; .$$

- This is a multitrace matrix model which can be analyzed.

Plane kinetic term effective action model

[Bukor, JT '23]

- The effect of the kinetic term

$$S(M) = \mathrm{Tr}\left(M\mathcal{K}M\right) - \mathrm{Tr}\left(g_r R M^2\right) - g_2\,\mathrm{Tr}\left(M^2\right) + g_4\,\mathrm{Tr}\left(M^4\right) \ .$$

- This leads to the effective action

$$S_{eff}(\Lambda) = N^2\left[\varepsilon t_2 - \varepsilon^2 \frac{2}{3} t_2^2 + \varepsilon^2 \frac{97}{120}\left(t_4 - 2t_2^2\right)\right] \ ,$$

where $t's$ are symmetrized models

$$t_n = \frac{1}{N}\,\mathrm{Tr}\left(\phi - \frac{\mathbb{1}}{N}\mathrm{tr}\left(\phi\right)\right)^n \ .$$

- This is a multitrace matrix model which can be analyzed, e.g. to obtain phase structure of the model.

Correlation functions in the GW model

- Going back to

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{Z}\int d\Lambda e^{-S_{pot}-S_{vdm}-S_{eff}}\left(I_2-\varepsilon I_4\right)\left(1+\varepsilon S_1\right)$$

inputs and saddles change, but the formula and the idea stays.

Take home message

Take home message and 2do list

- Plenty of interesting things happen on spaces with quantum structure.
- Among these are the properties of the correlation function.
- Doing field theory on fuzzy spaces is straightforward since they are essentially matrix models.

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  - Trick similar to the kinetic term effective?
  - Generating function $Z[J]$?
- Entanglement entropy on fuzzy spaces.

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Thank you for your attention!