Fuzzy onion strikes back

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CEEPUS

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Take home message

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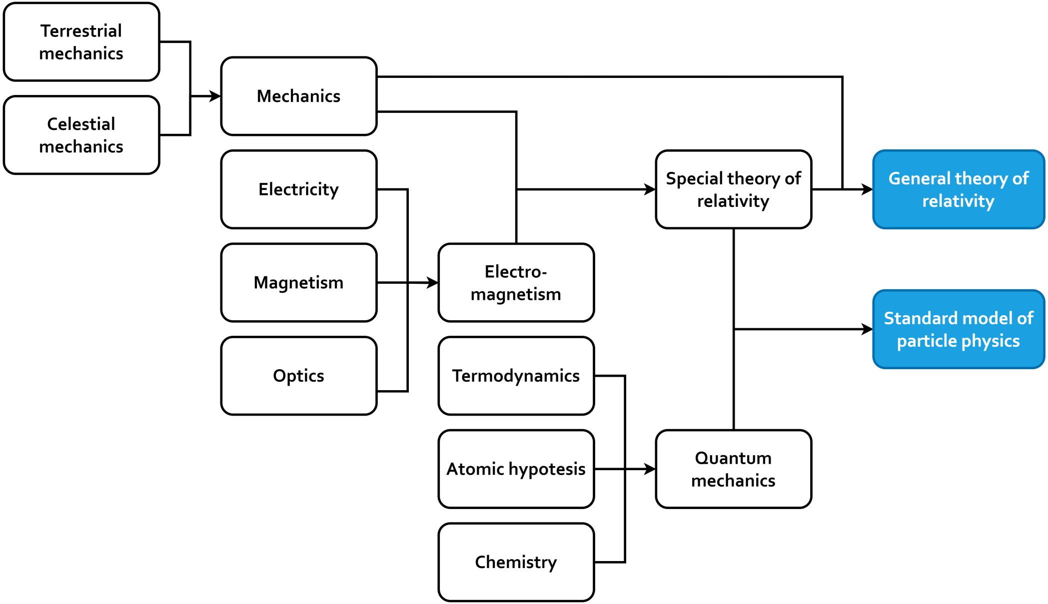
- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy onion is a three dimensional model of such a space.
- It is rather straightforward to work with so it is a nice toy model / playground to check the consequences of quantum structure.

Take home message

- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy onion is a three dimensional model of such a space.
- It is rather straightforward to work with so it is a nice toy model / playground to check the consequences of quantum structure.

- We do have PhD. and postdoc positions related to this and more in Bratislava.

Quick motivation


Terrestrial mechanics
Celestial mechanics
Mechanics
Electricity
Magnetism
Optics
Electro-magnetism
Termodynamics
Atomic hypotesis
Chemistry
Special theory of relativity
Quantum mechanics
General theory of relativity
Standard model of particle physics

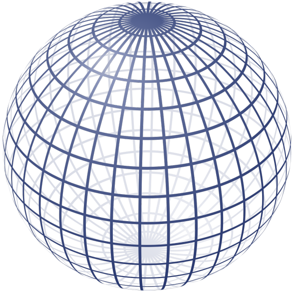
Quantum structure of spacetime

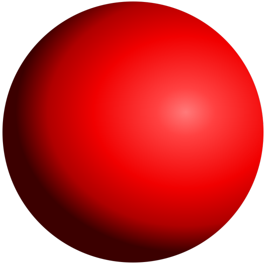
- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. [Hossenfelder 1203.6191]
- Reasons:
  - gravitational Heisenberg microscope,
  - instability of quantum gravitational vacuum, [Doplicher, Fredenhagen, Roberts '95]
  - emergent spacetime.

Fuzzy spaces

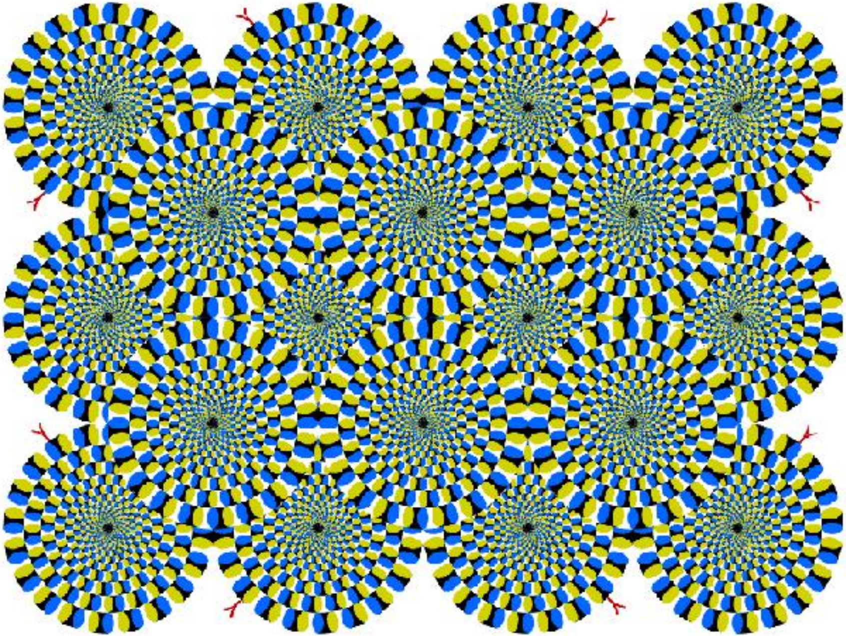
Fuzzy sphere

- The sphere is divided into N cells. Function on the fuzzy sphere is given by a matrix Φ and the eigenvalues of Φ represent the values of the function on these cells.





- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.



Fuzzy sphere

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i,j = 1,2,3 \ ,$$

which generate the algebra of functions.

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta \varepsilon_{ijk} \hat{x}_k \ , \ i,j = 1,2,3 \ .$$

- Such $\hat{x}_i$'s generate a different, non-commutative, algebra and S_N^2 is an object, which has this algebra as an algebra of functions.

Fuzzy sphere

- The conditions can be realized as an $N = 2s+1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- (Matrix) geometry encoded in the Laplacian

$$\mathcal{K}\Phi = \frac{1}{r^2}[L_i,[L_i,\Phi]] \ .$$

Analogue of the integral is the trace

$$\frac{4\pi r^2}{N} \mathrm{tr}_N (\Phi) \ .$$

- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy sphere

- Most importantly nonzero commutators

$$\hat{x}_i\hat{x}_i = \rho^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \ .$$

- Configuration space is analogous to phase space of quantum mechanics.
- In a similar fashion it is possible to construct an analogous deformation of the plane

$$\hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ij} = i\theta_{ij} \ , \ i = 1,2 \ .$$

Construction uses the $\star$-product

$$f \star g = f\, e^{\frac{i}{2}\overleftarrow{\partial}\,\theta\,\overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} + \cdots$$

Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
  [Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  [Steinacker 2013]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair 2006]
- Toy models of spaces with discrete quantum structure, which is expected to arise in quantum theory of gravity.

The/A fuzzy onion

The/A fuzzy onion

- Idea: construct a 3D non-commutative space as a series of concentric fuzzy spheres of increasing radius.

[Hammou, Lagraa, Sheikh-Jabbari 2002; Vitale, Wallet 2013; Scholtz et. al; Schupp, Solodukhin 2009; Prešnajder, Gáliková, Kováčik 2015]

The/A fuzzy onion

Construction of [Prešnajder, Gáliková, Kováčik 2015; Scholtz et. al].

- Start with
$$[x_i, x_j] = 2\lambda i \varepsilon_{ijk} x_k$$
- Explicitly realize by
$$\hat{x}_i = \frac{\theta}{2} a^\dagger_\alpha \sigma^i_{\alpha\beta} a_\beta \ .$$
where
$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \quad [a_\alpha, a_\beta] = [a^\dagger_\alpha, a^\dagger_\beta] = 0 \ .$$
- Free Hamiltonian, i.e. the Laplacian
$$H_0 \Psi = \frac{1}{2\lambda r}[a^\dagger_\alpha, [a_\alpha, \Psi]] \ , \ r = \lambda \left(a^\dagger_\alpha a_\alpha + 1 \right) \ .$$
- Spectrum of hydrogen atom
$$E^l_{\lambda\, n} = \frac{\hbar}{m_e \lambda^2} \left(1 - \sqrt{1 + \left(\frac{m_e q \lambda}{\hbar^2 n} \right)^2} \right) \ .$$

The/A fuzzy onion

- Ours is a bottom-up approach.
- Take M fuzzy spheres or radii $r = \lambda, 2\lambda, \dots, M\lambda$.
- Functions given by a matrix

$$\Psi = \begin{pmatrix} \Phi^{(1)} & & & \\ & \Phi^{(2)} & & \\ & & \ddots & \\ & & & \Phi^{(M)} \end{pmatrix} .$$

- Recall the single layer expression

$$\theta = \frac{2r}{\sqrt{N^2 - 1}} .$$

- The dimension of this space is

$$d = \sum_{N=1}^{M} N^2 = \frac{1}{6} M(M+1)(2M+1) .$$

The/A fuzzy onion

- Recall the standard three-dimensional integration of a function ψ

$$\int d^3x\ \psi = \int r^2\ dr \int d\Omega\ \psi\ .$$

- We change this to a version discrete in the radial direction

$$\sum_{N=1}^{M} (\lambda N)^2\, \lambda\, \frac{4\pi}{N}\ \mathrm{tr}_N \Phi^{(N)} = \mathrm{Tr}\left(4\pi\lambda^2 r\ \Psi\right)$$

where we have defined the radial distance matrix r as

$$r = \begin{pmatrix} \lambda\ \mathbb{1}_{1\times 1} & & & & \\ & 2\lambda\ \mathbb{1}_{2\times 2} & & & \\ & & 3\lambda\ \mathbb{1}_{3\times 3} & & \\ & & & \ddots & \\ & & & & M\lambda\ \mathbb{1}_{M\times M} \end{pmatrix}$$

THE/A FUZZY ONION

- The angular part of the kinetic term defined layerwise

$$\mathcal{K}_L \Psi = r^{-2} \begin{pmatrix} \mathcal{K}^{(1)}\Phi^{(1)} & & & & \\ & \mathcal{K}^{(2)}\Phi^{(2)} & & & \\ & & \mathcal{K}^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \mathcal{K}^{(M)}\Phi^{(M)} \end{pmatrix} .$$

- What about the radial direction?

The/A fuzzy onion

$$\text{for } \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \ , \ \Phi^{(N+1)} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)}$$

$$\mathcal{D} : \Phi^{(N+1)} \to \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \ , \ c_{lm}^{(N)} = c_{lm}^{(N+1)} \text{ for } l \leq N-1$$

$$\mathcal{U} : \Phi^{(N)} \to \Phi^{(N+1)} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)} \ , \ \begin{cases} c_{lm}^{(N+1)} = c_{lm}^{(N)} \text{ for } l \leq N-1 \\ c_{Nm}^{(N+1)} = 0 \end{cases} .$$

- Define the first and second radial derivatives

$$\partial_r^{(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda} \; ,$$

$$\partial_r^{2\,(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - 2\Phi^{(N)} + \mathcal{U}\phi^{(N-1)}}{\lambda^2} \; .$$

- Define the radial part of Laplacian

$$\mathcal{K}_R\Psi = \partial_r^2\Psi + 2r^{-1}\partial_r\Psi \;,\; \partial_r\Psi = \begin{pmatrix} \partial_r^{(1)}\Phi^{(1)} & & & & \\ & \partial_r^{(2)}\Phi^{(2)} & & & \\ & & \partial_r^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \partial_r^{(M)}\Phi^{(M)} \end{pmatrix} \; .$$

- Recall

$$\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \to f'(x), \; \frac{f(x+\varepsilon) - 2f(x) + f(x-\varepsilon)}{\varepsilon^2} \to f''(x), \; \Delta = r^{-2}\partial_r r^2 \partial_r + \Delta_\Omega \; .$$

The/A fuzzy onion

- Define the first and second radial derivatives

$$\partial_r^{(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda} \ ,$$

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$$\mathcal{K}_R\Psi = \partial_r^2\Psi + 2r^{-1}\partial_r\Psi \ , \ \partial_r\Psi = \begin{pmatrix} \partial_r^{(1)}\Phi^{(1)} & & & & \\ & \partial_r^{(2)}\Phi^{(2)} & & & \\ & & \partial_r^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \partial_r^{(M)}\Phi^{(M)} \end{pmatrix} \ .$$

- Laplace operator for the/a fuzzy onion

$$\mathcal{K} = \mathcal{K}_R + K_L \ .$$

Some onion physics

Quantum mechanical problems

- Hamiltonian

$$H = -\frac{\hbar^2}{2m_e}\mathcal{K} - qr^{-1}$$

with $\hbar = m_e = q = 1$, i.e. eigenvalue problem

$$H\Psi = E\Psi\ .$$

- We can express H as a matrix acting on vectors

$$\mathcal{C}^T = \left(c^{(1)}_{00}, c^{(2)}_{00}, c^{(2)}_{1-1}, c^{(2)}_{10}, c^{(2)}_{11}, \dots\right)^T$$

with c's from the decomposition

$$\Phi^{(N)} = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} c^{(N)}_{lm} Y^{(N)}_{lm}\ .$$

Quantum mechanical problems – hydrogen atom

- Hamiltonian is now expressed as a $d \times d$ matrix

$$\mathsf{H} = -\frac{1}{2}\mathsf{K} - \mathsf{r}^{-1} \ .$$

- We can make restriction to $l = 0$ sector thanks to rotational symmetry to make it $M \times M$.
- Look for eigenvalues of H.

$M = 50$ and $\lambda = 1$

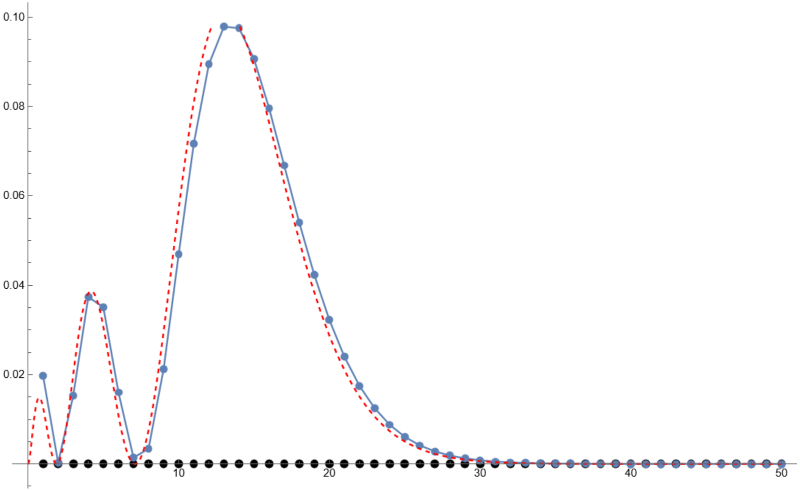
n	1	2	3	4	5	6
E_n	-0.4142	-0.1180	-0.0541	-0.0307	-0.0179	-0.0031
$E^I_{\lambda n}$	-0.4142	-0.1180	-0.0541	-0.0307	-0.0198	-0.0138
E_n^{CQM}	-0.5	-0.125	-0.0556	-0.0313	-0.02	-0.0139

$n = 1$

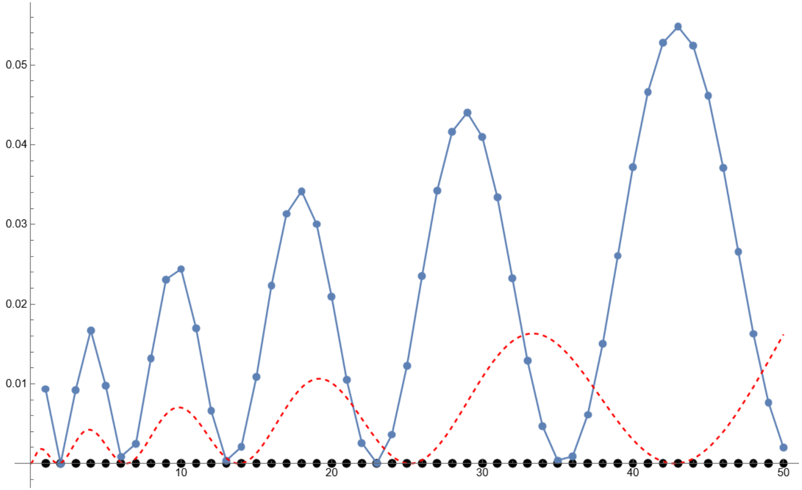
M	$\lambda = 0.1$	$\lambda = 0.01$	$\lambda = 0.001$
50	$6.24 \cdot 10^{-3}$	N/A	N/A
100	$1.27 \cdot 10^{-6}$	N/A	N/A
200	$1.97 \cdot 10^{-13}$	2.81	N/A
400	$1.56 \cdot 10^{-13}$	$3.41 \cdot 10^{-2}$	N/A
800	$5.22 \cdot 10^{-13}$	$4.9 \cdot 10^{-5}$	N/A
1600	$4.8 \cdot 10^{-14}$	$1.13 \cdot 10^{-11}$	N/A
3200	$9.02 \cdot 10^{-15}$	$5.75 \cdot 10^{-12}$	$1.26 \cdot 10^{-1}$

- It seems that in the limit $M \to \infty$ we recover the construction of [Prešnajder, Gáliková, Kováčik 2015].
- We can go further and look at the wavefunctions.

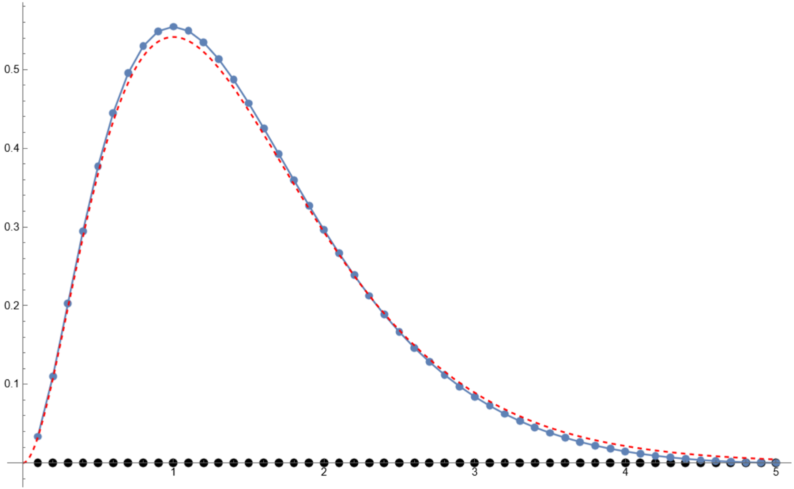
$M = 50, \lambda = 1, n = 3$



$M = 50, \lambda = 1, n = 6$

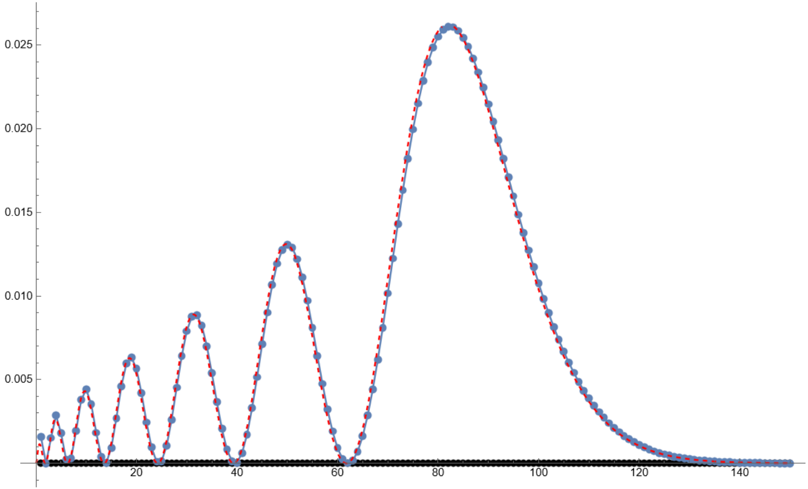


$M = 50, \lambda = 0.1, n = 1$



Quantum mechanical problems – hydrogen atom

$M = 150, \lambda = 1, n = 7$



Quantum mechanical problems – hydrogen atom

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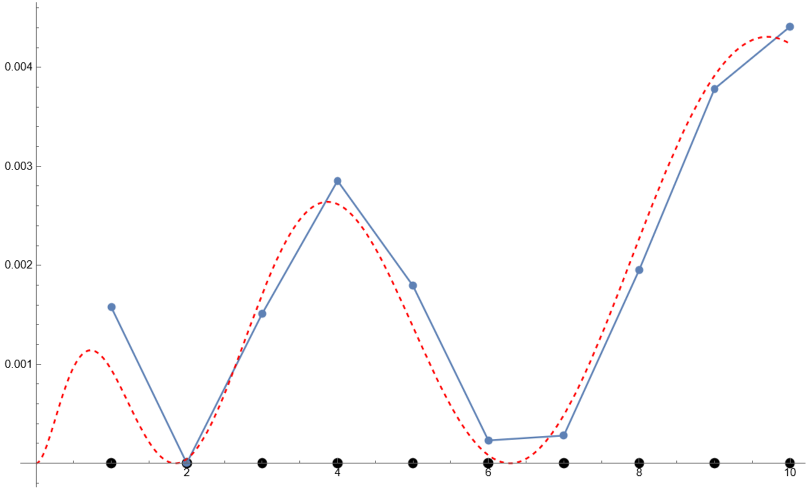
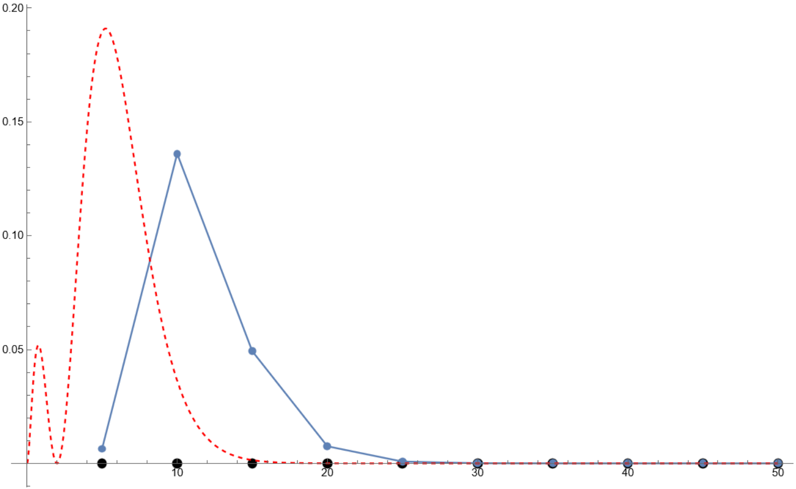
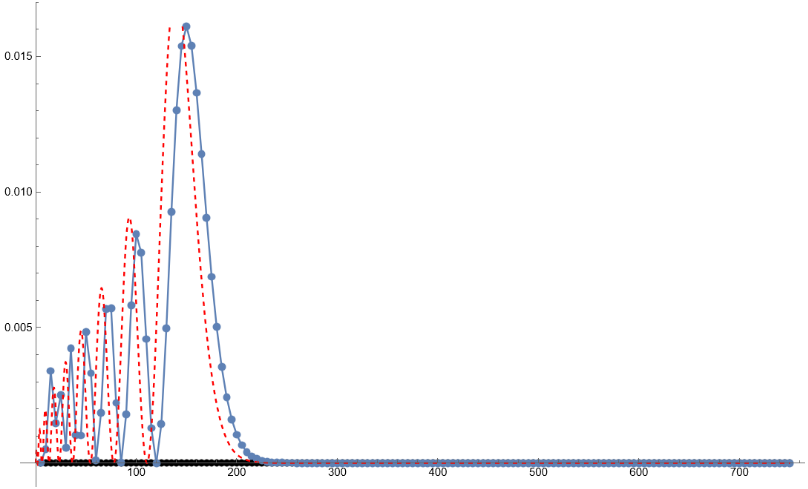


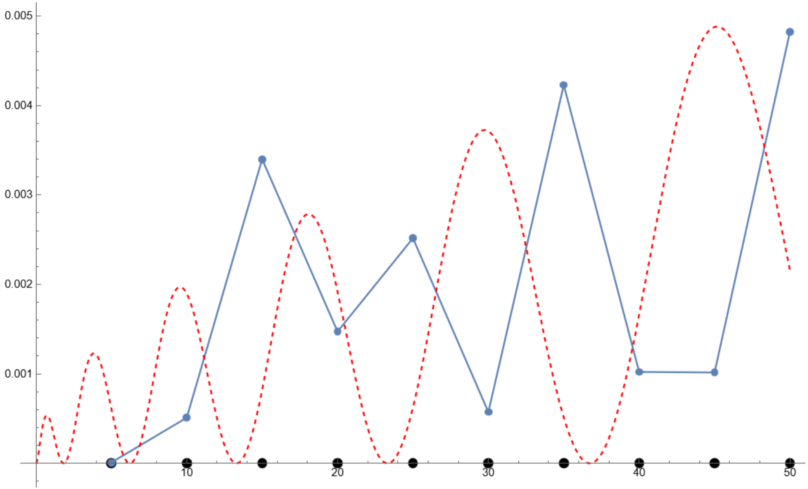
$M = 50, \lambda = 5, n = 2$



$M = 150, \lambda = 5, n = 9$



$M = 150, \lambda = 5, n = 9$



Quantum mechanical problems – harmonic oscillator

- The same Hamiltonian with
$$\mathsf{H} = -\frac{1}{2}\mathsf{K} + \frac{1}{2}r^2$$
describes three dimensional harmonic oscillator.
- This can not be solved analytically in the previous approach. $M = 100$, $\lambda = 1$, $l = 0, 1$

n	0	1	2	3	4	5
E_n	1.4984	3.4922	5.4809	7.4645	9.4431	11.4166
E_n^{CQM}	1.5	3.5	5.5	7.5	9.5	11.5

n	0	1	2	3	4	5
E_n	2.5005	4.4979	6.491	8.4795	10.4632	12.4422
E_n^{CQM}	2.5	4.5	6.5	8.5	10.5	12.5

Quantum mechanical problems

- In principle any potential can be analyzed exactly.
- The only problem is how to recover the limit of classical (and infinite) space.

Quantum mechanical problems – fuzzy cavity

- Fuzzy onion is a regularization of a spherical cavity. In the limit $M \to \infty, \lambda \to 0, M\lambda \to R$ we recover continuous cavity of radius R.
- In chemical literature this models atoms under pressure. [refs in [2]]

S orbital [2]

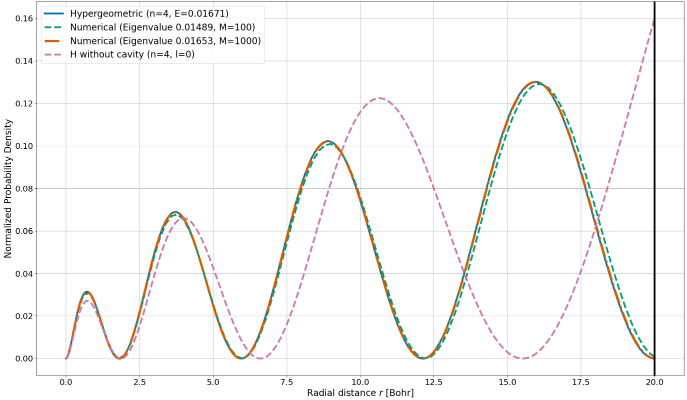
$r_0 \backslash M$	∞ as a reference	100	1000	10000
0.5	14.747970030350280	14.406037740780091	14.713401904425357	14.744509454556344
1	2.373990866103664	2.300565723232022	2.366554263053759	2.373246264259394
3	-0.423967287733454	-0.427225951376656	-0.424313148630359	-0.424002075109953
10	-0.499999263281525	-0.498755577647694	-0.499986776756742	-0.499999139626354
20	-0.499999999999994	-0.495097567963923	-0.499950009998093	-0.499999499991737

P orbitals [2]

$r_0 \backslash M$	∞ as a reference	100	1000	10000
0.5	72.672039190463577	71.154357099658682	72.520341020495181	72.656870688127995
1	16.570256093469736	16.206670008470599	16.533894253548414	16.566620019574611
3	1.111684737436364	1.078613638687640	1.108361257317863	1.111352239065044
10	-0.112806210295841	-0.113415153701996	-0.112878188197422	-0.112813520180460
20	-0.124987114312918	-0.124677720985899	-0.124984183578311	-0.124987102906836

$r_0 \backslash M$	∞ as a reference	100	1000	10000
0.5	36.658875880189399	35.160313617726310	36.505181049931707	36.643467765541793
1	8.223138316160864	7.866678332336676	8.186560245882733	8.219471198758979
3	0.481250312526643	0.449669060926531	0.478000341365532	0.480924423219991
10	-0.118859544853860	-0.119527885630338	-0.118934647870327	-0.118867073891238
20	-0.124994606647078	-0.124692831261815	-0.124995259742673	-0.124994633404707

[2]


Hypergeometric (n=4, E=0.01671)
Numerical (Eigenvalue 0.01489, M=100)
Numerical (Eigenvalue 0.01653, M=1000)
H without cavity (n=4, l=0)
Normalized Probability Density
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Radial distance r [Bohr]

Scalar field theory

Scalar field theory

- Defined by the (euclidean) action

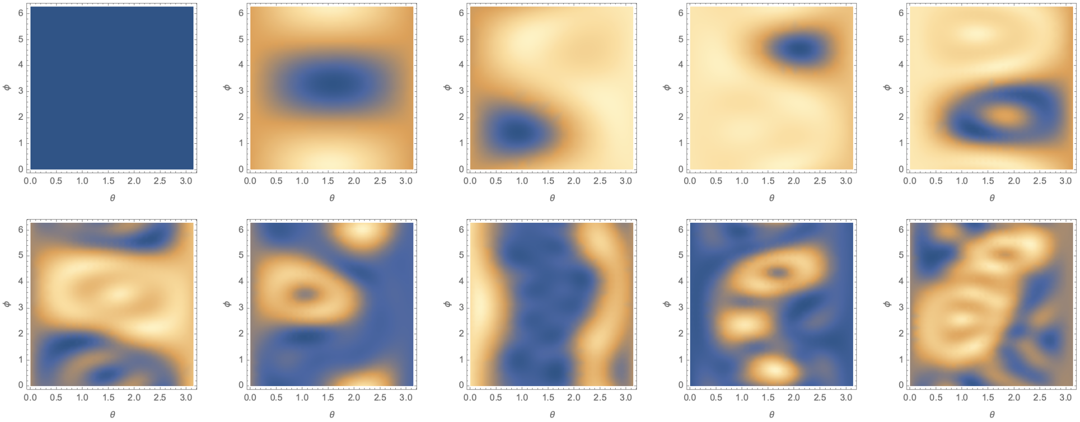
$$S[\Psi] = 4\pi\lambda^2 \mathrm{Tr}\left[r\left(a\ \Psi\mathcal{K}\Psi + b\ \Psi^2 + c\ \Psi^4\right)\right]$$

  and expectation values

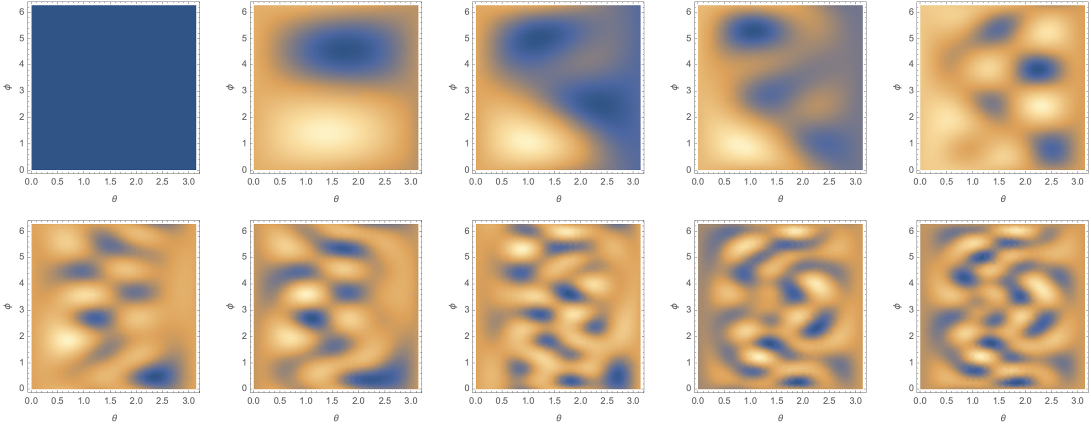
$$\langle\mathcal{O}(\Psi)\rangle = \frac{1}{Z}\int d\Psi e^{-S(\Psi)}\mathcal{O}(\Psi)\ ,\ d\Psi = \prod_{N=1}^{M} d\Phi^{(N)}\ .$$

- Hybrid Monte Carlo evolution of a field configuration.[1]

Scalar field theory



Scalar field theory



Scalar field theory

- Radial part of the Laplacian couples the oscillations in also in radial direction.

Scalar field theory

- Radial part of the Laplacian couples the oscillations in also in radial direction.
- What about symmetry breaking? How – if at all – do the phases on layers align? Is the derivative enough or do we need something further?

Different Laplacians

Different Laplacians

- Recall that the definition of Laplacian using

$$\partial_r^{(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda} ,$$

$$\partial_r^{2\,(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - 2\Phi^{(N)} + \mathcal{U}\phi^{(N-1)}}{\lambda^2} ,$$

and

$$\mathcal{K}_R\Psi = \partial_r^2\Psi + 2r^{-1}\partial_r\Psi$$

was in some sense arbitrary.

Different Laplacians

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$$\partial_r^{(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda},$$
$$\partial_r^{2\,(N)}\Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - 2\Phi^{(N)} + \mathcal{U}\phi^{(N-1)}}{\lambda^2},$$

and

$$\mathcal{K}_R\Psi = \partial_r^2\Psi + 2r^{-1}\partial_r\Psi$$

was in some sense arbitrary.

- What are consequences of other choices? Any preferred choice?

$$\mathcal{K}_R\frac{1}{r} \sim \delta$$

Fuzzy radial coordinate

Fuzzy radial coordinate

- Angular coordinates on layers are properly fuzzy. The radial coordinate is discrete and lattice-like.
- This calls for improvement. Several possible ways how to do this.

Fuzzy radial coordinate – smearing

[Kovacik, Prekrat, JT work in progress]

$$\mathcal{S}\phi^{(N)} = \frac{\phi^{(N)} + \sum_i \alpha_i \left(\mathcal{U}^i \phi^{(N-i)} + \mathcal{D}^i \phi^{(N+i)}\right)}{1 + \sum_i \alpha_i}$$

- This procedure simply smears the values of fields $\Phi^{(N)}$ over neighboring layers.

Fuzzy radial coordinate – smearing

[Kovacik, Prekrat, JT work in progress]

$$\mathcal{S}\phi^{(N)} = \frac{\phi^{(N)} + \sum_i \alpha_i \left(\mathcal{U}^i \phi^{(N-i)} + \mathcal{D}^i \phi^{(N+i)}\right)}{1 + \sum_i \alpha_i}$$

- This procedure simply smears the values of fields $\Phi^{(N)}$ over neighboring layers.
- What does this do?

Fuzzy radial coordinate – string states

- Functions on the fuzzy sphere are matrices acting on $\mathcal{H}$

$$\Phi = \sum_{m,n=-s}^{s} \Phi_{mn} \left| m \right\rangle \left\langle n \right| \ .$$

- We can express the matrix Φ in a similar fashion using the coherent states

$$\Phi = \left(\frac{N}{4\pi} \right)^2 \int d^2x \, d^2y \, \phi(x,y) \left| x \right\rangle \left\langle y \right| \ .$$

- Objects [Iso, Kawai, Kitazawa 2000; Steinacker 2016; Steinacker, JT '22]

$$\left| x \right\rangle \left\langle y \right| =: \left| \begin{matrix} x \\ y \end{matrix} \right)$$

form a basis of functions on the fuzzy sphere and we will call them the **string modes**.

Fuzzy radial coordinate – string states

- In onion construction, for x and y on the same layer these for matrices $\Phi^{(N)}$.
- For x and y on different layers these naturally fit into the off-diagonal blocks of

$$\Psi = \begin{pmatrix} \Phi^{(1)} & \bullet & & \\ \bullet & \Phi^{(2)} & & \\ & & \ddots & \\ & & & \Phi^{(M)} \end{pmatrix} .$$

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- What is the effect of off-diagonal blocks? What is their correct dynamics? Is this with or instead of the radial derivative?
- Can we define physics of the onion in terms of the whole matrix Ψ?

- A simplified model of two spheres with the same N

$$\Psi_F = \begin{pmatrix} \phi_1 & A \\ A^\dagger & \phi_2 \end{pmatrix} .$$

- We take A to be identity and get

$$\mathrm{Tr}\left[\frac{1}{2} r\Psi^2 + \Psi^4\right] = \frac{1}{2}(r+4)\mathrm{tr}_N\left(\Phi_1^2\right) + \mathrm{tr}_N\left(\Phi_1^4\right) + \frac{1}{2}(r+4)\mathrm{tr}_N\left(\Phi_2^2\right) + \mathrm{tr}_N\left(\Phi_2^4\right) + 4\mathrm{tr}_N\left(\Phi_1\Phi_2\right)$$

- This is a solvable two matrix model related to Ising model.

- There is a different and more mathematical description of noncommutative spaces.
- Uses notion of spectral triples

$$(\mathcal{A}, \mathcal{D}, \mathcal{H}) \ .$$

- Construction for fuzzy sphere available, construction of a lattice like set of points available. [Barrett '15]

Fuzzy radial coordinate – spectral triples

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$$(\mathcal{A}, \mathcal{D}, \mathcal{H}) \ .$$
- Construction for fuzzy sphere available, construction of a lattice like set of points available. [Barrett '15]
- Can we put these two together? Any other (better) version for radial direction? Does this lead to any canonical structures on the onion?

Onion as a spacetime solution in matrix models

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- Matrix models formulations of string theory usually have action along the lines

$$S = \frac{1}{g^2} \mathrm{Tr} \left[-[X_a, X_b][X^a, X^b] + \dots \right]$$

with matrices X_a describing spacetime degrees of freedom. [Steinacker '24]

- Equations of motion

$$[X_a, [X^a, X_b]] = \dots$$

lead to solutions in form of fuzzy spaces. The simplest case is set of fuzzy spheres of various radii

$$X_a = \begin{pmatrix} L_a^{(N_1)} & & & \\ & L_a^{(N_2)} & & \\ & & \ddots & \\ & & & L_a^{(N_M)} \end{pmatrix} .$$

- Hmmmmmmmm.

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- What does this tell us about the construction of fuzzy onion? What about the off-diagonal blocks?

Model of dynamical spacetime

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- In the construction λ is a constant. Does not need be and in principle we could have $\lambda(r)$ and a deformed onion – curvature.
- This deformation does not need to be constant in time.
- Perhaps also possibility of making λ dependent on the angular direction.

Model of dynamical spacetime

- In the construction λ is a constant. Does not need be and in principle we could have $\lambda(r)$ and a deformed onion – curvature.
- This deformation does not need to be constant in time.
- Perhaps also possibility of making λ dependent on the angular direction.
- What kind of dynamics of space can we define? What is the effect on physics happening on the onion?

Toy model of expanding universe

- If we interpret the radial coordinate as time, fuzzy onion is a model of expanding universe with quantized time.
- Each time step one cell of spacetime is created.

Toy model of expanding universe

- If we interpret the radial coordinate as time, fuzzy onion is a model of expanding universe with quantized time.
- Each time step one cell of spacetime is created.
- What are the consequences?

Phenomenology of infalling matter

- Properties of black holes in quantum spacetimes are different, no singularity present.
- We can analyze collapse of matter creating a black hole by writing the corresponding equations.

Phenomenology of infalling matter

- Properties of black holes in quantum spacetimes are different, no singularity present.
- We can analyze collapse of matter creating a black hole by writing the corresponding equations.
- Is the collapse stopped by outward pressure? What is the dissipation mechanism? What is fate of the horizon? Any bounces?
- Comparison with numerical results in loop quantum gravity.
  [Modesto '08; Husain, Kelly, Santacruz, Wilson-Ewing '22]

Classical applications

Classical applications

- The cells of the "quantum" space need to to arise from fundamental physics.
- Flow of granular materials or heat flow in such materials.[Saitou, Bamba, Sugamot '14]
- Structure of neutron stars.
- Applicable in situations where granularity is due to lack of precise knowledge – atmospheric physics.

Take home message and 2do list

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- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy onion is a three dimensional model of such a space.
- It is rather straightforward to work with so it is a nice toy model / playground to check the consequences of quantum structure.

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Thank you for your attention!

If time permits

- Reformulation in terms of vector $\mathcal{C}$ and operators as $d \times d$ matrices.
- Is this useful for the field theory in any way?
- Action turns out to be

$$\begin{aligned} S &= 4\pi \mathrm{Tr}\; r \left(a\Psi\mathcal{K}\Psi + b\,\Psi^2 + c\,\Psi^4 \right) = \\ &= \frac{1}{2}\mathcal{C}\cdot \mathrm{P}^{-1}\cdot\mathcal{C} + 4\pi\lambda^3 \sum_{N=1}^{M} cN \left[\frac{1}{4N}\left(\mathcal{C}^{(N)}\cdot\mathcal{C}^{(N)}\right)^2 + \frac{1}{8}\left(\mathcal{C}^{(N)}\cdot \mathrm{G}_a^{(N)}\cdot\mathcal{C}^{(N)}\right)^2\right] , \\ \mathrm{P} &= \frac{1}{4\pi\lambda^2}\left(2ar\mathrm{K} + 2b\mathrm{r}\right)^{-1} \end{aligned}$$

where

$$\mathrm{G}_a^{(N)} = \begin{pmatrix} 0 & (v_a^{(N)})^T \\ v_a^{(N)} & D_a^{(N)} \end{pmatrix} , \quad \left(D_a^{(N)}\right)_{ij} = 2\mathrm{tr}_N\left(\left\{T_i^{(N)}, T_j^{(N)}\right\} T_a^{(N)}\right) , \quad (v_a^{(N)})_b = \sqrt{\frac{2}{N}}\delta_{ab} .$$

Field theory as a vector model

- What are consequences of writing

$$S = \frac{1}{2}\mathcal{C}\cdot \mathsf{P}^{-1}\cdot\mathcal{C} + 4\pi c\lambda^3\left[\frac{1}{4}(\mathcal{C}\cdot\mathcal{C})^2 + \frac{1}{8}(\mathcal{C}\cdot \mathsf{G}_A\cdot\mathcal{C})^2\right]$$

instead of

$$S = \frac{1}{2}\mathcal{C}\cdot \mathsf{P}^{-1}\cdot\mathcal{C} + 4\pi\lambda^3\sum_{N=1}^{M} cN\left[\frac{1}{4N}\left(\mathcal{C}^{(N)}\cdot\mathcal{C}^{(N)}\right)^2 + \frac{1}{8}\left(\mathcal{C}^{(N)}\cdot \mathsf{G}_a^{(N)}\cdot\mathcal{C}^{(N)}\right)^2\right]?$$