Fuzzy Physics and Matrix Models

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Take home message

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- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy spaces are examples of such spacetimes.
- Physics on such spaces is described by random matrix ensembles.

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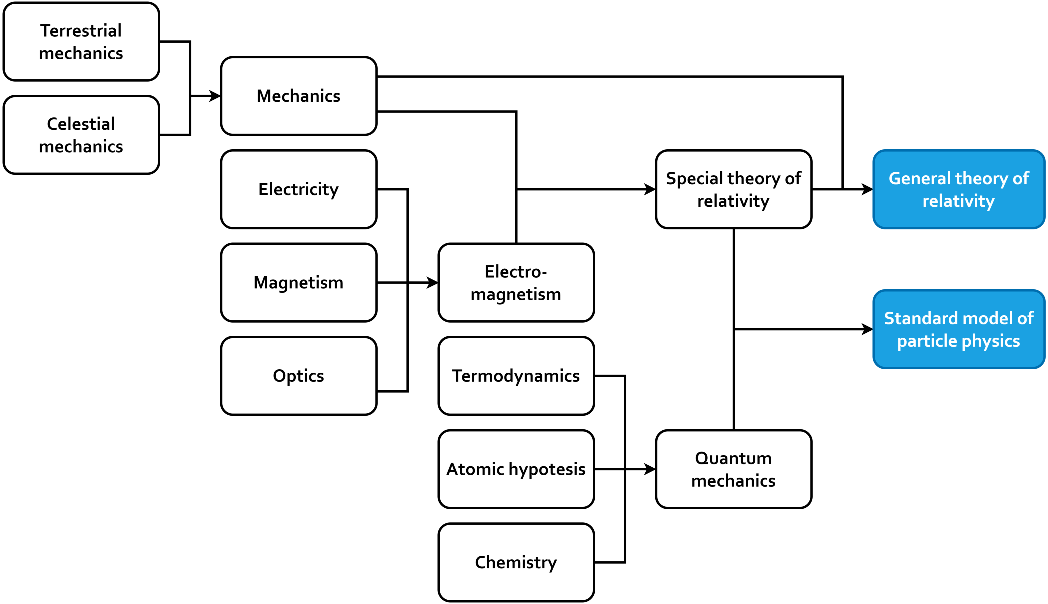
- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy spaces are examples of such spacetimes.
- Physics on such spaces is described by random matrix ensembles.

- We do have PhD. and postdoc positions related to this in Bratislava.

Quick motivation


Terrestrial mechanics
Celestial mechanics
Mechanics
Electricity
Magnetism
Optics
Electro-magnetism
Termodynamics
Atomic hypotesis
Chemistry
Special theory of relativity
Quantum mechanics
General theory of relativity
Standard model of particle physics

Quantum structure of spacetime

- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. [Hossenfelder 1203.6191]
- Reasons:
  - gravitational Heisenberg microscope,
  - instability of quantum gravitational vacuum, [Doplicher, Fredenhagen, Roberts '95]
  - emergent spacetime.

Fuzzy spaces

Fuzzy spaces

Fuzzy sphere [Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder 1990s]

- Functions on the usual sphere are given by

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta,\phi) \ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi) = l(l+1) Y_{lm}(\theta,\phi) \ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

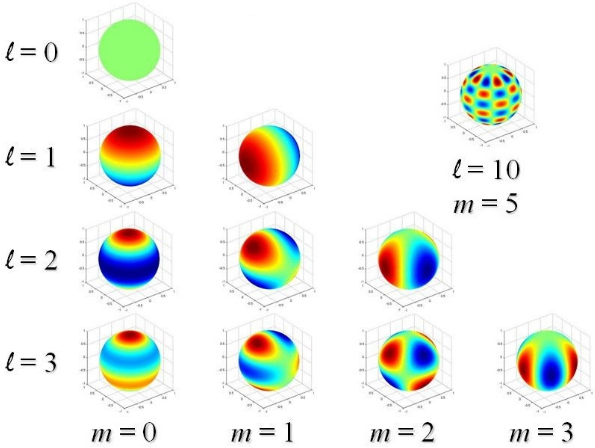


Image taken from http://principles.ou.edu/mag/earth.html

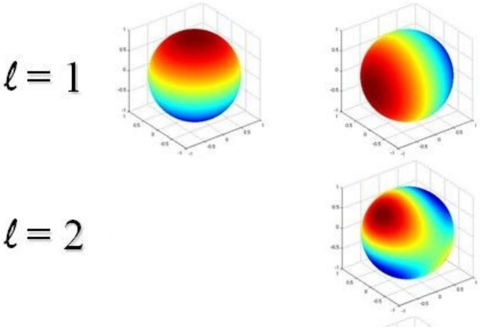
- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta,\phi) \ ,$$

we will not be able to see any features of functions under certain length scales.

- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces



Fuzzy spaces

- Number of independent functions with $l \leq L$ is N^2, the same as the number of $N \times N$ hermitian matrices.
  The idea is to map the former on the latter and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1} \left(\varphi\left(Y_{lm}\right) \varphi\left(Y_{l'm'}\right) \right) \ .$$

Fuzzy spaces

- We have a short distance structure, but the prize we had to pay was a noncommutative product $\star$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- In the limit N or $L \to \infty$ we recover the original sphere.

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i,j = 1,2,3 \ ,$$

which generate the algebra of functions.

- For the fuzzy sphere S^2_N we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i,j = 1,2,3 \ .$$

- Such $\hat{x}_i$'s generate a different, non-commutative, algebra and S^2_N is an object, which has this algebra as an algebra of functions.

- The conditions can be realized as an $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} s(s+1) = r^2 \; .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

- Most importantly nonzero commutators

$$\hat{x}_i\hat{x}_i = \rho^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \ .$$

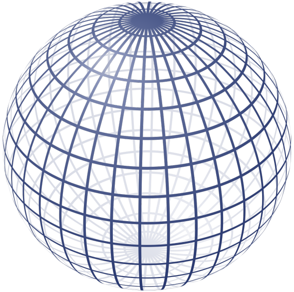
- Configuration space is analogous to phase space of quantum mechanics.
- In a similar fashion it is possible to construct an analogous deformation of the plane

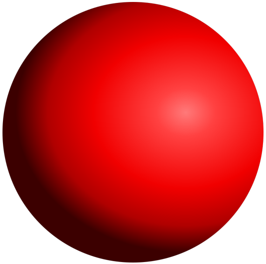
$$\hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ij} = i\theta_{ij} \ , \ i = 1,2 \ .$$

Construction uses the $\star$-product

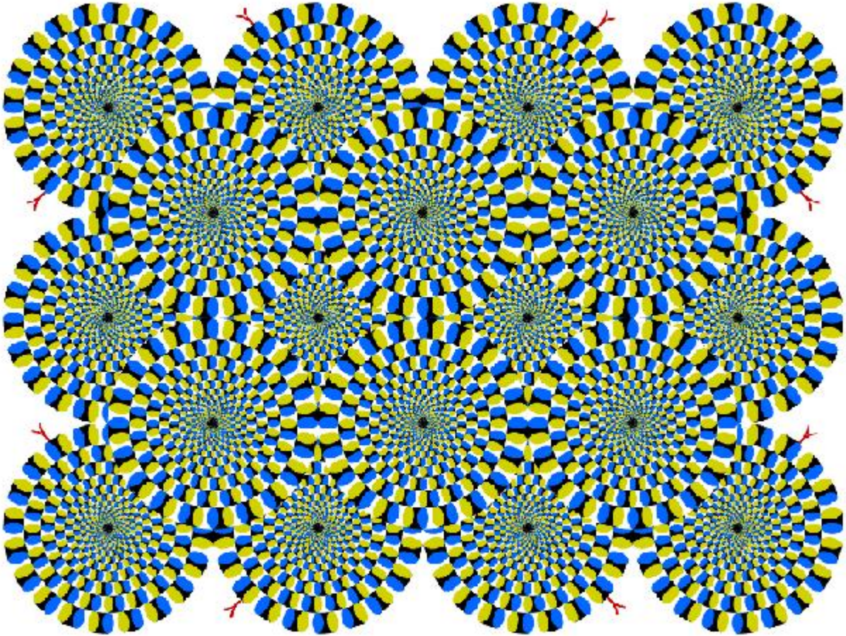
$$f \star g = f\, e^{\frac{i}{2}\overleftarrow{\partial}\,\theta\,\overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} + \cdots$$

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of ϕ represent the values of the function on these cells.





- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.



Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
  [Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  [Steinacker 2013]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair 2006]
- Toy models of spaces with discrete quantum structure, which is expected to arise in quantum theory of gravity.

Fuzzy field theories

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \ .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right] ,$$

$$\langle F\rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} .$$

- **Noncommutative** (for S_F^2)

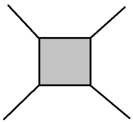
$$S(M) = \frac{4\pi R^2}{N}\mathrm{Tr}\left[\frac{1}{2}M\frac{1}{R^2}[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + V(M)\right] ,$$

$$\langle F\rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} .$$

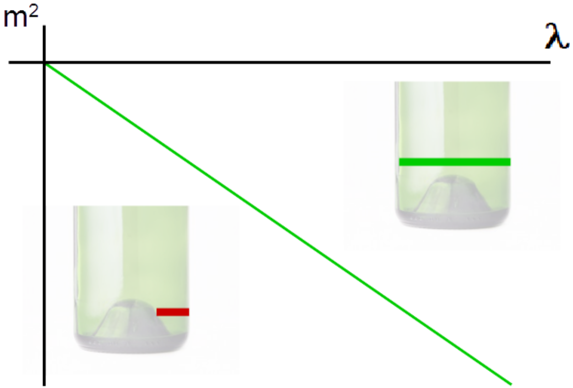
[Balachandran, Kürkçüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]

Fuzzy scalar field theory - UV/IR mixing

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  [Minwalla, Van Raamsdonk, Seiberg 2000; Vaidya 2001; Chu, Madore, Steinacker 2001]
- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- In terms of diagrams different properties of planar and non-planar ones.
  The (matrix) vertex is not invariant under permutation of incoming momenta.



$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\Phi\partial_i\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right)$$

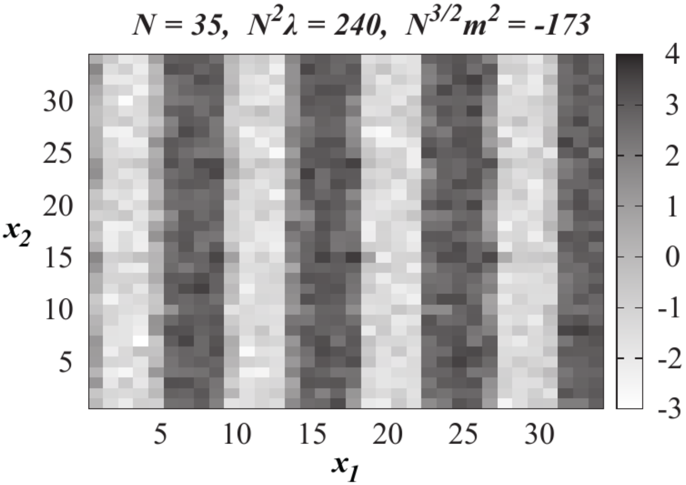


[Glimm, Jaffe 1974; Glimm, Jaffe, Spencer 1975; Chang 1976]
[Loinaz, Willey 1998; Schaich, Loinaz 2009]

Phases of fuzzy field theories

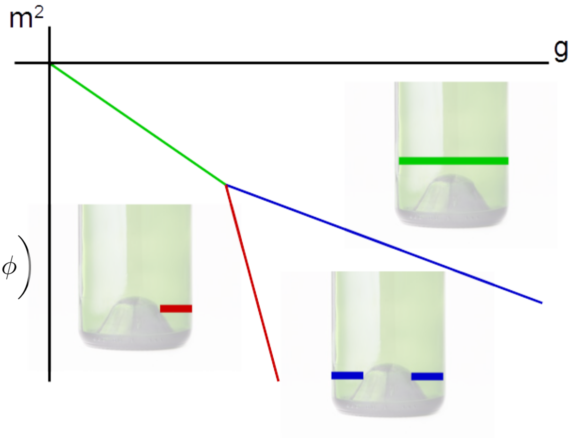
- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  [Gubser, Sondhi 2001; Chen, Wu 2002]
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  [Martin 2004; García Flores, Martin, O'Connor 2006, 2009; Panero 2006, 2007; Ydri 2014; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero 2014; Mejía-Díaz, Bietenholz, Panero 2014; Medina, Bietenholz, D. O'Connor 2008; Bietenholz, Hofheinz, Nishimura 2004; Lizzi, Spisso 2012; Ydri, Ramda, Rouag 2016; Kováčik, O'Connor 2018]
  [Panero 2015]

[Mejía-Díaz, Bietenholz, Panero 2014] for $\mathbb{R}^2_\theta$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$

$$S = \int d^2x\left(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right)$$



Random matrices ...

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Matrix model = ensemble of random matrices.
- An important example - ensemble of $N \times N$ hermitian matrices with

$$P(M) \sim e^{-N\mathrm{Tr}(V(M))}\ , \ \text{usually}\ V(x) = \frac{1}{2} r\, x^2 + g\, x^4$$

and

$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij}\, Im\, M_{ij}\right] .$$

- Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$.
- Requirement of such invariance is very restrictive. One is usually interested in the distribution of eigenvalues.

- If we ask invariant questions, we can turn

$$\langle f\rangle = \frac{1}{Z}\int dM\, f(M)P(M)$$

into an eigenvalue problem by diagonalization $M = U\Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \dots, \lambda_N)$, the integration measure becomes

$$dM = dU\left(\prod_{i=1}^{N} d\lambda_i\right) \times \prod_{i<j}(\lambda_i - \lambda_j)^2$$

- We are to compute integrals like

$$\langle f\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) f(\lambda_i)\, e^{-\left[\sum_i V(\lambda_i) - 2\sum_{i<j}\log|\lambda_i - \lambda_j|\right]} \times \int dU$$

- Term
$$2\sum_{i<j}\log|\lambda_i-\lambda_j|$$
is of order N^2 if $\lambda_i\sim 1$. Potential term
$$\sum_i V(\lambda_i)$$
is of order N.
- We need to enhance the probability measure by a factor of N to
$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i)-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
- This makes the N^2 dependence explicit.

Random matrices - eigenvalue decomposition

- We introduce eigenvalue distribution

$$\rho(\lambda) = \frac{1}{N}\sum_j \delta(\lambda - \lambda_j)$$

which gives for the averages

$$\langle f \rangle = \int d\lambda\, \rho(\lambda) f(\lambda)\ .$$

- The question is, how does do probability measure

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i) - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]}$$

translate into eigenvalue distribution ρ.

Random matrices - large N

- For finite N - orthogonal polynomials method.
- For $N \to \infty$ the question simplifies due to the factor N^2

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i) - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]}.$$

- For large N only configurations with small exponent contribute significantly to the integral. In the limit $N \to \infty$ only the extremal configuration

$$V'(\lambda_i) - \frac{2}{N}\sum_{j \neq i}\frac{1}{\lambda_i - \lambda_j} = 0 \quad \forall i$$

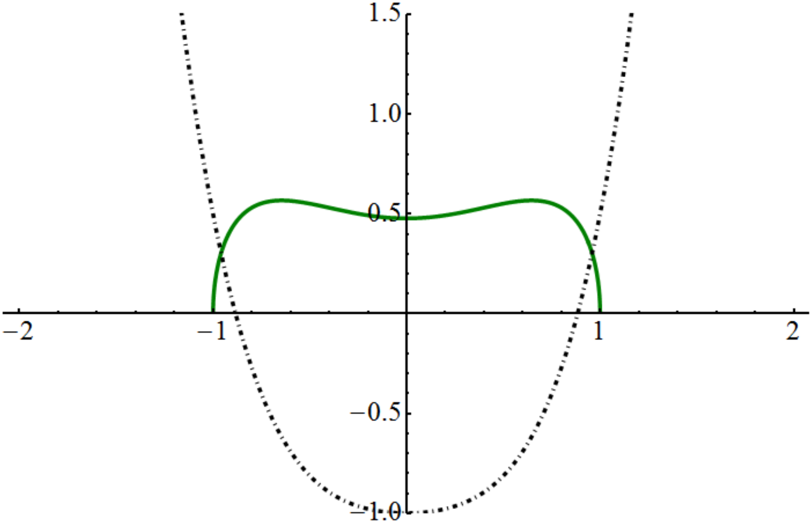
- Like a gas of particles with logarithmic repulsion. This gives us nice intuition.

- The simplest case

$$V(x) = \frac{1}{2}rx^2 + gx^4$$

Random matrices - quartic potential

$V(x) = rx^2/2 + gx^4$ and $r > 0$



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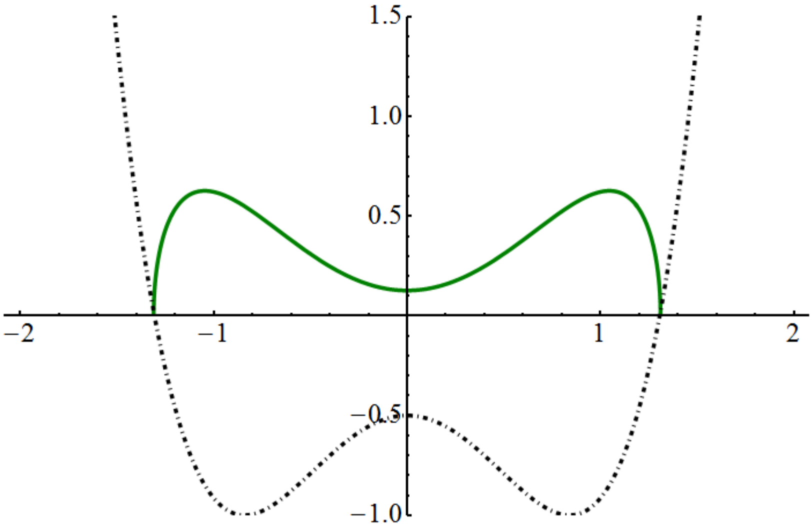

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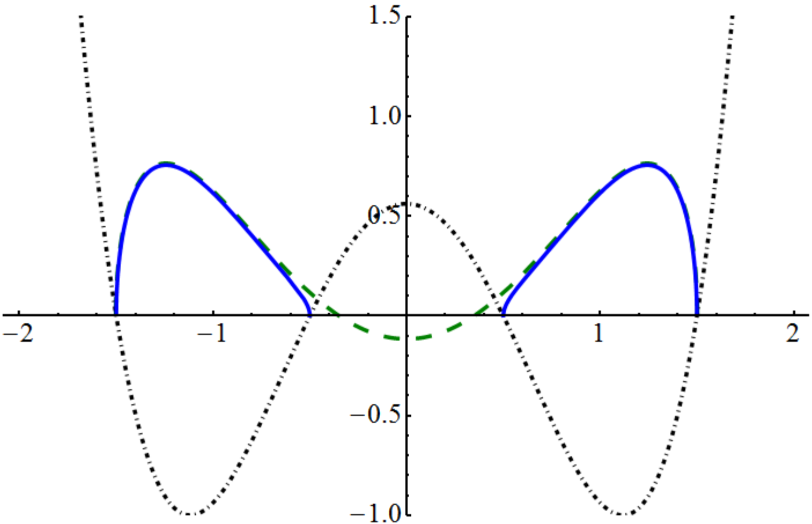

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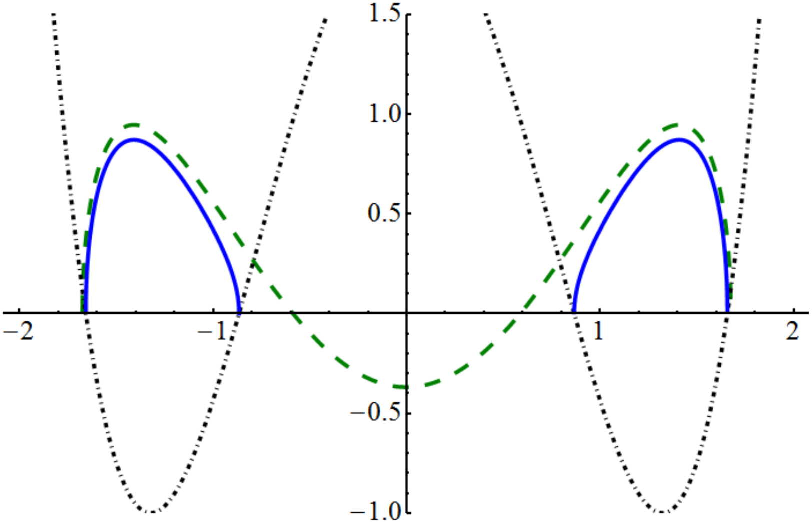

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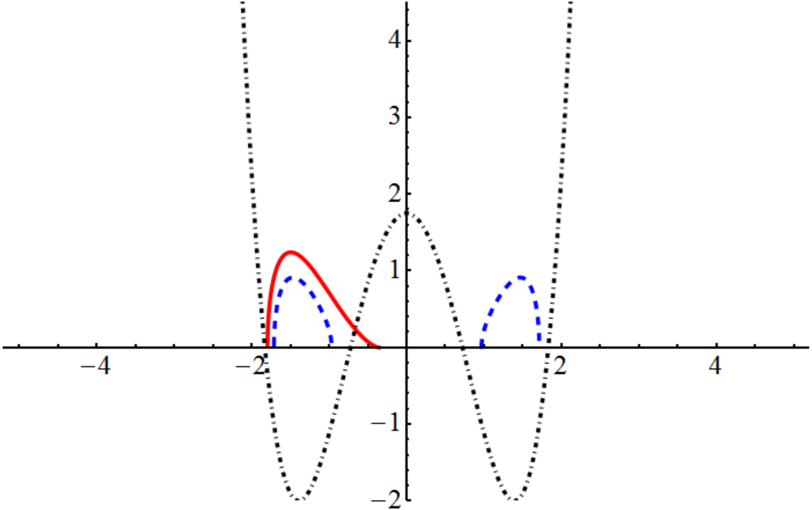

Random matrices - quartic potential

- If more than one solution is possible, the one with lower energy

$$\mathcal{F} = -N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]$$

  is the preferred one.
- The probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]}$$

  i.e. the more probable solution.

Random matrices and fuzzy field theories

Kinetic term effective action

- Recall the action of the fuzzy scalar field theory

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \ .$$

  This is a particular case of a matrix model since we need

$$\int dM\, F(M) e^{-S(M)} \ .$$

- The large N limit of the model with the kinetic term is not well understood.
  The key issue being that diagonalization $M = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ no longer straightforward.
- Integrals like

$$\langle F\rangle \sim \int d\Lambda \int dU\ \ F(\lambda_i,U)\ e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
$$\times\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)} \ .$$

Perturbative calculation

$$e^{-N^2 S_{eff}(\Lambda)} = \int dU\, e^{-\varepsilon \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M.
  [O'Connor, Sämann 2007; Sämann 2010]
- The most recent result is
  [Sämann 2015]

$$S_{eff}(\Lambda) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n$$

- Standard treatment of such multitrace matrix model yields a very unpleasant behaviour close to the origin of the parameter space.

Perturbative calculation

$$e^{-N^2 S_{eff}(\Lambda)} = \int dU\, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

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- More reasonable for large values of m^2, g.
  [Rea, Sämann 2015]

Second moment approximation

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. [Steinacker 2005]
- There is a unique parameter independent effective action that reconstructs this rescaling. [Polychronakos 2013]

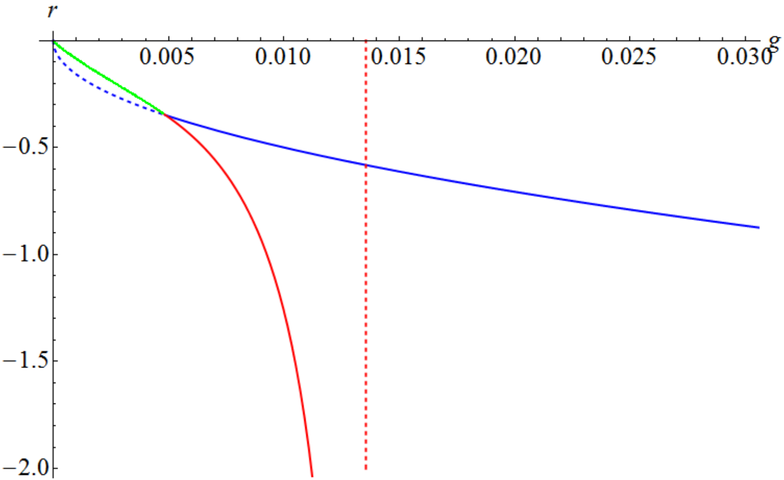
$$S_{eff}(\Lambda) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}\ .$$

  Can be generalized to more a more complicated kinetic term $\mathcal{K}$.
- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

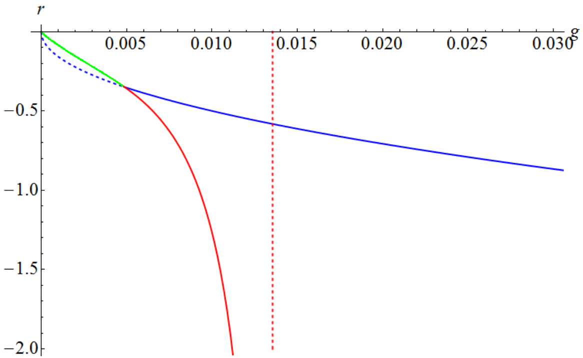
$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)\ .$$

[Šubjaková, JT PoS CORFU2019]

Second moment approximation



[JT '18; Šubjaková, JT 2020]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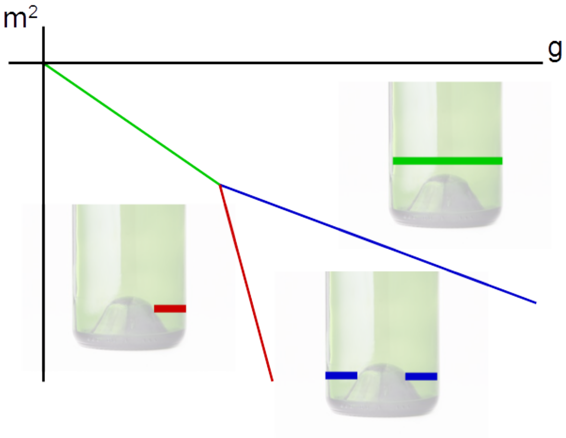

m^2
g

Truncated Heisenberg algebra and Grosse-Wulkenhaar model

GW MODEL

- Grosse-Wulkenhaar model [2000's]

$$S_{GW} = \int d^2x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) ,$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu \ .$$

- This model is renormalizable.
- Described by a matrix model in terms of truncated Heisenberg algebra. [Burić, Wohlgenannt 2010]

Truncated Heisenberg algebra

- The NC plane coordinates can be realized by

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} , \quad Y = \frac{i}{\sqrt{2}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} ,$$

then

$$[X, Y] = i \ .$$

- This algebra is then truncated to a finite dimension.

Truncated Heisenberg algebra

- Define finite matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \sqrt{N-1} \\ & & & \sqrt{N-1} & \end{pmatrix} \ , \ Y = \dots \ ,$$

which gives

$$[X, Y] = i(1 - Z) \ , \ Z = \mathrm{diag}\,(0, \dots, N) \ .$$

- Original algebra is recovered in the $N \to \infty$ limit or under the $Z = 0$ condition.
- The kinetic term becomes

$$\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi \to [X, M][X, M] + [Y, M][Y, M] \ .$$

GW matrix model

- The harmonic potential becomes

$$\frac{1}{2}\Omega^2(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) \to RM^2 \ ,$$

where R is a fixed external matrix

$$R = \frac{15}{2} - 4Z^2 - 8\left(X^2 + Y^2\right) = \frac{31}{2} - 16\,\mathrm{diag}\,(1, 2, \ldots, N-1, 8N) \ .$$

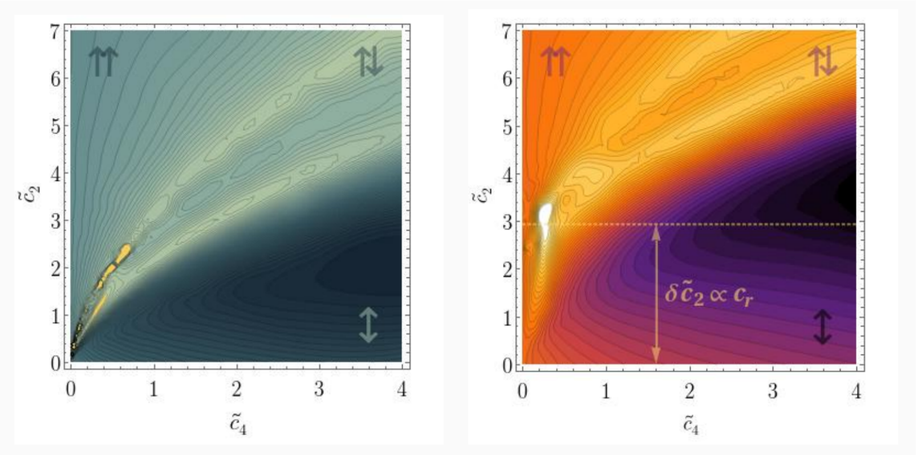
- Interpretation of coupling to the curvature of the space.
- We are thus left with a matrix model with action

$$S = \mathrm{Tr}\,(M[X,[X,M]] + M[Y,[Y,M]]) - g_r\mathrm{Tr}\left(RM^2\right) - g_2\mathrm{Tr}\left(M^2\right) + g_4\mathrm{Tr}\left(M^4\right) \ .$$

Removal of stripes – GW model

[Prekrat, Todorović-Vasović, Ranković '21; Prekrat '21]

- Numerical investigation of this matrix model leads to



[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]

- The effect of the curvature term

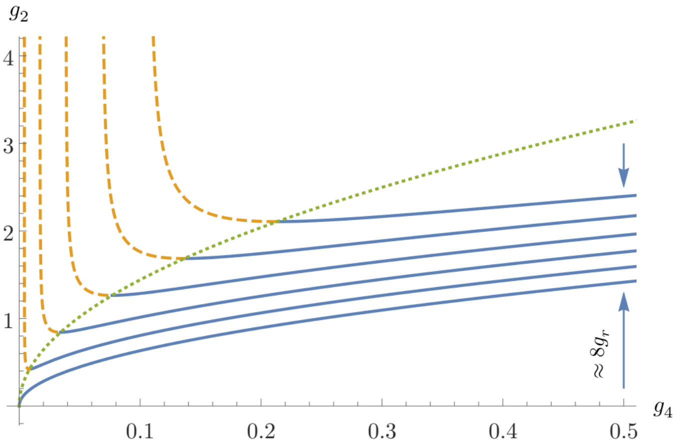
$$S(M) = \mathrm{Tr}\left(M\mathcal{K}M\right) - \mathrm{Tr}\left(g_r R M^2\right) - g_2\,\mathrm{Tr}\left(M^2\right) + g_4\,\mathrm{Tr}\left(M^4\right) \ .$$

- Effective action up to g_r^4

$$\begin{aligned} S(\Lambda) &= N\,\mathrm{Tr}\Big(-g_2\Lambda^2 + 8g_r\Lambda^2 + g_4\Lambda^4 - \frac{32}{3}g_r^2\Lambda^4\Big) + \frac{1024}{45}g_r^4\Lambda^8 + \\ &+ \frac{32}{3}g_r^2\left(\mathrm{Tr}\left(\Lambda^2\right)\right)^2 + \frac{1024}{15}g_r^4\left(\mathrm{Tr}\left(\Lambda^4\right)\right)^2 - \frac{4096}{45}g_r^4\,\mathrm{Tr}\left(\Lambda^6\right)\mathrm{Tr}\left(\Lambda^2\right) \ . \end{aligned}$$

- This is a multitrace matrix model which can be analyzed.

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT ’22]



Removal of stripes – GW model

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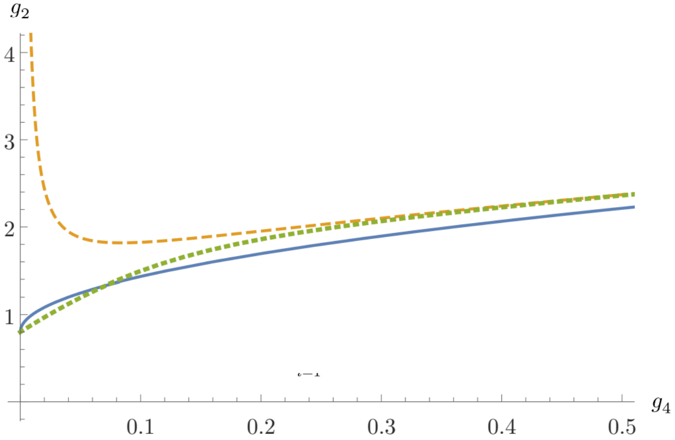

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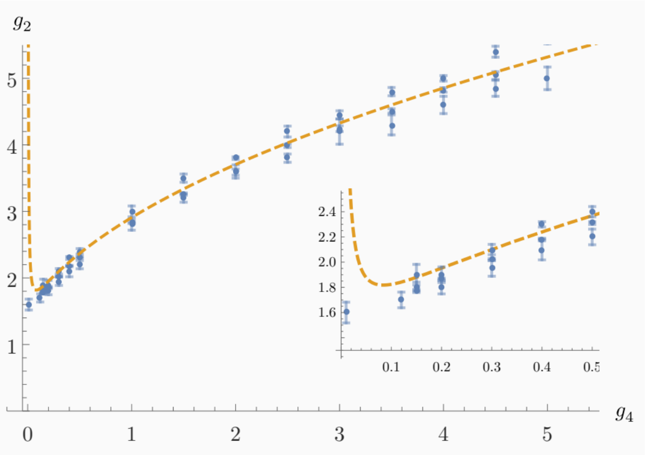


[Bukor, JT '23]

- The effect of the kinetic term

$$S(M) = \mathrm{Tr}\left(M\mathcal{K}M\right) - \mathrm{Tr}\left(g_r R M^2\right) - g_2\,\mathrm{Tr}\left(M^2\right) + g_4\,\mathrm{Tr}\left(M^4\right) \ .$$

- This leads to the effective action

$$S_{eff}(\Lambda) = N^2\left[\varepsilon t_2 - \varepsilon^2\frac{2}{3}t_2^2 + \varepsilon^2\frac{97}{120}\left(t_4 - 2t_2^2\right)\right] \ ,$$

where $t's$ are symmetrized models

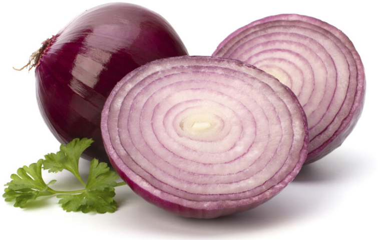
$$t_n = \frac{1}{N}\mathrm{Tr}\left(\phi - \frac{\mathbb{1}}{N}\mathrm{tr}\left(\phi\right)\right)^n \ .$$

- This is a multitrace matrix model which can be analyzed, e.g. to obtain phase structure of the model.

The/A fuzzy onion

The/A fuzzy onion

- Idea: construct a 3D non-commutative space as a series of concentric fuzzy spheres of increasing radius.



[Hammou, Lagraa, Sheikh-Jabbari 2002; Vitale, Wallet 2013; Scholtz et. al; Schupp, Solodukhin 2009; Prešnajder, Gáliková, Kováčik 2015]

- Ours is a bottom-up approach.

The/A fuzzy onion

- Take M fuzzy spheres or radii $r = \lambda, 2\lambda, \ldots, M\lambda$.
- Functions given by a matrix

$$\Psi = \begin{pmatrix} \Phi^{(1)} & & & \\ & \Phi^{(2)} & & \\ & & \ddots & \\ & & & \Phi^{(M)} \end{pmatrix} .$$

- Recall the single layer expression

$$\theta = \frac{2r}{\sqrt{N^2 - 1}} .$$

- The dimension of this space is

$$d = \sum_{N=1}^{M} N^2 = \frac{1}{6} M(M+1)(2M+1) .$$

- Recall the standard three-dimensional integration of a function ψ

$$\int d^3x\ \psi = \int r^2\ dr \int d\Omega\ \psi\ .$$

- We change this to a version discrete in the radial direction

$$\sum_{N=1}^{M} (\lambda N)^2\, \lambda\, \frac{4\pi}{N}\ \mathrm{tr}_N \Phi^{(N)} = \mathrm{Tr}\left(4\pi\lambda^2 r\ \Psi\right)$$

where we have defined the radial distance matrix r as

$$r = \begin{pmatrix} \lambda\ \mathbb{1}_{1\times 1} & & & & \\ & 2\lambda\ \mathbb{1}_{2\times 2} & & & \\ & & 3\lambda\ \mathbb{1}_{3\times 3} & & \\ & & & \ddots & \\ & & & & M\lambda\ \mathbb{1}_{M\times M} \end{pmatrix}$$

The/A fuzzy onion

- The angular part of the kinetic term defined layerwise

$$\mathcal{K}_L \Psi = r^{-2} \begin{pmatrix} \mathcal{K}^{(1)}\Phi^{(1)} & & & & \\ & \mathcal{K}^{(2)}\Phi^{(2)} & & & \\ & & \mathcal{K}^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \mathcal{K}^{(M)}\Phi^{(M)} \end{pmatrix} .$$

- What about the radial direction?

The/A fuzzy onion

$$\text{for } \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \ , \ \Phi^{(N+1)} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)}$$

$$\mathcal{D} : \Phi^{(N+1)} \to \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \ , \ c_{lm}^{(N)} = c_{lm}^{(N+1)} \text{ for } l \leq N-1$$

$$\mathcal{U} : \Phi^{(N)} \to \Phi^{(N+1)} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)} \ , \ \begin{cases} c_{lm}^{(N+1)} = c_{lm}^{(N)} \text{ for } l \leq N-1 \\ c_{Nm}^{(N+1)} = 0 \end{cases} .$$

- Define the first and second radial derivatives

$$\partial_r^{(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda} \,,$$
$$\partial_r^{2\,(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - 2\Phi^{(N)} + \mathcal{U}\phi^{(N-1)}}{\lambda^2} \,.$$

- Define the radial part of Laplacian

$$\mathcal{K}_R\Psi = \partial_r^2\Psi + 2r^{-1}\partial_r\Psi \,,\ \partial_r\Psi = \begin{pmatrix} \partial_r^{(1)}\Phi^{(1)} & & & & \\ & \partial_r^{(2)}\Phi^{(2)} & & & \\ & & \partial_r^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \partial_r^{(M)}\Phi^{(M)} \end{pmatrix} .$$

- Recall

$$\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \to f'(x),\ \frac{f(x+\varepsilon) - 2f(x) + f(x-\varepsilon)}{\varepsilon^2} \to f''(x),\ \Delta = r^{-2}\partial_r r^2\partial_r + \Delta_\Omega \,.$$

The/A fuzzy onion

- Define the first and second radial derivatives

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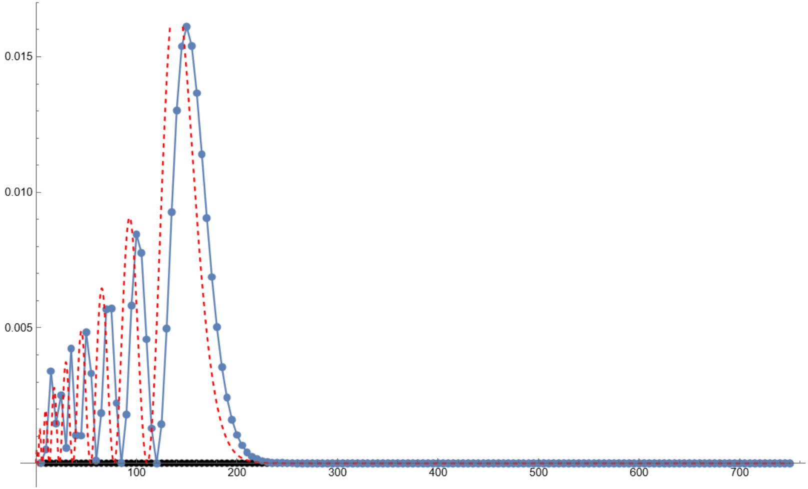
- Laplace operator for the/a fuzzy onion

$$\mathcal{K} = \mathcal{K}_R + K_L .$$

Some onion physics

Quantum mechanical hydrogen atom

$M = 150, \lambda = 5, n = 9$



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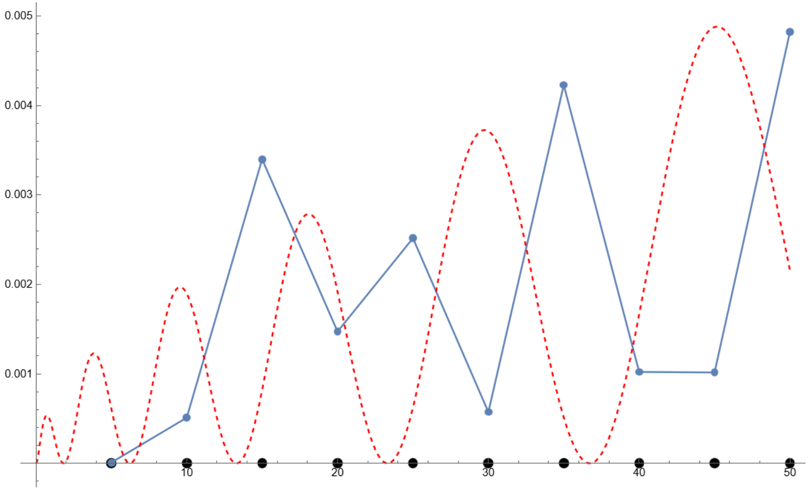

Take home message

Take home message and 2do list

- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy spaces are examples of such spacetimes.
- Physics on such spaces is described by random matrix ensembles.

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Thank you for your attention!