Fuzzy field theories and matrix models

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Take home message

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- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy spaces are examples of such spacetimes.
- Plenty of interesting things happen on spaces with quantum structure.
- Physics on such spaces is described by random matrix ensembles.

Quick motivation

Quantum structure of spacetime

- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. [Hossenfelder 1203.6191]
- Reasons:
  - gravitational Heisenberg microscope,
  - instability of quantum gravitational vacuum, [Doplicher, Fredenhagen, Roberts '95]
  - emergent spacetimes.
- Fuzzy spaces are examples of such spacetimes.

Fuzzy spaces

Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
  [Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BNM).
  [Steinacker 2013]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair 2006]
- Toy models of spaces with discrete quantum structure, which is expected to arise in quantum theory of gravity.

Fuzzy spaces

Fuzzy sphere [Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder 1990s]

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

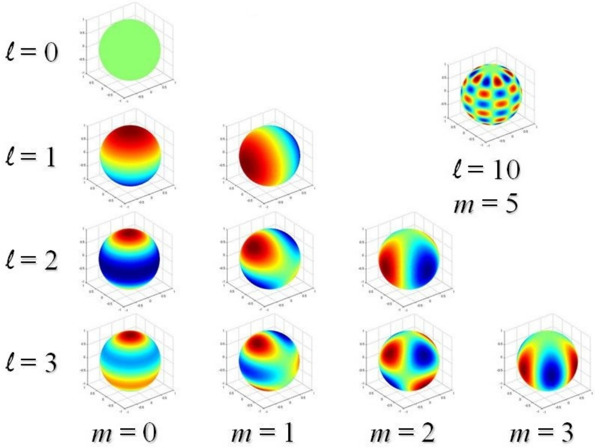


Image taken from http://principles.ou.edu/mag/earth.html

Fuzzy spaces

- If we truncate the possible values of l in the expansion

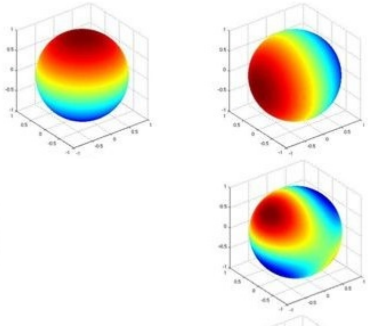
$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

  we will not be able to see any features of functions under certain length scales.
- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces

$\ell = 1$

$\ell = 2$



- Number of independent functions with $l \leq L$ is $(L+1)^2$, the same as the number of $N \times N$ hermitian matrices.
- We have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- Opposing to some lattice discretization this space still possess a full rotational symmetry.
- In the limit N or $L \to \infty$ we recover the original sphere.

Fuzzy spaces - an alternative construction

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i\hat{x}_i = r^2 \quad , \quad [\hat{x}_i, \hat{x}_j] = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ \ i,j = 1,2,3 \ .$$

- The conditions can be realized as an $N = 2s+1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}}L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1}s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry. Most importantly nonzero commutators imply uncertainty relations for positions $\Delta x_i \Delta x_j \neq 0$.
- In a similar fashion it is possible to construct an analogous deformation of the plane

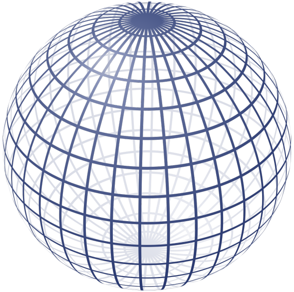
$$\hat{x}_i, \hat{x}_j] = i\theta\varepsilon_{ij} = i\theta_{ij} \ , \ \ i = 1,2 \ .$$

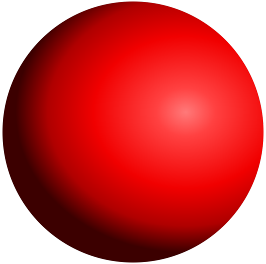
Construction uses the $\star$-product

$$f \star g = f\, e^{\frac{i}{2}\overleftarrow{\partial}\,\theta\,\overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} + \cdots$$

Fuzzy spaces

- We have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of ϕ represent the values of the function on these cells.





- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Fuzzy field theories

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

  and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} \ .$$

- We construct the noncommutative theory as an analogue with
  - field $\to$ matrix,
  - functional integral $\to$ matrix integral,
  - spacetime integral $\to$ trace,
  - derivative $\to$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right] ,$$

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} .$$

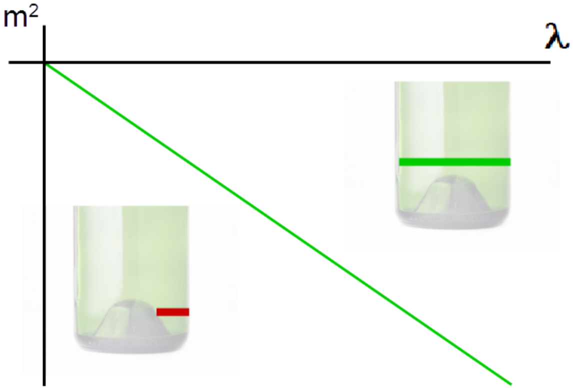
- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr}\left[\frac{1}{2} M \frac{1}{R^2}[L_i,[L_i,M]] + \frac{1}{2}m^2 M^2 + V(M)\right] ,$$

$$\langle F \rangle = \frac{\int dM \, F(M) e^{-S(M)}}{\int dM \, e^{-S(M)}} .$$

[Balachandran, Kürkçüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\Phi\partial_i\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right)$$

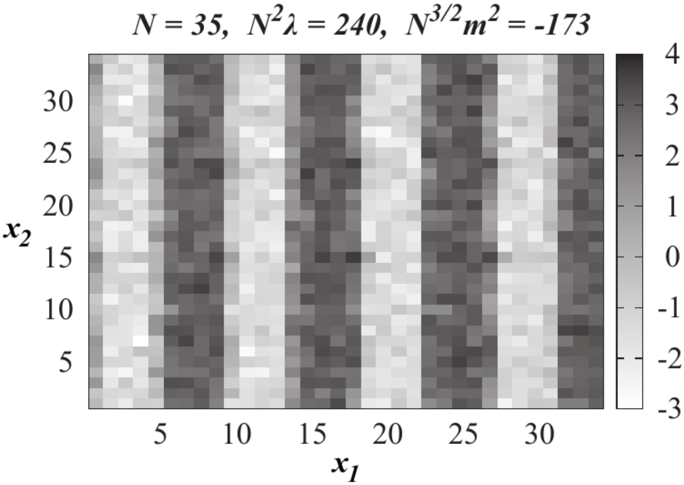


[Glimm, Jaffe 1974; Glimm, Jaffe, Spencer 1975; Chang 1976]
[Loinaz, Willey 1998; Schaich, Loinaz 2009]

Phases of fuzzy field theories

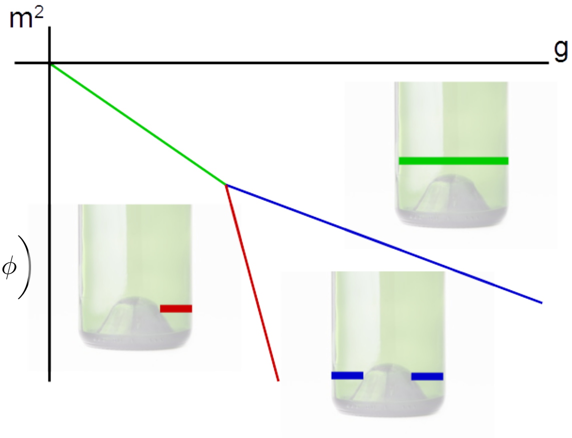
- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  [Gubser, Sondhi 2001; Chen, Wu 2002]
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is spontaneously broken.
- This has been established in numerous numerical works for variety different spaces.
  [Martin 2004; García Flores, Martin, O'Connor 2006, 2009; Panero 2006, 2007; Ydri 2014; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero 2014; Mejía-Díaz, Bietenholz, Panero 2014; Medina, Bietenholz, D. O'Connor 2008; Bietenholz, Hofheinz, Nishimura 2004; Lizzi, Spisso 2012; Ydri, Ramda, Rouag 2016; Kováčik, O'Connor 2018]
  [Panero 2015]

[Mejía-Díaz, Bietenholz, Panero 2014] for $\mathbb{R}^2_\theta$

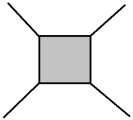


$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$

$$S = \int d^2x\left(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right)$$



- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  [Minwalla, Van Raamsdonk, Seiberg 2000; Vaidya 2001; Chu, Madore, Steinacker 2001]
- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- In terms of diagrams different properties of planar and non-planar ones.
  The (matrix) vertex is not invariant under permutation of incoming momenta.

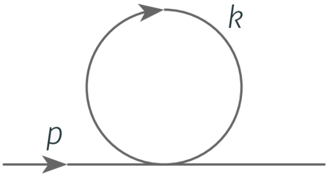


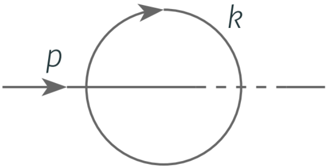
[Chu, Madore, Steinacker '01]

$$I^{NP} - I^{P} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} \left[(-1)^{l+j+N-1} \begin{Bmatrix} l & s & s \\ j & s & s \end{Bmatrix} - 1 \right]$$

- $N \to \infty$ limit of the effective action is different from the standard S^2 effective action.
- The space (geometry) forgets where it came from, but the field theory (physics) remembers its fuzzy origin.

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)$$





Random matrices and fuzzy field theories

Kinetic term effective action

- Recall the action of the fuzzy scalar field theory

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \ .$$

This is a particular case of a matrix model since we need

$$\int dM\, F(M) e^{-S(M)} \ .$$

- The large N limit of the model with the kinetic term is not well understood. The key issue being that diagonalization $M = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ no longer straightforward.
- Integrals like

$$\langle F\rangle \sim \int d\Lambda \int dU\ \ F(\lambda_i, U)\ e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$
$$\times\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)} \ .$$

Perturbative calculation

$$e^{-N^2 S_{eff}(\Lambda)} = \int dU\, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. [O'Connor, Sämann 2007; Sämann 2010]
- The most recent result is [Sämann 2015]

$$S_{eff}(\Lambda) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$-\varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$-\varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n$$

- Standard treatment of such multitrace matrix model yields a very unpleasant behaviour close to the origin of the parameter space.

The first set of results

Second moment approximation

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. [Steinacker 2005]
- There is a unique parameter independent effective action that reconstructs this rescaling. [Polychronakos 2013]

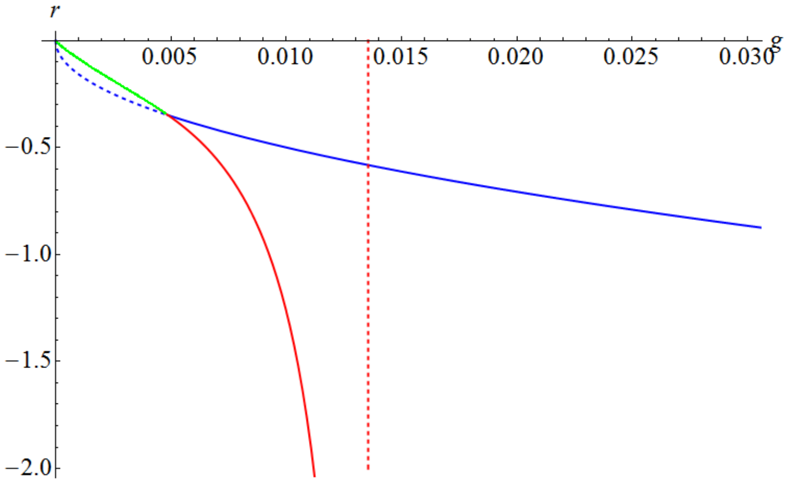
$$S_{eff}(\Lambda) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R} \ .$$

  Can be generalized to more a more complicated kinetic term $\mathcal{K}$.
- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

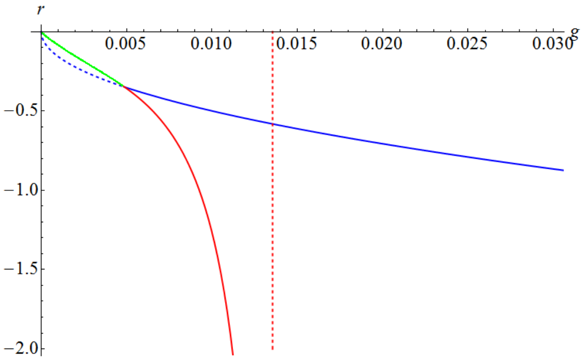
$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right) \ .$$

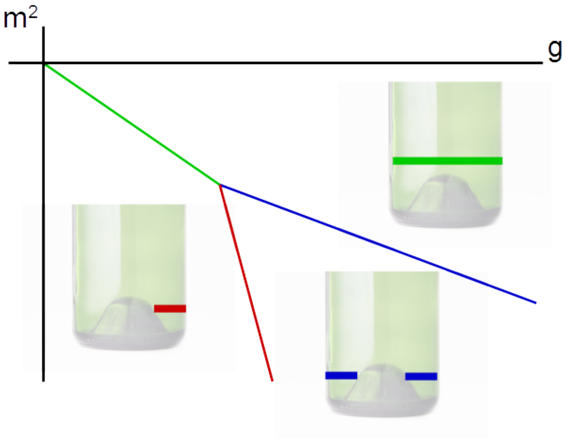
[Šubjaková, JT PoS CORFU2019; JT '14 '15 '18; Šubjaková, JT '20]

Second moment approximation



[JT '18; Šubjaková, JT 2020]


r
g
0.005
0.010
0.015
0.020
0.025
0.030
−0.5
−1.0
−1.5
−2.0


m²
g

Beyond the second moment approximation

- Taking a lesson from

$$S(M)=\frac{1}{2}F(c_2-c_1^2)+\frac{1}{2}r\,\mathrm{Tr}\left(M^2\right)+g\,\mathrm{Tr}\left(M^4\right)\quad,\quad F(t)=\log\left(\frac{t}{1-e^{-t}}\right)$$

  we could try to complete the perturbative action

$$S_{eff}=F\left[c_1,t_2,t_3,t_4-2t_2^2\right]=\frac{1}{2}\log\left(\frac{t_2}{1-e^{-t_2}}\right)+F_3(t_3)+F_4(t_4-2t_2^2)$$

  and

$$F_4(y_4)=\alpha_0\log(y_4)+\alpha_1+\frac{\alpha_2}{y_4}+\frac{\alpha_3}{y_4^2}+\ldots\ .$$

- Any attempt to complete the perturbative expansion in the spirit of the non-perturbative model is not capable of solving the above problems and does not lead to a phase diagram that is in complete agreement with the numerical simulations. Most importantly the location of the triple point can not be brought closer to the numerical value. [Šubjaková, JT '22]

The second set of results

- We would like to analyse the more complicated model

$$S = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + 12gMQM + \frac{1}{2}rM + gM^4\right) \ ,$$

where

$$QT_{lm} = \underbrace{-\left(\sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+r}\left[(-1)^{l+j+N-1}\left\{\begin{array}{ccc} l & s & s \\ j & s & s \end{array}\right\} - 1\right]\right)}_{Q(l)} T_{lm} \ .$$

- This removes the UV/IR mixing in the theory, essentially by removing the problematic part by brute force.

[Dolan, O'Connor, Prešnajder '01]

Removal of stripes – fuzzy sphere

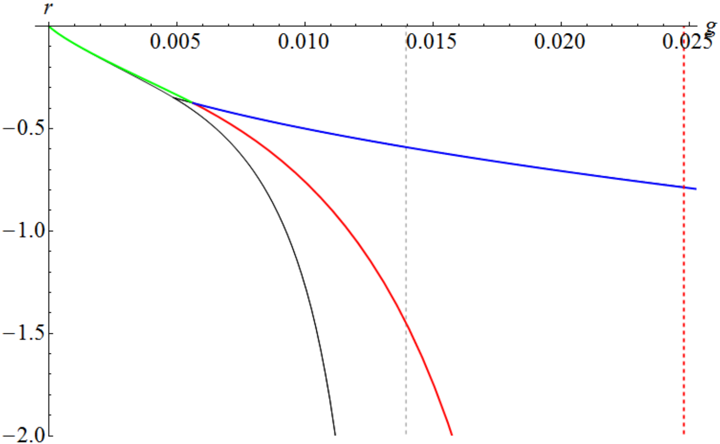
- Operator Q can be expressed as a power series in $C_2 = [L_i, [L_i, \cdot]]$

$$Q = q_1 C_2 + q_2 C_2^2 + \dots \ .$$

- As a starting point, it is interesting to see the phase structure of such simplified model. [O'Connor, Säman '07]
- This is the case of

$$\mathcal{K} = (1 + ag) C_2 \quad \text{or} \quad \mathcal{K} = (1 + ag) C_2 + bg\, C_2^2 \ .$$

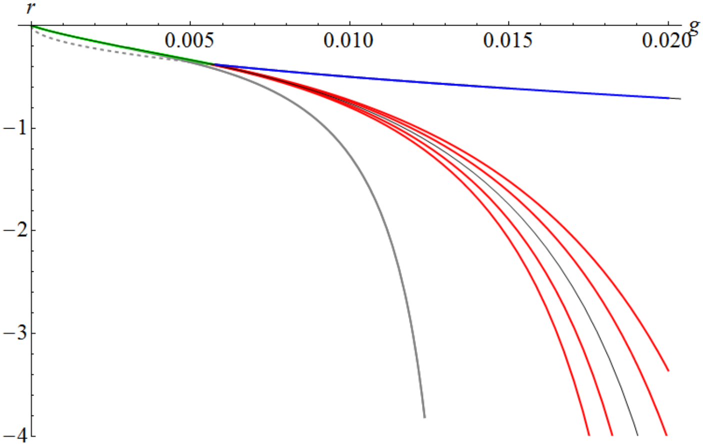
[Šubjaková, JT '20]



$$a = 3e^{3/2} \ , \ b = 0 \ .$$

Removal of stripes – fuzzy sphere

[Šubjaková, JT '20]



$$a = 3e^{3/2}\ ,\ \ b = -4, -2, 0, 2, 4\ .$$

Removal of stripes – GW model

- Grosse-Wulkenhaar model ['00's]

$$S_{GW} = \int d^2x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) ,$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu \ .$$

- This model is renormalizable.
- Described by a matrix model in terms of truncated Heisenberg algebra. [Burić, Wohlgenannt '10]

- The kinetic term becomes

$$\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi\to[X,M][X,M]+[Y,M][Y,M]\ .$$

and the harmonic potential becomes

$$\frac{1}{2}\Omega^2(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)\to RM^2\ ,$$

where X,Y,R are fixed external matrices.
- Interpretation of R coupling to the curvature of the space.
- We are thus left with a matrix model with action

$$S=\mathrm{Tr}\left(M[X,[X,M]]+M[Y,[Y,M]]\right)-g_r\mathrm{Tr}\left(RM^2\right)-g_2\mathrm{Tr}\left(M^2\right)+g_4\mathrm{Tr}\left(M^4\right)\ .$$

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]

- We concentrate on the effect of the curvature term and discard the kinetic term

$$S(M) = \mathrm{Tr}\left(M\mathcal{K}M\right) - \mathrm{Tr}\left(g_r R M^2\right) - g_2\,\mathrm{Tr}\left(M^2\right) + g_4\,\mathrm{Tr}\left(M^4\right) \ .$$

- This leads to the angular integral

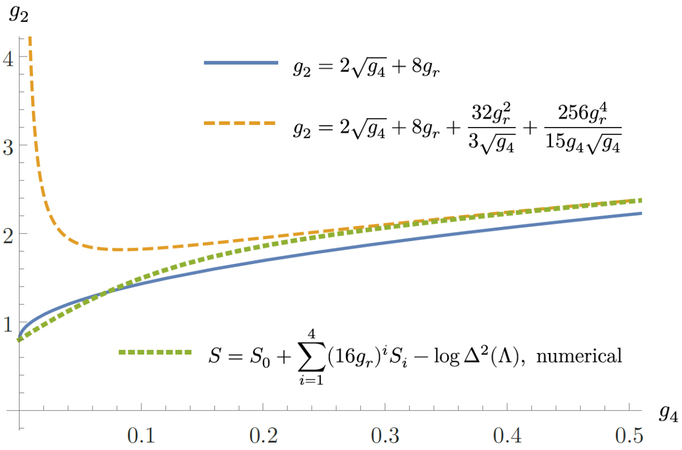
$$\int dU\, e^{g_r \mathrm{Tr}\left(URU^\dagger \Lambda^2\right)} \ ,$$

which gives up to g_r^4

$$S(\Lambda) = N\,\mathrm{Tr}\left(-g_2\Lambda^2 + 8g_r\Lambda^2 + g_4\Lambda^4 - \frac{32}{3}g_r^2\Lambda^4\right) + \frac{1024}{45}g_r^4\Lambda^8 + \\ + \frac{32}{3}g_r^2\left(\mathrm{Tr}\left(\Lambda^2\right)\right)^2 + \frac{1024}{15}g_r^4\left(\mathrm{Tr}\left(\Lambda^4\right)\right)^2 - \frac{4096}{45}g_r^4\mathrm{Tr}\left(\Lambda^6\right)\mathrm{Tr}\left(\Lambda^2\right) \ .$$

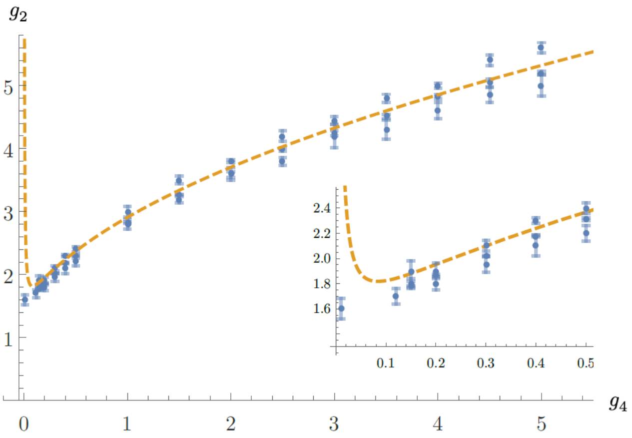
- This is a multitrace matrix model which can be analyzed.

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]



Removal of stripes – GW model

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]



The third set of results

String modes - representation of functions on S_F^2

- Functions on the fuzzy sphere are matrices acting on $\mathcal{H}$

$$M = \sum_{m,n=-s}^{s} M_{mn} \left|m\right\rangle \left\langle n\right| \ .$$

- We can express the matrix M in a similar fashion using the coherent states

$$M = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y\, \phi(x,y) \left|x\right\rangle \left\langle y\right| \ .$$

- Objects [Iso, Kawai, Kitazawa 2000; Steinacker 2016]

$$\left|x\right\rangle \left\langle y\right| =: \left|\begin{matrix} x \\ y \end{matrix}\right)$$

form a basis of functions on the fuzzy sphere and we will call them the **string modes**.

String modes - representation of functions on S_F^2

$$\left| \begin{matrix} x \\ y \end{matrix} \right)$$

- **Short modes** for $|x-y| < 1/\sqrt{N}$ can be shown to represent localized wave-packets with momentum $\sim N|x-y|$.
  This is the classical regime.
- Particularly string mode $\left| \begin{matrix} x \\ x \end{matrix} \right)$ represents a maximal localized function around point x, i.e. a fuzzy version of δ-function.
  Functions with $\phi(x,y) = \phi(x)\delta(x,y)$ are local and become the standard functions on S^2 in the commutative limit.
- **Long modes** for $|x-y| > 1/\sqrt{N}$ are non-local and have no classical analogue.
  This is the non-commutative regime.

String modes - representation of operators on functions

- A general representation of operators on matrices in terms of the string modes is straightforward

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2x'\, d^2y\, d^2y' \left|{}^x_y\right) \mathcal{O}(x, y; x', y') \left({}^{x'}_{y'}\right| \,.$$

- For the propagator

$$\frac{1}{\Box + m^2} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{}^x_y\right) \mathcal{O}_P^D(x, y) \left({}^x_y\right|$$

where

$$\left({}^x_y\right| \frac{1}{\Box + m^2} \left|{}^x_y\right) \approx \frac{1}{\frac{N^2}{4}|x - y|^2 + m^2} \,.$$

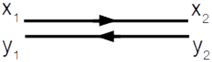
- For any function of the $\Box$ operator $f(\Box)$ we have

$$\left({}^x_y\right| f(\Box) \left|{}^x_y\right) = \frac{1}{N} \sum_{k,l} (2k+1)(2l+1)(-1)^{l+k+2s} f\left(k(k+1)\right) \left\{ \begin{matrix} l & s & s \\ k & s & s \end{matrix} \right\} e^{-l^2/N} P_l(\cos\vartheta)$$

Loop computations and (non)locality in fuzzy QFT

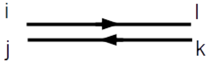
[Steinacker 2016; Steinacker, JT 2022]

- Feynman rules in string modes formalism - propagator



$$= \left({x_2 \atop y_2} \middle| \frac{1}{\Box + m^2} \middle| {x_1 \atop y_1} \right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \delta(x_1, x_2)\delta(y_1, y_2)$$

- Compare with the pure matrix models propagator

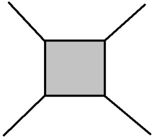


$$\sim \frac{1}{m^2}\delta_{il}\delta_{jk} \ .$$

and field theory action

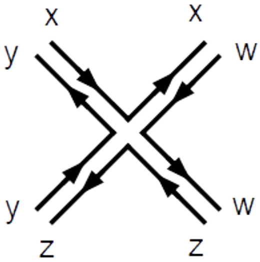
$$S(M) = \frac{4\pi}{N}\mathrm{Tr}\left[\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2 M^2 + g\, M^4\right]$$

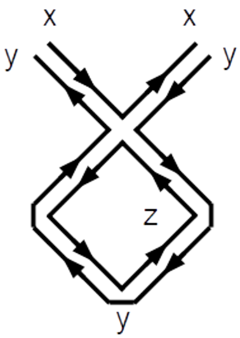
- Feynman rules in string modes formalism - vertex



$$= g\,\langle y_1|x_2\rangle\,\langle y_2|x_3\rangle\,\langle y_3|x_4\rangle\,\langle y_4|x_1\rangle \approx g\,\delta(y_1,x_2)\delta(y_2,x_3)\delta(y_3,x_4)\delta(y_4,x_1)$$

One-loop two-point function and effective action

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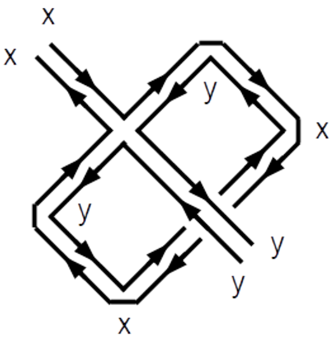
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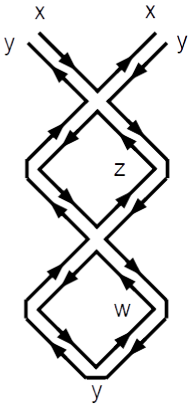
- We obtain the one-loop effective action for the classical fields $\phi(x,y)=\phi(x)\delta(x,y)$

$$S_{\text{eff}} = \int dx \phi(x) \frac{1}{2}(\Box + \mu^2)\phi(x) + \frac{g}{3}\frac{1}{4\pi}\int dx\, \phi(x)^2 \mu_N^2 + \\ + \frac{g}{6}\left(\frac{N}{4\pi}\right)^2 \int dx\, dy\, \phi(x)\phi(y) \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \ .$$

- It this is equivalent to the previous formula, but with a different interpretation.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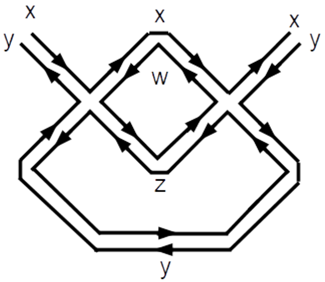
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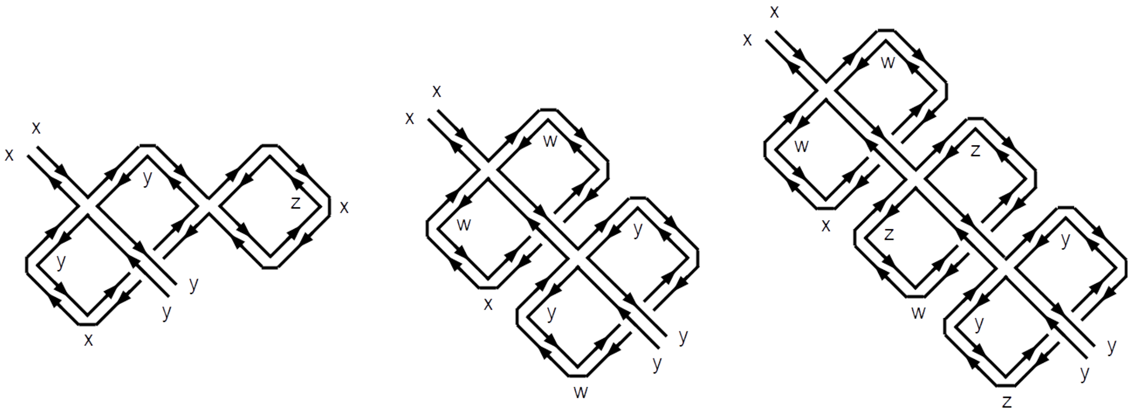
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Take home message

Take home message and 2do list

- Quantization of gravity seems to lead to quantized spacetimes.
- Fuzzy spaces are examples of such spacetimes.
- Plenty of interesting things happen on spaces with quantum structure.
- Physics on such spaces is described by random matrix ensembles.

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- Beyond fuzzy sphere.
- More on kinetic term effective action.
- Correlation functions, entanglement entropy.
- Multitrace models in emergent and random fuzzy geometries.
- Some other things that Samuel will talk about.

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...

Fuzzy spaces - an alternative construction

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i,j = 1,2,3 \ ,$$

  which generate the algebra of functions.

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i,j = 1,2,3 \ .$$

- Such $\hat{x}_i$'s generate a different, non-commutative, algebra and S_N^2 is an object, which has this algebra as an algebra of functions.

Fuzzy spaces - an alternative construction

- The conditions can be realized as an $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} s(s+1) = r^2 \ .$$

- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces - an alternative construction

- Most importantly nonzero commutators

$$\hat{x}_i\hat{x}_i = \rho^2 \quad , \quad \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i = 1,2,3 \ .$$

imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \ .$$

- Configuration space is analogous to phase space of quantum mechanics.
- In a similar fashion it is possible to construct an analogous deformation of the plane

$$\hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = i\theta\varepsilon_{ij} = i\theta_{ij} \ , \ i = 1,2 \ .$$

Construction uses the $\star$-product

$$f \star g = f\, e^{\frac{i}{2}\overleftarrow{\partial}\,\theta\,\overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} + \cdots$$

Random matrices ...

Random matrices

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Matrix model = ensemble of random matrices.
- An important example - ensemble of $N \times N$ hermitian matrices with

$$P(M) \sim e^{-N\mathrm{Tr}(V(M))} \ , \ \text{usually} \ V(x) = \frac{1}{2} r\, x^2 + g\, x^4$$

  and

$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij}\, Im\, M_{ij}\right] .$$

- Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$.
- Requirement of such invariance is very restrictive. One is usually interested in the distribution of eigenvalues.

- If we ask invariant questions, we can turn

$$\langle f \rangle = \frac{1}{Z} \int dM\, f(M) P(M)$$

into an eigenvalue problem by diagonalization $M = U \Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \dots, \lambda_N)$, the integration measure becomes

$$dM = dU \left(\prod_{i=1}^{N} d\lambda_i \right) \times \prod_{i<j} (\lambda_i - \lambda_j)^2$$

- We are to compute integrals like

$$\langle f \rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i \right) f(\lambda_i)\, e^{-\left[\sum_i V(\lambda_i) - 2 \sum_{i<j} \log|\lambda_i - \lambda_j|\right]} \times \int dU$$

- Term

$$2\sum_{i<j}\log|\lambda_i-\lambda_j|$$

is of order N^2 if $\lambda_i \sim 1$. Potential term

$$\sum_i V(\lambda_i)$$

is of order N.

- We need to enhance the probability measure by a factor of N to

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i)-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

- This makes the N^2 dependence explicit.

- We introduce eigenvalue distribution

$$\rho(\lambda) = \frac{1}{N} \sum_j \delta(\lambda - \lambda_j)$$

which gives for the averages

$$\langle f \rangle = \int d\lambda \, \rho(\lambda) f(\lambda) \ .$$

- The question is, how does do probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]}$$

translate into eigenvalue distribution ρ.

- For finite N - orthogonal polynomials method.
- For $N \to \infty$ the question simplifies due to the factor N^2

$$e^{-N^2\left[\frac{1}{N}\sum_i V(\lambda_i)-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}\,.$$

- For large N only configurations with small exponent contribute significantly to the integral. In the limit $N \to \infty$ only the extremal configuration

$$V'(\lambda_i)-\frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}=0 \quad \forall i$$

- Like a gas of particles with logarithmic repulsion. This gives us nice intuition.

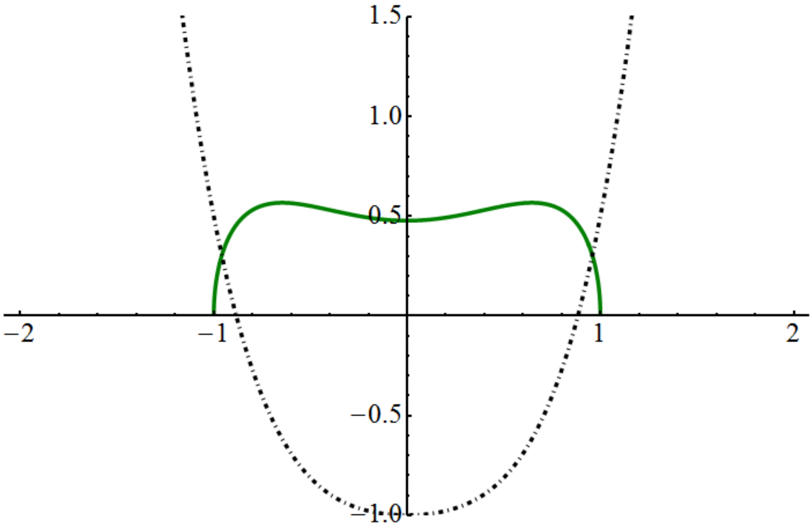
Random matrices - quartic potential

- The simplest case

$$V(x) = \frac{1}{2} r x^2 + g x^4$$

Random matrices - quartic potential

$V(x) = rx^2/2 + gx^4$ and $r > 0$



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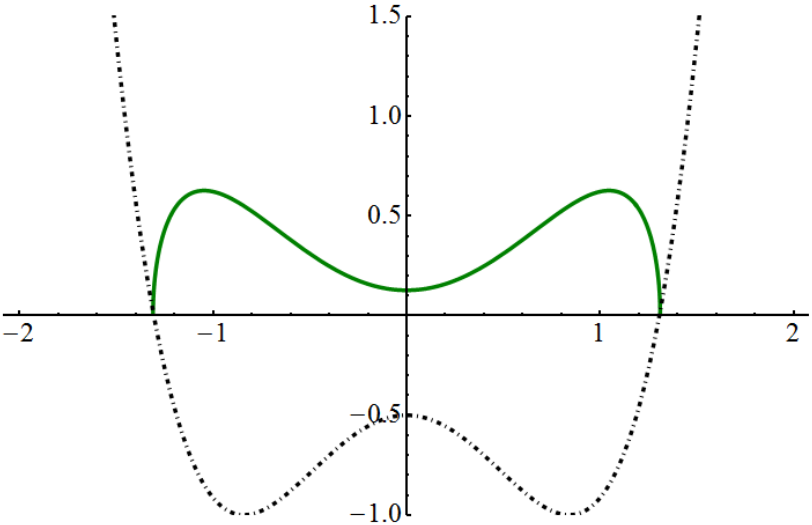

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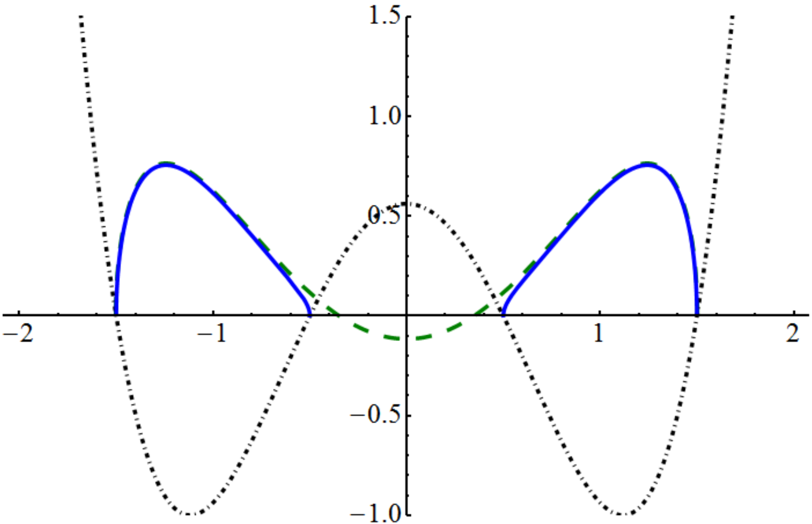

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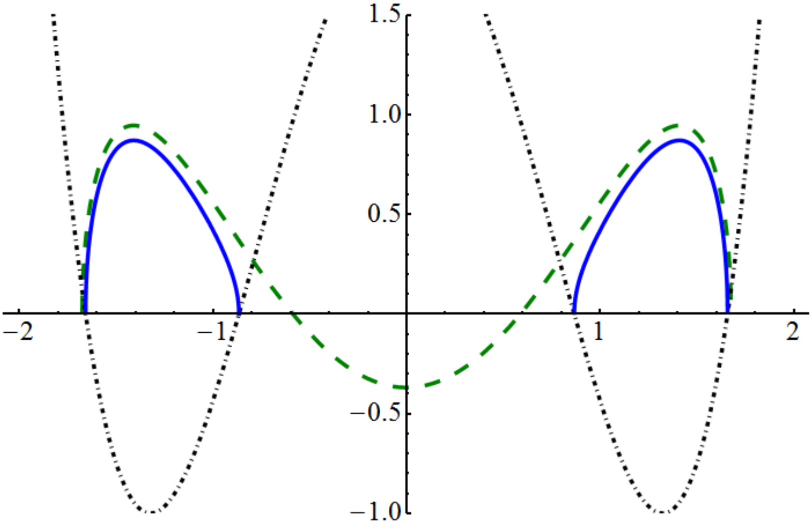

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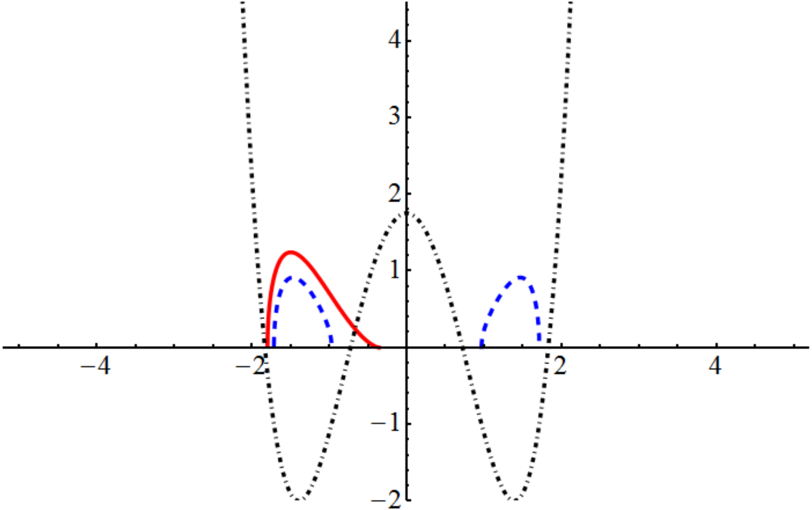


- If more than one solution is possible, the one with lower energy

$$\mathcal{F} = -N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]$$

  is the preferred one.
- The probability measure

$$e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]}$$

  i.e. the more probable solution.

Second moment approximation

$$\frac{4-3\delta^2 g}{\delta} - r - F'\left(\frac{4\delta+\delta^3 g}{16}\right) = 0 \ ,$$

$$4Dg + r + F'(D) = 0 \ , \ \delta^2 = \frac{1}{g} = 0 \ ,$$

$$4\frac{4+15\delta^2 g + 2r\delta}{\delta(4+9\delta^2 g)} - F'\left(\frac{\delta\left(64+160\delta^2 g+144\delta^4 g^2+81\delta^6 g^3+36\delta^3 gr+27\delta^5 g^2 r\right)}{64(4+9\delta^2 g)}\right) = 0 \ .$$

More on GW model

- The NC plane coordinates can be realized by

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} , \; Y = \frac{i}{\sqrt{2}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} ,$$

then

$$[X, Y] = i \; .$$

- This algebra is then truncated to a finite dimension.

- Define finite matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \sqrt{N-1} \\ & & & \sqrt{N-1} & \end{pmatrix} \ , \ Y = \dots \ ,$$

which gives

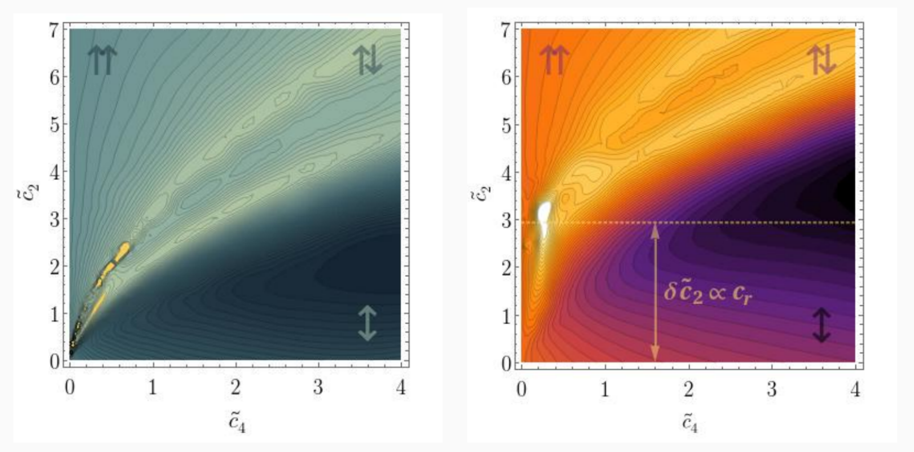
$$[X, Y] = i(1 - Z) \ , \ Z = \mathrm{diag}\,(0, \dots, N) \ .$$

- Original algebra is recovered in the $N \to \infty$ limit or under the $Z = 0$ condition.

$$R = \frac{15}{2} - 4Z^2 - 8\left(X^2 + Y^2\right) = \frac{31}{2} - 16\,\mathrm{diag}\,(1, 2, \dots, N-1, 8N) \ .$$

[Prekrat, Todorović-Vasović, Ranković '21; Prekrat '21]

- Numerical investigation of this matrix model leads to



Fuzzy field theories in the string modes formalism

String modes - coherent states

- Natural basis in the auxiliary hilbert space $\mathcal{H}$ is the "spin" basis

$$|n\rangle = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \ , \ n = -s, \ldots, s \ ,$$

  derived from the highest weight state $|s\rangle$.

- For any $x \in S^2$ with radius 1, choose some $g_x \in SO(3)$ such that $x = g_x \cdot p$, where p is the north pole on S^2. We define [Perelomov 1986]

$$|x\rangle = g_x \cdot |s\rangle \ , \ g_x \in SU(2)$$

  and call the set of all $|x\rangle$ the coherent states.

- $|x\rangle$ is located around x, but is an element of $\mathcal{H}$, and is a noncommutative analogue of the point x. [Steinacker 2020]

String modes - coherent states

- They form an over-complete set in $\mathcal{H}$ and

$$\mathbb{1} = \frac{N}{4\pi} \int d^2x \, |x\rangle \langle x| \quad , \quad \mathbb{1} = \sum_n |n\rangle \langle n| \; .$$

- They are orthogonal only in the large N limit

$$|\langle x|y\rangle|^2 = \left(\frac{1 + x \cdot y}{2} \right)^{N-1} \quad .$$

- Coherent states can be used to map (quantize) functions on S^2 on matrices

$$\phi(x) \to M = \int d^2x\, \phi(x) |x\rangle\langle x| \ .$$

and matrices on functions (de-quantize)

$$M \to \phi(x) = \langle x| M |x\rangle \ .$$

- This maps T_{lm} on Y_{lm} up to normalization

$$T_{lm} \to \langle x| T_{lm} |x\rangle = \frac{1}{c_l} Y_{lm}(x) \ , \ c_l^2 = \frac{1}{4\pi} \frac{(N-1-l)!(N+l)!}{((N-1)!)^2} \sim \frac{N}{4\pi} e^{\frac{l^2}{N}} \ .$$

- For $l < \sqrt{N}$ coefficients c_l are approximately constant, quantization and de-quantization are inverse of each other.
  For $l > \sqrt{N}$ coefficient c_l grows extremely fast and the-quantized matrices are misleading.

String modes - representation of functions on S_F^2

- Functions on the fuzzy sphere are matrices acting on $\mathcal{H}$

$$M = \sum_{m,n=-s}^{s} M_{mn} \left| m \right\rangle \left\langle n \right| \ .$$

- We can express the matrix M in a similar fashion using the coherent states

$$M = \left(\frac{N}{4\pi} \right)^2 \int d^2x \, d^2y \, \phi(x,y) \left| x \right\rangle \left\langle y \right| \ .$$

- Objects [Iso, Kawai, Kitazawa 2000; Steinacker 2016]

$$\left| x \right\rangle \left\langle y \right| =: \left| \begin{matrix} x \\ y \end{matrix} \right)$$

form a basis of functions on the fuzzy sphere and we will call them the **string modes**.

String modes - representation of functions on S_F^2

- Such representation of matrix M by function $\phi(x, y)$ seems to be not unique (way more functions than matrices).
  But one can show that derivatives of $\phi(x, y)$ are bounded by $\sqrt{N}$, which means that the Fourier modes of ϕ to be restricted by $l_x, l_y \leq \sqrt{N}$.
- Functions $\phi(x, y)$ that represent functions on the fuzzy sphere have rather mild behavior. The coherent states are spread out over an area $\sim 4\pi/N$ and average out any larger oscillations.
- Large momentum UV wavelengths are smoothed out on the fuzzy sphere. But the price we pay is non-local string modes.

String modes - representation of functions on S_F^2

$$\left| \begin{matrix} x \\ y \end{matrix} \right)$$

- **Short modes** for $|x-y| < 1/\sqrt{N}$ can be shown to represent localized wave-packets with momentum $\sim N|x-y|$.
  This is the classical regime.
- Particularly string mode $\left| \begin{matrix} x \\ x \end{matrix} \right)$ represents a maximal localized function around point x, i.e. a fuzzy version of δ-function.
  Functions with $\phi(x,y) = \phi(x)\delta(x,y)$ are local and become the standard functions on S^2 in the commutative limit.
- **Long modes** for $|x-y| > 1/\sqrt{N}$ are non-local and have no classical analogue.
  This is the non-commutative regime.

String modes - representation of operators on functions

- When working with functions we encounter operators

$$\mathcal{O} : M \to \mathcal{O}(M) \ .$$

- For example the kinetic term of the field theory or the propagator of the theory

$$[L_i, [L_i, M]] =: \Box M \quad , \quad \frac{1}{\Box + m^2} \ .$$

- String modes are eigenfunctions of $\Box$

$$\Box \left| \begin{smallmatrix} x \\ y \end{smallmatrix} \right) = \left(\frac{N^2}{4} |x - y|^2 + N \right) \left| \begin{smallmatrix} x \\ y \end{smallmatrix} \right) \ .$$

[Steinacker, T work in progress]

- A general representation of such operators in terms of the string modes is straightforward

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2x'\, d^2y\, d^2y' \left|{x \atop y}\right)\mathcal{O}(x,y;x',y')\left({x' \atop y'}\right| .$$

- There are two special cases
  - Local

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop x}\right)\mathcal{O}^L(x,y)\left({y \atop y}\right| .$$

  - Non-local, but diagonal,

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop y}\right)\mathcal{O}^D(x,y)\left({x \atop y}\right| .$$

- Functions $\mathcal{O}_L$ and $\mathcal{O}_D$ may have very different behavior for different operators (oscillation, singularity). Local representations are typically highly oscillatory, non-local representations are better behaved.

- Operator traces

$$\mathrm{Tr}\,\mathcal{O} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle|\mathcal{O}\middle|\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \ .$$

[used in the "I don't have time to show you details" part of Harold's talk @ Humboldt Kolleg]

String modes - representation of operators on functions

- For the propagator

$$\frac{1}{\Box + m^2} = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|{x \atop y}\right) \mathcal{O}_P^D(x,y) \left({x \atop y}\right|$$

where

$$\mathcal{O}_P^D(x,y) = \left({x \atop y}\right| \frac{1}{\Box + m^2} \left|{x \atop y}\right) \ .$$

- For any function of the $\Box$ operator $f(\Box)$ we have

$$\left({x \atop x} \middle| f(\Box) \middle| {y \atop y} \right) = \frac{1}{N} \sum_l (2l+1) f\big(k(k+1)\big) e^{-l^2/N} P_l(\cos\vartheta)$$

$$\left({x \atop y} \middle| f(\Box) \middle| {x \atop y} \right) = \frac{1}{N} \sum_{k,l} (2k+1)(2l+1)(-1)^{l+k+2s} f\big(k(k+1)\big) \left\{ \begin{array}{ccc} l & s & s \\ k & s & s \end{array} \right\} e^{-l^2/N} P_l(\cos\vartheta)$$

where the curly bracket is the $6j$-symbol and $\cos\vartheta = x \cdot y$.

- For the propagator we obtain

$$\left({x \atop y} \middle| \frac{1}{\Box + m^2} \middle| {x \atop y} \right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \ .$$

- Trace of propagator

$$\mathrm{Tr}\,\frac{1}{\Box+m^2}=\frac{N^2}{(4\pi)^2}\int d^2x\,d^2y\left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle|\frac{1}{\Box+m^2}\middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\frac{N^2}{(4\pi)^2}\int\frac{1}{\frac{N^2}{4}|x-y|^2+m^2}=$$

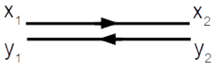
$$=\frac{N^2}{2}\int_{-1}^{1}du\,\frac{1}{\frac{N^2}{2}(1-u)+m^2}\sim 2\log(N)\ .$$

- This is consistent with

$$\mathrm{Tr}\,\frac{1}{\Box+m^2}=\sum_{l=0}^{N-1}\frac{2l+1}{l(l+1)+m^2}\sim N\int_0^1\frac{2Nx}{N^2x^2+m^2}\sim 2\log(N)\ .$$

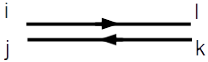
[Steinacker 2016; Steinacker, T work in progress]

- Feynman rules in string modes formalism - propagator



$$= \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix} \middle| \frac{1}{\Box + m^2} \middle| \begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix} \right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} \delta(x_1, x_2)\delta(y_1, y_2)$$

- Compare with the pure matrix models propagator

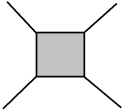


$$\sim \frac{1}{m^2}\delta_{il}\delta_{jk} \ .$$

and field theory action

$$S(M) = \frac{4\pi}{N}\mathrm{Tr}\left[\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2 M^2 + g\, M^4\right]$$

- Feynman rules in string modes formalism - vertex

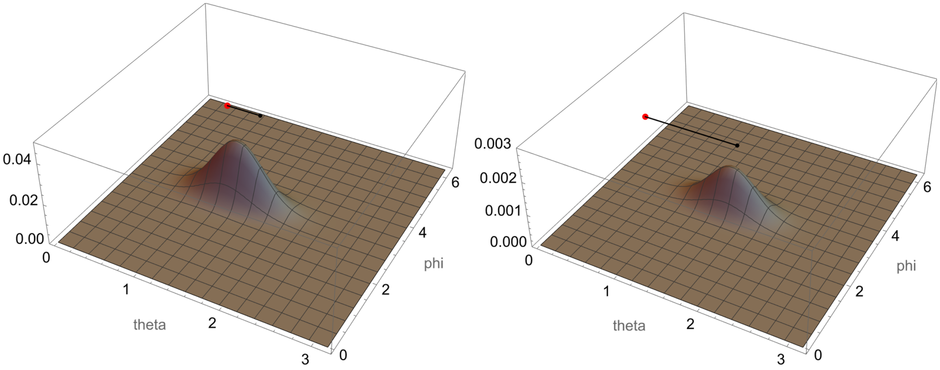


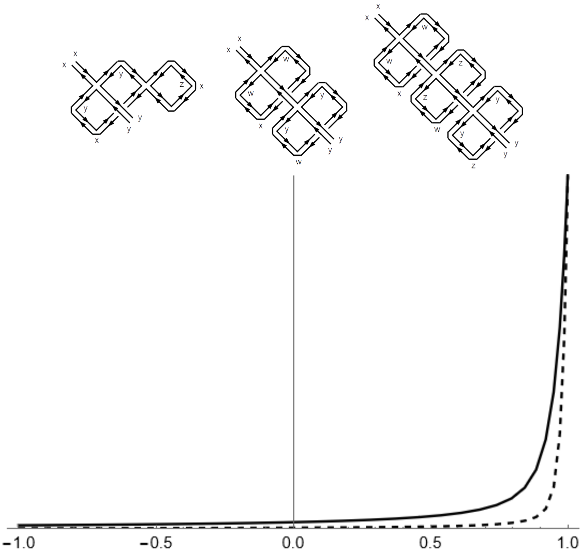
$$= g \langle y_1|x_2\rangle \langle y_2|x_3\rangle \langle y_3|x_4\rangle \langle y_4|x_1\rangle \approx g\, \delta(y_1, x_2)\delta(y_2, x_3)\delta(y_3, x_4)\delta(y_4, x_1) \ .$$

Loop computations and (non)locality in fuzzy QFT

- String modes bring, in the large N limit, the best from the two worlds. They diagonalize the kinetic term and keep a simple structure of the vertices.
- Similar to the standard QFT calculations, but regular thanks to the effective noncommutative cutoff. No singularities and no issues when computing loop diagrams in position space.

Loop computations and (non)locality in fuzzy QFT




-1.0
-0.5
0.0
0.5
1.0

Correlation functions

Correlation functions

- Analogues of points in the NC setting are coherent states $|\vec{x}\rangle$.
- "Value" of field ϕ at "point" $\vec{x}$ given by

$$\langle\vec{x}|\,\phi\,|\vec{x}\rangle = \phi(x)\ .$$

- Behaviour of

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{Z}\int d\phi\,\langle\vec{x}|\,\phi\,|\vec{x}\rangle\,\langle\vec{y}|\,\phi\,|\vec{y}\rangle\,e^{-S(M)}$$

  in the matrix model can be studied numerically.
  [Hatakeyama, Tsuchiya '17; Hatakeyama, Tsuchiya, Yamashiro '18 '18]
- At the "standard" phase transition, the behaviour of the correlation functions at short distances differs from the commutative theory and seems to agree with the tricritical Ising model.
  A different behaviour at long distances.

Entanglement entropy

Entanglement entropy

- In local theories $S(A) \sim A$.
  [Ryu, Takayanagi '06]
- In non-local theories this can change.
  [Barbon, Fuertes '08; Karczmarek, Rabideau '13; Shiba, Takayanagi '14]
- Problem on the fuzzy sphere has been studied numerically.
  [Karczmarek, Sabella-Garnier '13; Sabella-Garnier '14; Okuno, Suzuki, Tsuchiya '15; Suzuki, Tsuchiya '16; Sabella-Garnier '17; Chen, Karczmarek '17]
- For free fields, the EE follows volume law rather than area law.
  In the interacting case much smaller EE than in the free case.

Challenges

Correlation functions

- Quantity $\langle\phi(x)\phi(y)\rangle$ is U dependent, so we need to figure out what to do with

$$\int dU\ F(\Lambda, U)\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}\ .$$

Entanglement entropy

- We need to extended the model to $\mathbb{R}\times S_F^2$, i.e. $M(t)$

$$S(M) = \int dt \mathrm{Tr}\left(-\frac{1}{2}M\partial_t^2 M + \frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$

[Medina, Bietenholz, O'Connor '07; Ihl, Sachse, Sämann '10]
Also the U dependence will play a role, but free theory where $\mathcal{R}=0$, is enough.