

MATRIX ENSEMBLES FROM FUZZY PHYSICS

THE GOOD, THE BAD, THE UGLY

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cost
EUROPEAN COOPERATION
IN SCIENCE & TECHNOLOGY



Funded by
the European Union

work with: M. Šubjaková, D. Prekrat, S. Kováčik, N. Pagliaroli,
B. Bukor, K. Magdolenová, P. Rusnák, A. Kubiš

10. 6. 2026, Faculty of Physics Seminar, Belgrade

supported by CEEPUS M-RS-1514-2526 - *Quantum Spacetime, Gravitation and Cosmology*

Matrix models, quantum geometry and beyond

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$



intensive course for PhD. students by CEEPUS

16. - 22. august 2026

Preliminary list of speakers:

Fedele Lizzi, Naples
Harold Steinacker, Vienna
Dragan Prekrat, Belgrade
Samuel Kováčik, Bratislava
Juraj Tekel, Bratislava



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Quick motivation



- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
No distinction between points under certain length scales. [[Hossenfelder 1203.6191](#)]
- Reasons:
 - gravitational Heisenberg microscope,
 - instability of quantum gravitational vacuum, [[Doplicher, Fredenhagen, Roberts '95](#)]
 - emergent spacetimes.
- Fuzzy spaces are very important examples of such spacetimes.



Take home message



TAKE HOME MESSAGE

- Fuzzy spaces are examples of spacetimes with quantum structure.
- Plenty of interesting things happen on such spaces.
- Physics is described by a wide array of random matrix ensembles. Analyzing these is technically challenging, but doable.



TAKE HOME MESSAGE

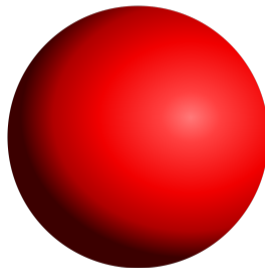
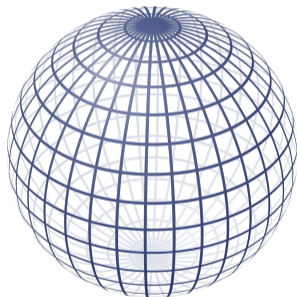
- Fuzzy spaces are examples of spacetimes with quantum structure.
- Plenty of interesting things happen on such spaces.
- Physics is described by a wide array of random matrix ensembles. Analyzing these is technically challenging, but doable.
- We can study toy models of path integrals over geometries.



Fuzzy spaces

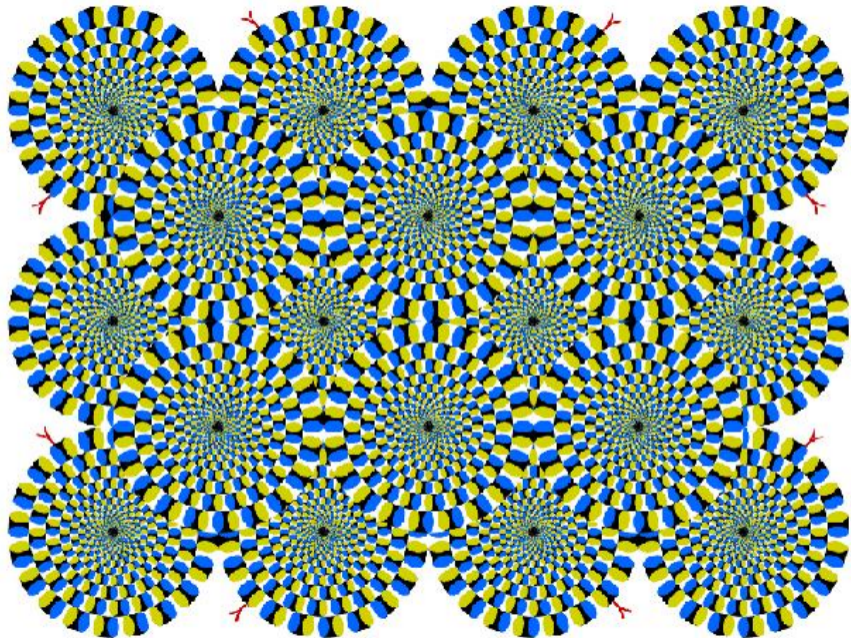


- We divide the space into N cells. Function on the fuzzy space is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.



- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.







- Regularization of infinities in the standard QFT.
[Heisenberg ~1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
[Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
[Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BMN).
[Steinacker 2013, 2024]
- Geometric unification of the particle physics and theory of gravity.
[van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
[Karabali, Nair 2006]
- Toy models of spaces with discrete quantum structure.



Fuzzy sphere [Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder 1990s]

- Functions on the usual sphere are given by

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm} Y_{lm}(\theta, \phi) ,$$

where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi) .$$

- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^L \sum_{m=-l}^l c_{lm} Y_{lm}(\theta, \phi) ,$$

we will not be able to see any features of functions under certain length scales.

- Expressions defined in this way are not closed under multiplication.



- Number of independent functions with $l \leq L$ is $(L + 1)^2$, the same as the number of $N \times N$ hermitian matrices.
- We have a map $\varphi : Y_{lm} \rightarrow M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1}(\varphi(Y_{lm})\varphi(Y_{l'm'})) .$$

- Opposing to some lattice discretization this space still possess a full rotational symmetry.
- In the limit N or $L \rightarrow \infty$ we recover the original sphere.



- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad [\hat{x}_i, \hat{x}_j] = i\theta \varepsilon_{ijk} \hat{x}_k \quad , \quad i, j = 1, 2, 3 .$$

- The conditions can be realized as an $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} s(s + 1) = r^2 .$$

- The group $SU(2)$ still acts on \hat{x}_i 's and this space enjoys a full rotational symmetry. Most importantly nonzero commutators imply uncertainty relations for positions $\Delta x_i \Delta x_j \neq 0$.
- In a similar fashion it is possible to construct an analogous deformation of the plane

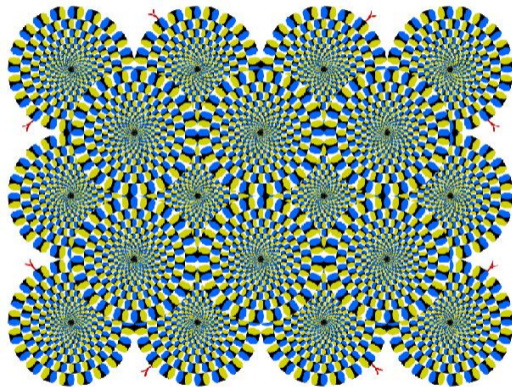
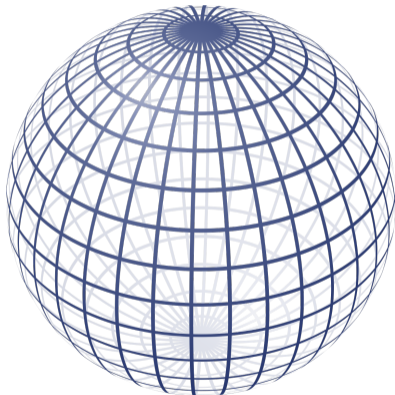
$$[\hat{x}_i, \hat{x}_j] = i\theta \varepsilon_{ij} = i\theta_{ij} \quad , \quad i = 1, 2 .$$

Construction uses the \star -product

$$f \star g = f e^{\frac{i}{2} \vec{\partial} \theta \vec{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} + \dots$$



FUZZY SPACES



Fuzzy field theories



- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right],$$

$$\langle F \rangle = \frac{\int d\Phi F(\Phi) e^{-S(\Phi)}}{\int d\Phi e^{-S(\Phi)}}.$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \text{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right],$$

$$\langle F \rangle = \frac{\int dM F(M) e^{-S(M)}}{\int dM e^{-S(M)}}.$$

[Balachandran, Kürkcüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]



Random matrices ensembles



RANDOM MATRICES

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Matrix model = ensemble of random matrices, e.g. ensemble of $N \times N$ hermitian matrices with

$$P(M) \sim e^{-N\text{Tr}(V(M))}, \text{ usually } V(x) = \frac{1}{2}r x^2 + g x^4$$

- Expectation values

$$\langle f \rangle = \frac{1}{Z} \int dM P(M) f(M)$$

can be analyzed

- numerically using Hamiltonian Monte Carlo,
 - analytically in the large N limit using saddle point equation.
- Key word is diagonalization.

$$M = U \Lambda U^\dagger, \quad dM = dU d\Lambda \prod_{i < j} (\lambda_i - \lambda_j)^2.$$

- One usually looks for eigenvalue distribution $\rho(x)$.

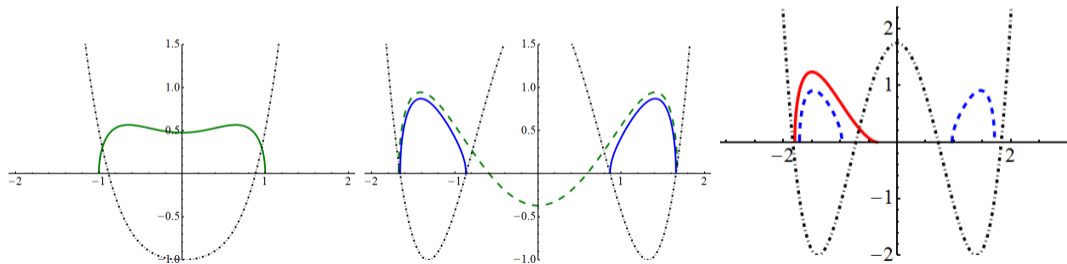


[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Saddle point equation

$$V'(\lambda_i) = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} .$$

- Equilibrium of a gas of interacting particles in external potential.



Fuzzy field theories ensembles I

Full matrix model



- Recall the action of the fuzzy scalar field theory

$$S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} m^2 \text{Tr} (M^2) + g \text{Tr} (M^4) . \quad (1)$$

This is a particular case of a matrix model since we need

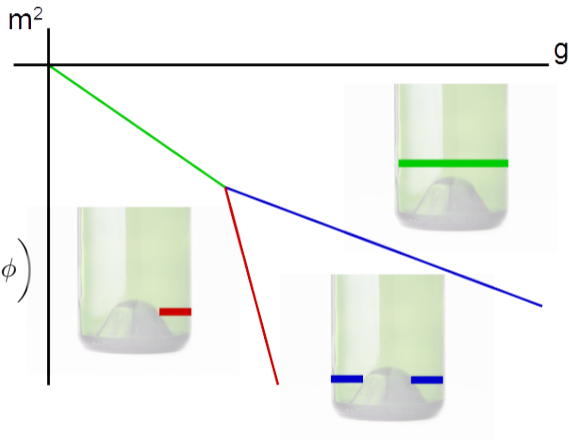
$$\int dM F(M) e^{-S(M)} .$$

- "Matrix model begs to be put on a computer".



$$S[M] = \text{Tr} \left(\frac{1}{2} M [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + g M^4 \right)$$

$$S = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)$$

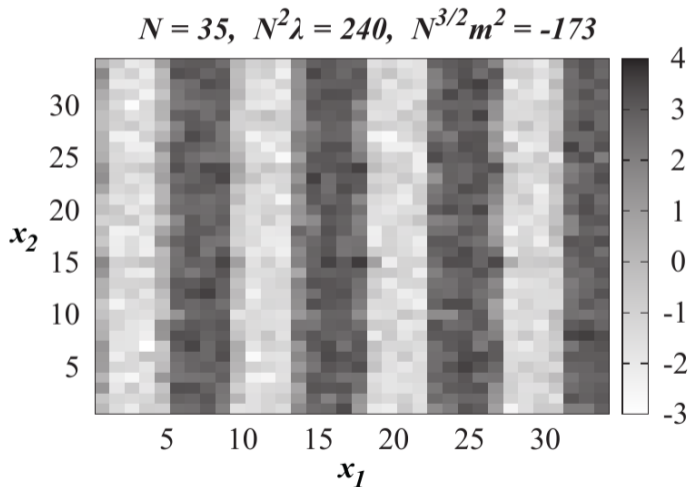


- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
[Gubser, Sondhi 2001; Chen, Wu 2002]
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is spontaneously broken.
- This has been established in numerous numerical works for variety different spaces.
[Martin 2004; García Flores, Martin, O'Connor 2006, 2009; Panero 2006, 2007; Ydri 2014; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero 2014; Mejía-Díaz, Bietenholz, Panero 2014; Medina, Bietenholz, D. O'Connor 2008; Bietenholz, Hofheinz, Nishimura 2004; Lizzi, Spisso 2012; Ydri, Ramda, Rouag 2016; Kováčik, O'Connor 2018]
[Panero 2015]



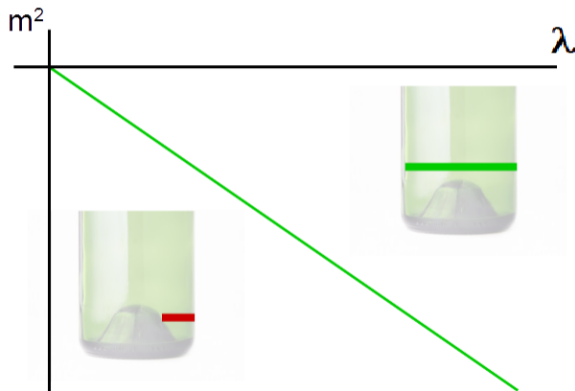
PHASES OF FUZZY FIELD THEORIES

[Mejía-Díaz, Bietenholz, Panero 2014] for \mathbb{R}_θ^2



PHASES OF FUZZY FIELD THEORIES

$$S[\phi] = \int d^2x \left(\frac{1}{2} \partial_i \phi \partial_i \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$



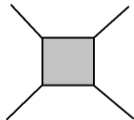
[Glimm, Jaffe 1974; Glimm, Jaffe, Spencer 1975; Chang 1976]

[Loinaz, Willey 1998; Schaich, Loinaz 2009]



FUZZY SCALAR FIELD THEORY - UV/IR MIXING

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
[Minwalla, Van Raamsdonk, Seiberg 2000; Vaidya 2001; Chu, Madore, Steinacker 2001]
- In terms of diagrams different properties of planar and non-planar ones.
The (matrix) vertex is not invariant under permutation of incoming momenta.



- Very energetic fluctuations (UV physics) have consequences at large distances (IR physics).
- $N \rightarrow \infty$ limit of the effective action is different from the standard S^2 effective action.
- Situation much more severe for NC plane.
- The space (geometry) forgets where it came from, but the field theory (physics) remembers its fuzzy origin. [[Steinacker, JT '22](#)]



Fuzzy field theories ensembles II

Perturbative model



$$e^{-N^2 S_{\text{eff}}(\Lambda)} = \int dU e^{-\varepsilon \frac{1}{2} \text{Tr}(U \Lambda U^\dagger [L_i, [L_i, U \Lambda U^\dagger]])}$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M .
[O'Connor, Sämann 2007; Sämann 2010]
- The most recent result is
[Sämann 2015]

$$\begin{aligned} S_{\text{eff}}(\Lambda) = & \frac{1}{2} \left[\varepsilon \frac{1}{2} (c_2 - c_1^2) - \varepsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \varepsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] - \\ & - \varepsilon^4 \frac{1}{3456} \left[(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4) - 2(c_2 - c_1^2)^2 \right]^2 - \\ & - \varepsilon^3 \frac{1}{432} \left[c_3 - 3c_1c_2 + 2c_1^3 \right]^2, \quad \text{where } c_n = \frac{1}{N} \sum_i \lambda_i^n \end{aligned} \quad (2)$$



[Das, Dhar, Sengupta, Wadia '90; Cicuta, Montaldi '90]

- Varying multitrace action leads to saddle point equation

$$\sum_n \frac{\partial f}{\partial c_n} n \lambda_i^{n-1} + r \lambda_i + 4g \lambda_i = \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} .$$

- At large N solved by effective single trace model with selfconsistency conditions on moments c_n .
- Multitrace terms introduce a new kind of interaction among the eigenvalues.



- Model [Ydri '14, '15]

$$\begin{aligned}
 S_{\text{eff}}(\Lambda) = & \frac{1}{2} \left[\varepsilon \frac{1}{2} (c_2 - c_1^2) - \varepsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \varepsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] - \\
 & - \varepsilon^4 \frac{1}{3456} \left[(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4) - 2(c_2 - c_1^2)^2 \right]^2 - \\
 & - \varepsilon^3 \frac{1}{432} \left[c_3 - 3c_1c_2 + 2c_1^3 \right]^2, \quad \text{where } c_n = \frac{1}{N} \sum_i \lambda_i^n
 \end{aligned}$$

yields a very unpleasant behaviour close to the origin of the parameter space. [JT '15]



Fuzzy field theories ensembles III

Nonperturbative model



SECOND MOMENT APPROXIMATION

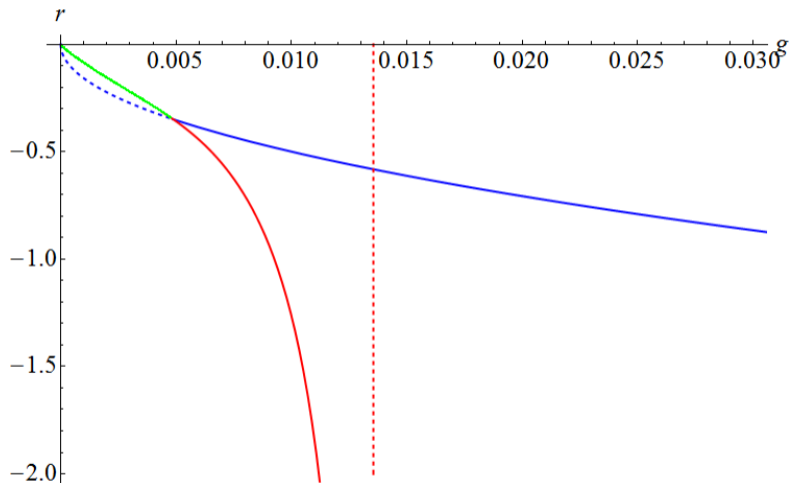
- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues.
[Steinacker 2005]
- There is a unique parameter independent effective action that reconstructs this rescaling.
[Polychronakos 2013] Can be generalized to more a more complicated kinetic term \mathcal{K} .
- Introducing the asymmetry $c_2 \rightarrow c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \mathcal{R} + \frac{1}{2}r \text{Tr}(M^2) + g \text{Tr}(M^4) \quad , \quad F(t) = \log\left(\frac{t}{1 - e^{-t}}\right) \quad . \quad (3)$$

[Šubjaková, JT PoS CORFU2019; JT '14 '15 '18; Šubjaková, JT '20]

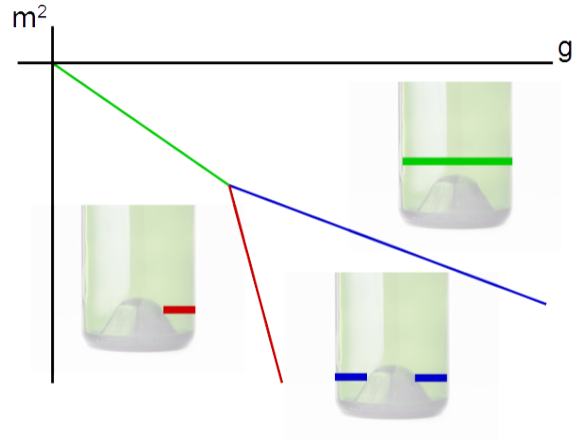
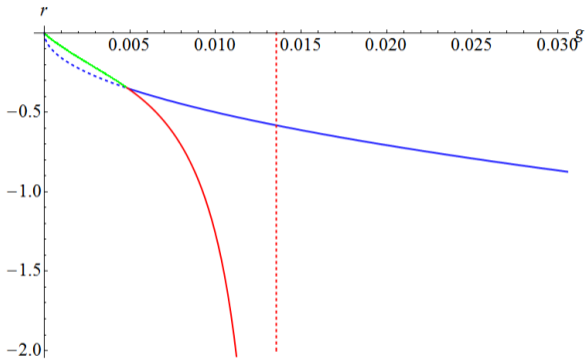


SECOND MOMENT APPROXIMATION



[JT '18; Šubjaková, JT 2020]





BEYOND THE SECOND MOMENT APPROXIMATION

- Taking a lesson from

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r \operatorname{Tr}(M^2) + g \operatorname{Tr}(M^4) \quad , \quad F(t) = \log\left(\frac{t}{1 - e^{-t}}\right)$$

we could try to complete the perturbative action

$$S_{\text{eff}} = F[c_1, t_2, t_3, t_4 - 2t_2^2] = \frac{1}{2} \log\left(\frac{t_2}{1 - e^{-t_2}}\right) + F_3(t_3) + F_4(t_4 - 2t_2^2) \quad (4)$$

and

$$F_4(y_4) = \alpha_0 \log(y_4) + \alpha_1 + \frac{\alpha_2}{y_4} + \frac{\alpha_3}{y_4^2} + \dots$$

- Any attempt to complete the perturbative expansion in the spirit of the non-perturbative model is not capable of solving the above problems and does not lead to a phase diagram that is in complete agreement with the numerical simulations. Most importantly the location of the triple point can not be brought closer to the numerical value. [Šubjaková, JT '22]





Fuzzy field theories ensembles IV

Removal of stripes



- Naive fuzzy field theory leads to strange commutative limit.
- Is there a different theory that leads to the expected commutative limit?



- Naive fuzzy field theory leads to strange commutative limit.
- Is there a different theory that leads to the expected commutative limit?
- Yes. [Dolan, O'Connor, Prešnajder '01; Grosse, Wulkenhaar '00; Burić, Wohlgenannt '10]



- We would like to analyse the more complicated model

$$S = \text{Tr} \left(\frac{1}{2} M [L_i, [L_i, M]] + 12gMQM + \frac{1}{2} rM + gM^4 \right), \quad (5)$$

where

$$QT_{lm} = - \underbrace{\left(\sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+r} \left[(-1)^{l+j+N-1} \left\{ \begin{matrix} l & s & s \\ j & s & s \end{matrix} \right\} - 1 \right] \right)}_{Q(l)} T_{lm}.$$

- This removes the UV/IR mixing in the theory, essentially by removing the problematic part by brute force.

[Dolan, O'Connor, Prešnajder '01]



- Operator Q can be expressed as a power series in $C_2 = [L_i, [L_i, \cdot]]$

$$Q = q_1 C_2 + q_2 C_2^2 + \dots$$

- As a starting point, it is interesting to see the phase structure of such simplified model.
[O'Connor, Säman '07]

- This is the case of

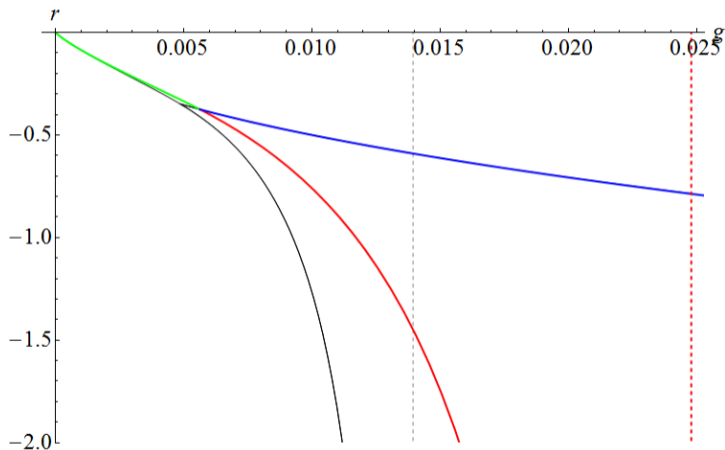
$$\mathcal{K} = (1 + ag)C_2 \quad \text{or} \quad \mathcal{K} = (1 + ag)C_2 + bg C_2^2 . \quad (6)$$

with some complicated form of $F(t)$.



REMOVAL OF STRIPES – FUZZY SPHERE

[Šubjaková, JT '20]

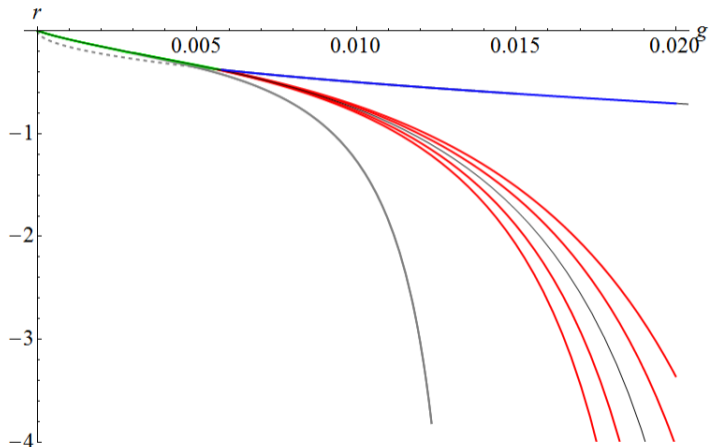


$$a = 3e^{3/2}, \quad b = 0.$$



REMOVAL OF STRIPES – FUZZY SPHERE

[Šubjaková, JT '20]



$$a = 3e^{3/2}, \quad b = -4, -2, 0, 2, 4.$$



- Grosse-Wulkenhaar model [‘00’s]

$$S_{GW} = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{1}{2} \Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right),$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu .$$

- This model is renormalizable.
- Described by a matrix model in terms of truncated Heisenberg algebra.
[Burić, Wohlgenannt ‘10]



- The NC plane coordinates can be realized by

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} +\sqrt{1} & +\sqrt{1} & & & \\ & +\sqrt{2} & +\sqrt{2} & & \\ & & & \ddots & \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix}, \quad Y = \frac{i}{\sqrt{2}} \begin{pmatrix} +\sqrt{1} & -\sqrt{1} & & & \\ & +\sqrt{2} & -\sqrt{2} & & \\ & & & \ddots & \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix},$$

then

$$[X, Y] = i.$$

- This algebra is then truncated to a finite dimension.



- Define finite matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & & \\ +\sqrt{1} & & +\sqrt{2} & & & \\ & +\sqrt{2} & & \ddots & & \\ & & \ddots & & \sqrt{N-1} & \\ & & & \sqrt{N-1} & & \end{pmatrix}, Y = \dots,$$

which gives

$$[X, Y] = i(1 - Z), \quad Z = \text{diag}(0, \dots, N).$$

- Original algebra is recovered in the $N \rightarrow \infty$ limit or under the $Z = 0$ condition.



REMOVAL OF STRIPES – GW MODEL

- The kinetic term becomes

$$\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi \rightarrow [X, M][X, M] + [Y, M][Y, M] .$$

and the harmonic potential becomes

$$\frac{1}{2} \Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) \rightarrow RM^2 ,$$

where X, Y, R are fixed external matrices.

- Interpretation of R coupling to the curvature of the space

$$R = \frac{15}{2} - 4Z^2 - 8(X^2 + Y^2) = \frac{31}{2} - 16 \operatorname{diag}(1, 2, \dots, N-1, 8N) .$$

- We are thus left with a matrix model with action

$$S = \operatorname{Tr} (M[X, [X, M]] + M[Y, [Y, M]]) - g_r \operatorname{Tr} (RM^2) - g_2 \operatorname{Tr} (M^2) + g_4 \operatorname{Tr} (M^4) .$$

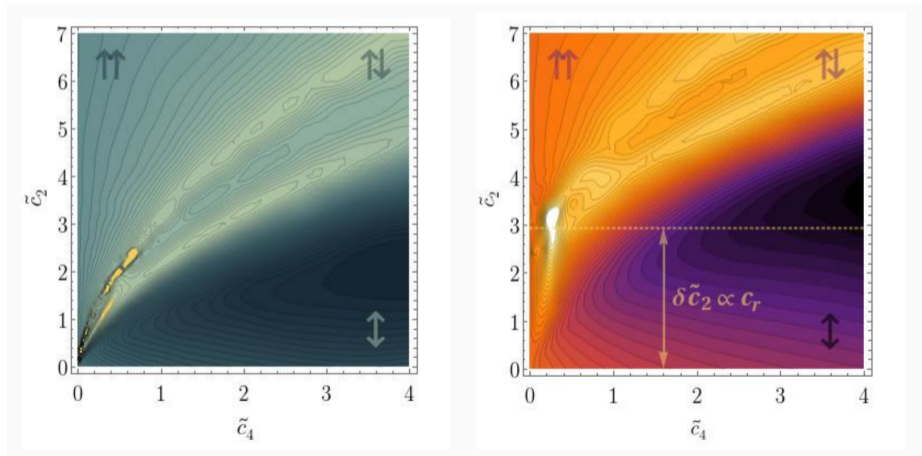


(7)

REMOVAL OF STRIPES – GW MODEL

[Prekrat, Todorović-Vasović, Ranković '21; Prekrat '21]

- Numerical investigation of this matrix model leads to



[Bukor, JT '23]

- The effect of the kinetic term

$$S(M) = \text{Tr}(MKM) - \text{Tr}(g_r RM^2) - g_2 \text{Tr}(M^2) + g_4 \text{Tr}(M^4) .$$

- This leads to the effective action

$$S_{\text{eff}}(\Lambda) = N^2 \left[\varepsilon t_2 - \varepsilon^2 \frac{2}{3} t_2^2 + \varepsilon^2 \frac{97}{120} (t_4 - 2t_2^2) \right] , \quad (8)$$

where t 's are symmetrized models

$$t_n = \frac{1}{N} \text{Tr} \left(M - \frac{1}{N} \text{Tr}(M) \right)^n .$$

- The same structure as before.



REMOVAL OF STRIPES – GW MODEL

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22; Bukor, Prekrat, JT '25]

- We concentrate on the effect of the curvature term and discard the kinetic term

$$S(M) = \text{Tr}(MKM) - \text{Tr}(g_r RM^2) - g_2 \text{Tr}(M^2) + g_4 \text{Tr}(M^4) . \quad (9)$$

- This leads to the angular integral

$$\int dU e^{g_r \text{Tr}(URU^\dagger \Lambda^2)} ,$$

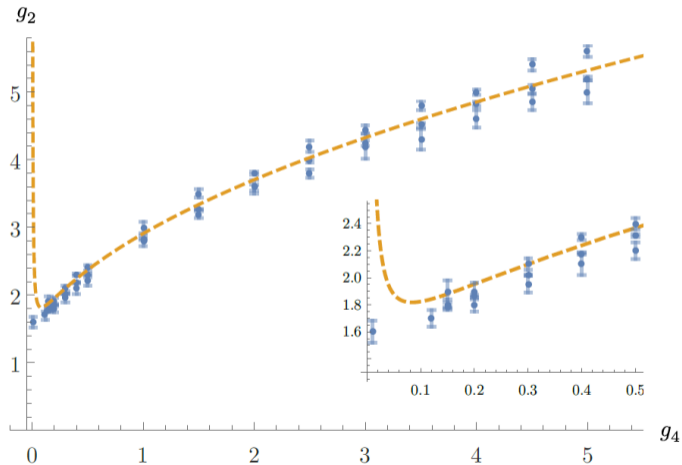
which gives up to g_r^6

$$S(\Lambda) = N \left(-g_2 c_2 + 8g_r c_2 + g_4 c_4 - \frac{32}{3} g_r^2 c_4 + \frac{1024}{45} g_r^4 c_8 - \frac{(8g_r)^6}{2835} c_{12} \right) + \\ + \frac{32}{3} g_r^2 c_2^2 + \frac{1024}{15} g_r^4 c_4^2 - \frac{4096}{45} g_r^4 c_6 c_2 + \frac{2(8g_r)^6}{945} c_2 c_{10} - \frac{(8g_r)^6}{189} c_4 c_8 + \frac{2(8g_r)^6}{567} c_6^2 . \quad (10)$$



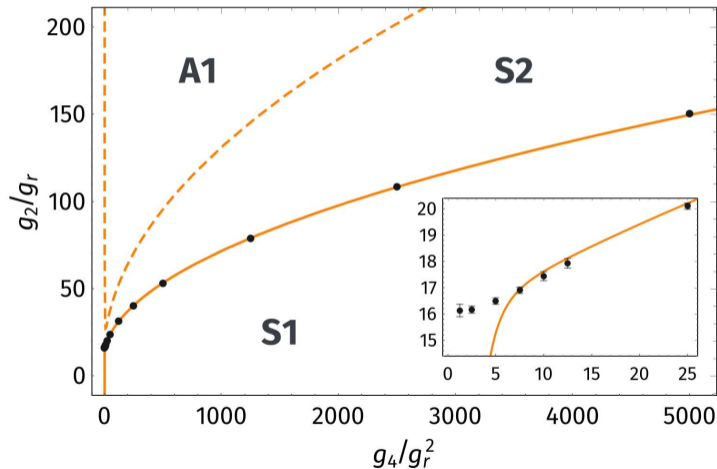
REMOVAL OF STRIPES – GW MODEL

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22]



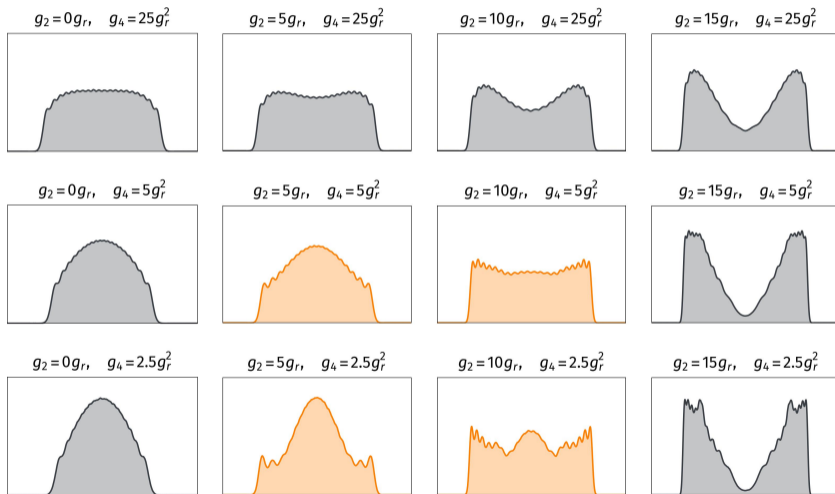
REMOVAL OF STRIPES – GW MODEL

[Bukor, Prekrat, JT '25]



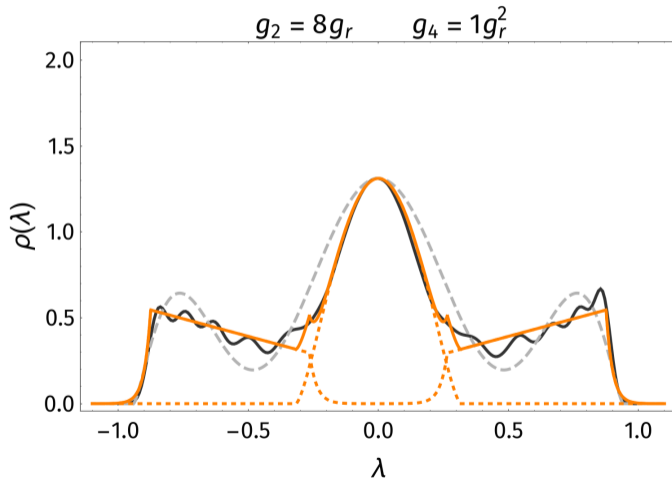
NEW PHASE IN GW MODEL?

[Bukor, Prekrat, JT '25]



NEW PHASE IN GW MODEL?

[Bukor, Prekrat, JT '25]



MODEL WITH BOTH THE KINETIC AND CURVATURE TERMS

$$S(M) = \text{Tr}(MKM) - g_r \text{Tr}(RM^2) - g_2 \text{Tr}(M^2) + g_4 \text{Tr}(M^4) .$$

[Bukor, Kubiš, Prekrat, JT work in progress]

$$S_{\text{GW}}(\Phi) = N^2 \left[\varepsilon (c_2 - c_1^2) - \frac{1}{120} \varepsilon^2 (565c_1^4 - 1130c_1^2c_2 + 388c_1c_3 + 274c_2^2 - 97c_4) \right. \\ \left. - 8g_r c_2 + \frac{32}{3} g_r^2 (c_2^2 - c_4) + \frac{8}{3} \varepsilon g_r (c_4 - c_2^2 + 2c_1^2c_2 - 2c_1c_3) + g_2 c_2 + g_4 c_4 \right] \quad (11)$$



- Correlation functions and their behavior at the phase transition.
- Entanglement entropy of field theories.
- Spaces more complicated than the fuzzy sphere.
- $U(1)$ gauge theory on NC plane with GW like mechanism.





Dirac ensembles and random fuzzy geometries



- Noncommutative geometry can be described by a spectral triple [Connes '94]

$$(\mathcal{A}, \mathcal{D}, \mathcal{H}) .$$

- For certain finite geometries the Dirac operator can be constructed using (anti)commutators with p hermitian and q antihermitian matrices (and some Clifford module baggage) to form a (p, q) geometry [Barrett '15].
- Path integral over geometries given by weight

$$\int d\mathcal{D} e^{-S(\mathcal{D})}$$

and becomes (multi)matrix integral. The simplest nontrivial choice is $S(\mathcal{D}) = \text{Tr} (g \mathcal{D}^2 + \mathcal{D}^4)$.
[Barrett, Glaser '16; Khalkhali '20s; D'Arcangelo '22; Glaser '23]

- Toy model of fluctuating dynamical geometry.



[Khalkhali, Pagliaroli '21; Bukor, Kováčik, Magdolenová, Pagliaroli, JT work in progress]

- In the simplest (0, 1) case the Dirac operator is given by

$$\mathcal{D} \cdot = [M, \cdot] .$$

- Then simply $d\mathcal{D} = dM$. The action for M is given by

$$S(M) = N(2g c_2 + 2c_4) - 2g c_1^2 - 8c_1 c_3 + 6c_2^2 . \quad (12)$$

- Nothing interesting happens, no asymmetric regime.



(1, 0) DIRAC ENSEMBLE

[Khalkhali, Pagliaroli '21; Bukor, Kováčik, Magdolenová, Pagliaroli, JT work in progress]

- In the simplest (1, 0) case the Dirac operator is given by

$$\mathcal{D} \cdot = \{M, \cdot\} .$$

- Then simply $d\mathcal{D} = dM$. The action for M is given by

$$S(M) = N(2g c_2 + 2c_4) + 2g c_1^2 + 8c_1 c_3 + 6c_2^2 . \quad (13)$$

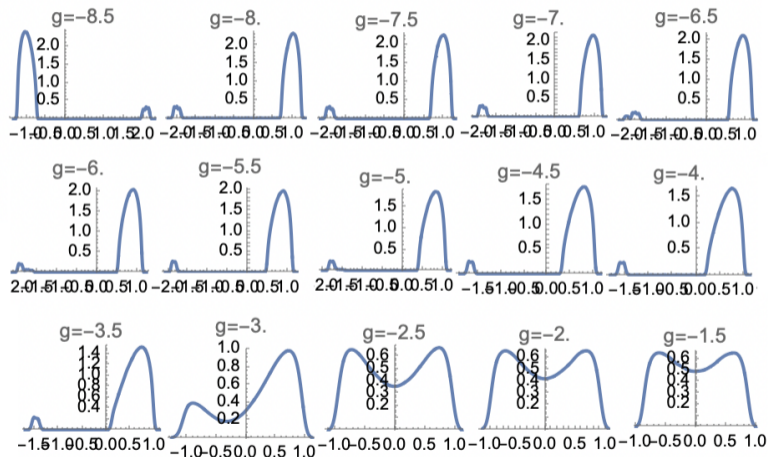
Simpler $c_1 c_3$ model analyzed before [Bukor, JT '25].

- Can be analyzed ...



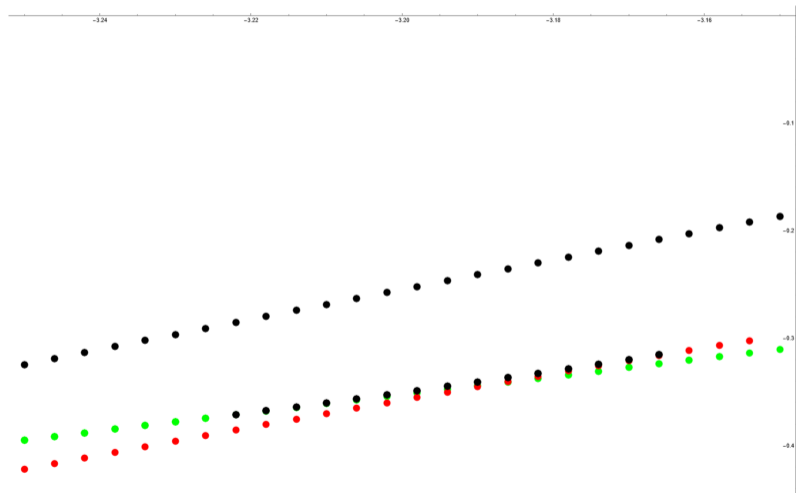
(1, 0) DIRAC ENSEMBLE

... numerically ...



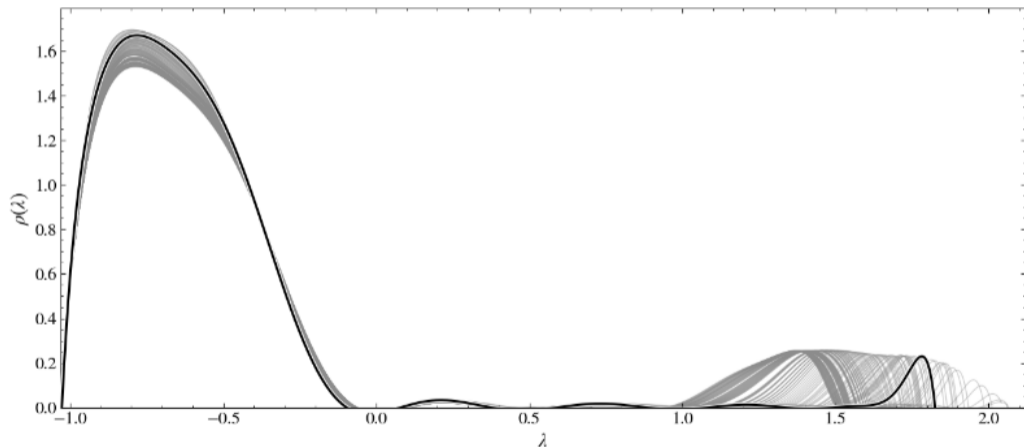
$(1, 0)$ DIRAC ENSEMBLE

... analytically ...



$(1, 0)$ DIRAC ENSEMBLE

... and using bootstrap.



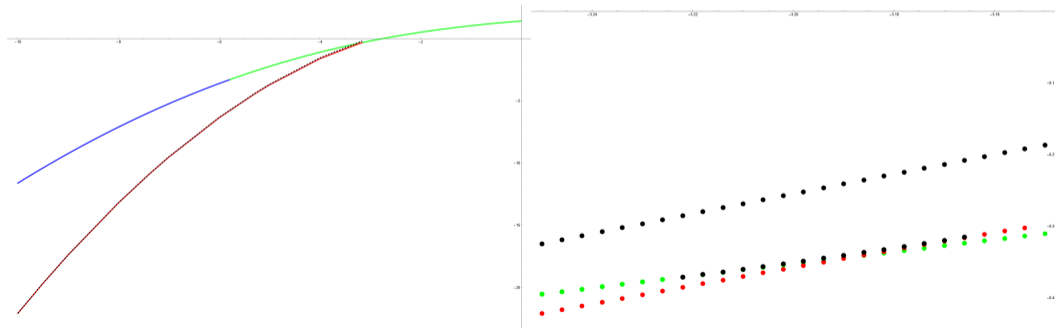
(1, 0) DIRAC ENSEMBLE

- The ensemble

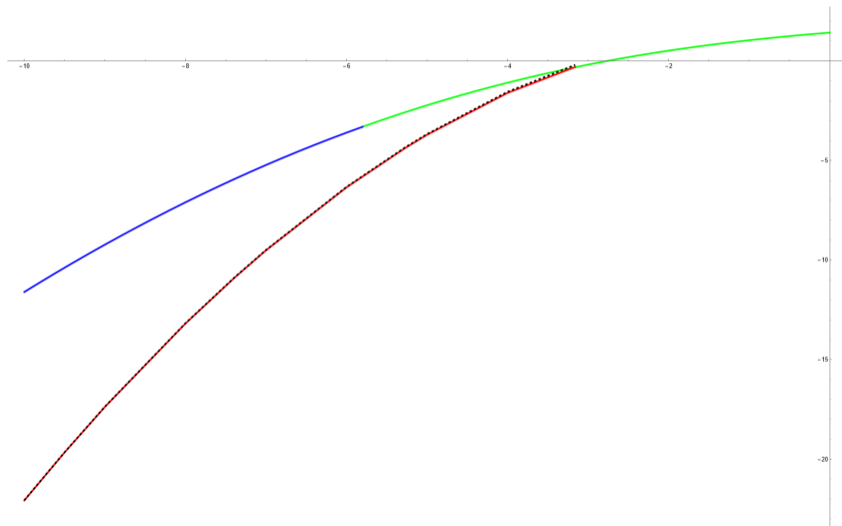
$$S(M) = N(2gc_2 + 2c_4) + 2gc_1^2 + 8c_1c_3 + 6c_2^2$$

has a stable asymmetric 2-cut regime for $g < -3.18703$.

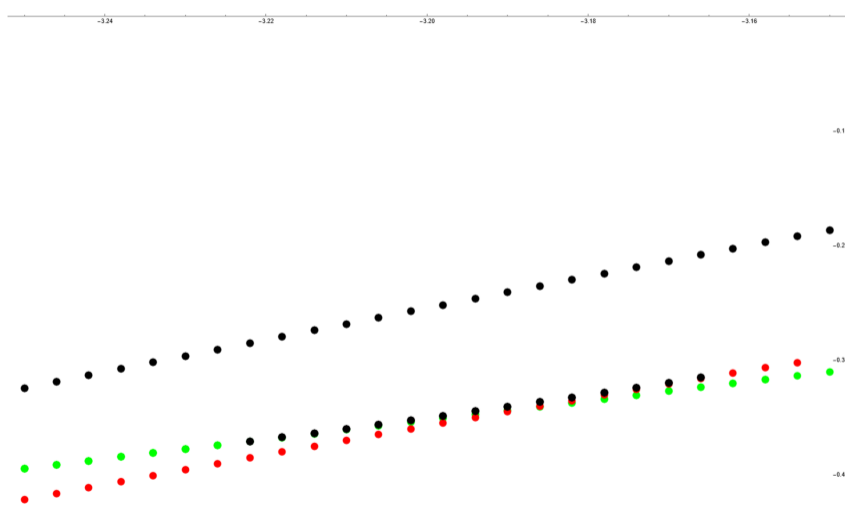
- Similar results obtained numerically before [D'Arcangelo '22] and analytically recently [D'Arcangelo, Gnutzmann '26].



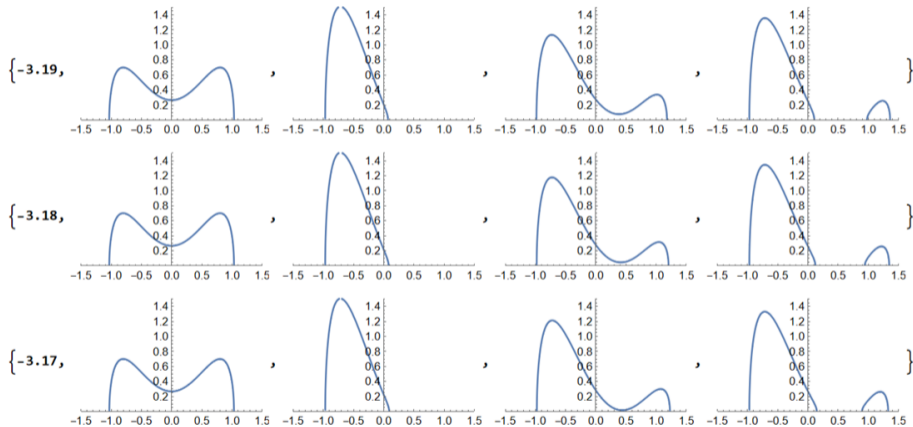
$(1, 0)$ DIRAC ENSEMBLE



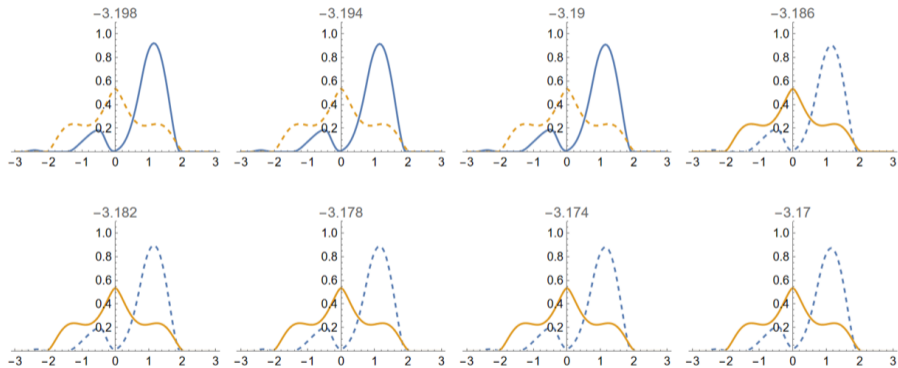
$(1, 0)$ DIRAC ENSEMBLE



(1, 0) DIRAC ENSEMBLE



$(1, 0)$ DIRAC ENSEMBLE



$(1, 0)$ DIRAC ENSEMBLE

- What does this mean for the geometry?



$(1, 0)$ DIRAC ENSEMBLE

- What does this mean for the geometry?
- Not sure. Nobody knows.



(1, 0) DIRAC ENSEMBLE

- What does this mean for the geometry?
- Not sure. Nobody knows.
- Points are pure states of the algebra

$$\phi : \mathcal{C} \rightarrow \mathbb{C} , \|\phi\| = 1 .$$

- Distance determined by \mathcal{D} in terms of

$$d(\phi_1, \phi_2) = \sup_{a \in \mathcal{A}} \{ |\phi_1(a) - \phi_2(a)| , \|[D, a]\| \leq 1 \} .$$

- Other geometrical properties given by

$$\mathrm{Tr} \left(e^{-t\mathcal{D}^2} \right) = t^{-d/2} \sum_{k=1}^{\infty} a_k t^k .$$

- Some preliminary results: a one dimensional spectral geometry with no metric structure.
[Rusnák, JT work in progress]



(1, 3) DIRAC ENSEMBLE IS FUZZY SPHERE

- More complicated spaces are described by multi matrix models. Symetric regime has been analyzed before, but no results for asymmetric regime.



$$(1, 3) \tag{14}$$

geometry is the fuzzy sphere!

- A four matrix model, not much hope for analytical results. But bootstrap might be useful.



- A general one-matrix model

$$D = a[H, \cdot] + b\{H, \cdot\} , \quad S(D) = g\text{Tr} (D^2) + \text{Tr} (D^4) . \quad (15)$$

- Multitrace generalization

$$S(D) = g\text{Tr} (D^2) + \text{Tr} (D^4) + h\text{Tr} (D^2)^2 . \quad (16)$$

- Field theory on a random geometry

$$S(D, \phi) = \text{Tr} (g D^2 + D^4) + \text{Tr} \left(\frac{1}{2} \phi D D \phi + \frac{1}{2} g_2 \phi^2 + g_4 \phi^4 \right) . \quad (17)$$



Take home message



TAKE HOME MESSAGE AND 2DO LIST

- Fuzzy spaces are examples of spacetimes with quantum structure.
- Plenty of interesting things happen on such spaces.
- Physics is described by a wide array of random matrix ensembles. Analyzing these is technically challenging, but doable.
- We can analyze toy models of path integrals over geometries.



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- Beyond fuzzy sphere.
 - Correlation functions, entanglement entropy.
 - Dirac ensembles and random fuzzy geometries.
 - $U(1)$ gauge theory on NC plane.
 - More on kinetic term effective action.



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Come to Bratislava and thank you for your attention!



Fuzzy field theories ensembles V

Beyond phase structure



CORRELATION FUNCTIONS

- Analogues of points in the NC setting are coherent states $|\vec{x}\rangle$.
- "Value" of field ϕ at "point" \vec{x} given by

$$\langle \vec{x} | \phi | \vec{x} \rangle = \phi(x) .$$

- Behaviour of

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{Z} \int d\phi \langle \vec{x} | \phi | \vec{x} \rangle \langle \vec{y} | \phi | \vec{y} \rangle e^{-S(\phi)} \quad (18)$$

in the matrix model can be studied numerically.

[Hatakeyama, Tsuchiya '17; Hatakeyama, Tsuchiya, Yamashiro '18 '18]

- At the "standard" phase transition, the behaviour of the correlation functions at short distances differs from the commutative theory and seems to agree with the tricritical Ising model. A different behaviour at long distances.
- Quantity $\phi(x)\phi(y)$ is U dependent, so we need to figure out what to do with

$$\int dU F(\Lambda, U) e^{-\frac{1}{2} \text{Tr}(U \Lambda U^\dagger [L_i, [L_i, U \Lambda U^\dagger]])} .$$



- In local theories $S(A) \sim A$.
[Ryu, Takayanagi '06]
- In non-local theories this can change.
[Barbon, Fuertes '08; Karczmarek, Rabideau '13; Shiba, Takayanagi '14]
- Problem on the fuzzy sphere has been studied numerically.
[Karczmarek, Sabella-Garnier '13; Sabella-Garnier '14; Okuno, Suzuki, Tsuchiya '15; Suzuki, Tsuchiya '16; Sabella-Garnier '17; Chen, Karczmarek '17]
- For free fields, the EE follows volume law rather than area law.
In the interacting case much smaller EE than in the free case.



Fuzzy field theories ensembles VI

Other spaces



- To study entanglement entropy, we need to extend the model to $\mathbb{R} \times S_F^2$, i.e. $M(t)$

$$S(M) = \int dt \text{Tr} \left(-\frac{1}{2} M \partial_t^2 M + \frac{1}{2} M [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + g M^4 \right) \quad (19)$$

[Medina, Bietenholz, O'Connor '07; Ihl, Sachse, Sämann '10]

- This is matrix quantum mechanics, different but similar methods apply. [Jevicki, Sakita '80]
- For EE free theory where $\mathcal{R} = 0$, is enough.



- The field theory on other spaces differs in the definition of the kinetic term.

$$S(M) = \text{Tr} \left(\frac{1}{2} MKM + \frac{1}{2} m^2 M^2 + gM^4 \right) .$$

Second moment approximation applicable.

- Numerical results available for fuzzy disc [Lizzi, Spisso '12] and torus [Mejía-Díaz, Bietenholz, Panero '14].
- Perturbative models have been derived for $\mathbb{C}P^2, \mathbb{C}P^3$ [Sämann '10], disc [Rea, Sämann '15].
- Analytic second moment approximation available for general $\mathbb{C}P^n$ [JT '14].



$c_1 c_3$ model



- Simple cubic mutitrace model

$$S(M) = N \left(\frac{1}{2} r c_2 + g c_4 \right) + t c_1 c_3 \quad (20)$$

with effective potential

$$V_{\text{eff}}(M) = \frac{1}{2} r M^2 + g M^4 + t c_1 M^3 + t c_3 M .$$

- Analyzed numerically in the context of emerging NC geometry [Ydri, Ahlam, Khaled '16; Ydri, Khaled, Soudani '21; Khaled '22].
- Interesting things happen for $t < 0$ and we set $t = -1$.



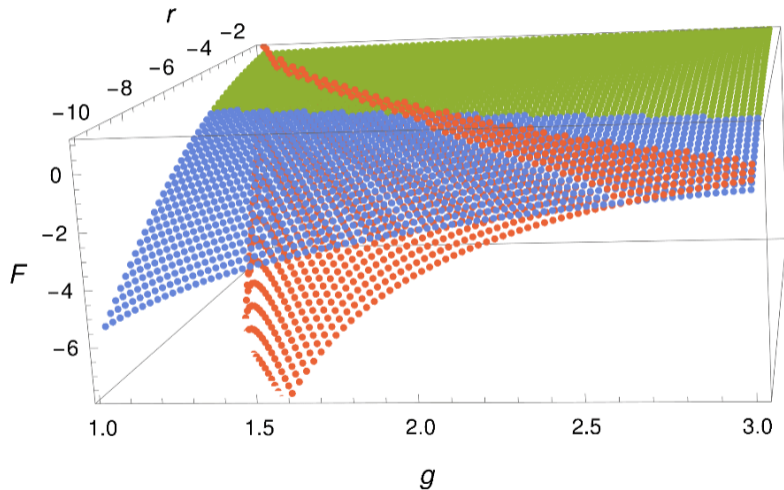
[Bukor, JT '25]

$$\begin{aligned}0 &= 2D^3g + 3D\delta g + \frac{1}{2}Dr + \left(\frac{1}{2}c_3 + \frac{3}{2}c_1D^2 + \frac{3}{4}c_1\delta\right)t, \\1 &= 3D^2\delta g + \frac{3}{4}\delta^2g + \frac{1}{4}\delta r + \frac{3}{2}c_1D\delta t, \\c_1 &= 3D^3\delta g + \frac{3}{2}D\delta^2g + \frac{1}{4}D\delta r + \left(\frac{3}{2}c_1D^2\delta + \frac{3}{16}c_1\delta^2\right)t, \\c_3 &= 3D^5\delta g + \frac{21}{4}D^3\delta^2g + \frac{9}{8}D\delta^3g + \frac{1}{4}D^3\delta r + \frac{3}{16}D\delta^2r + \\&\quad + \left(\frac{3}{2}c_1D^4\delta + \frac{27}{16}c_1D^2\delta^2 + \frac{3}{32}c_1\delta^3\right)t.\end{aligned}$$



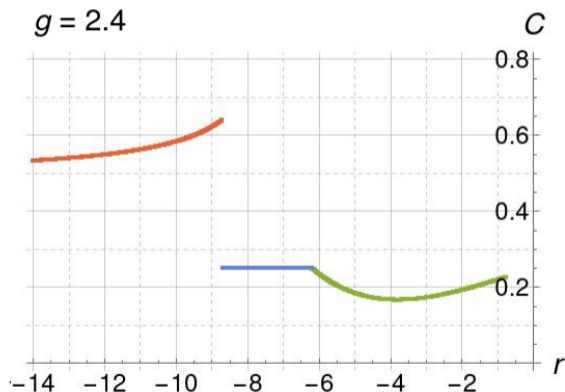
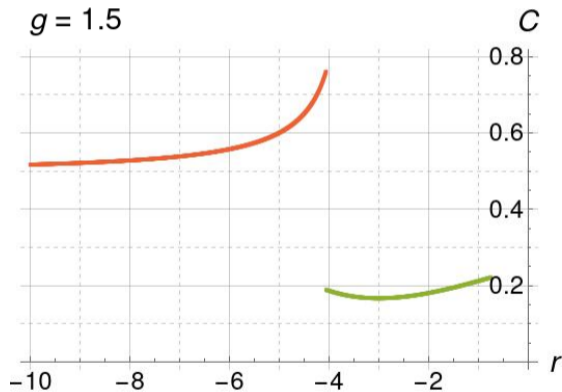
$C_1 C_3$ MODEL - PHASE STRUCTURE

[Bukor, JT '25]



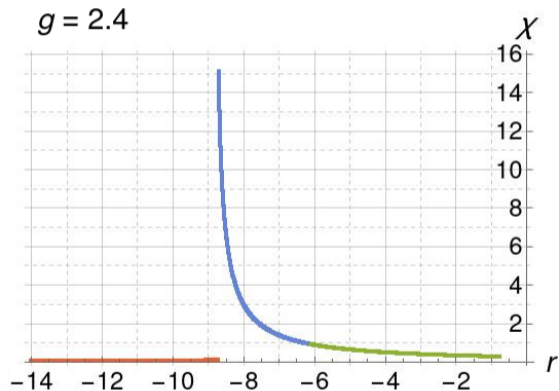
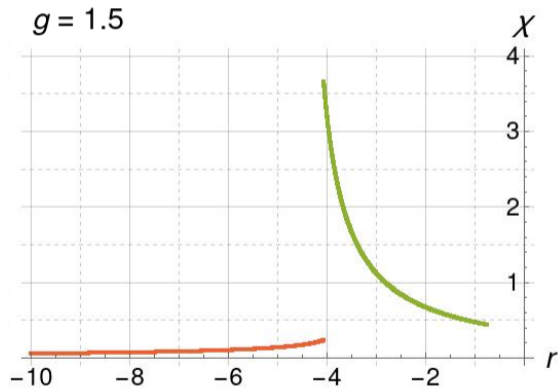
$C_1 C_3$ MODEL - RESPONSE FUNCTIONS

[Bukor, JT '25]



$C_1 C_3$ - RESPONSE FUNCTIONS

[Bukor, JT '25]



One final ensemble - gauge theory on NC plane



[Buric, Grosse, Madore '10]

- Three matrix model A_1, A_2, ϕ and

$$S_{\text{YM}} = \frac{1}{2} \text{tr} \left(\frac{1 - \epsilon^2}{g^2} F_{12}^2 + (D\phi)^2 + (5 - \epsilon^2)\mu^2\phi^2 - 2(1 - \epsilon^2)\frac{\mu}{g} F_{12}\phi - 4\epsilon F_{12}\phi^2 + \epsilon^2\{P + gA, \phi\}^2 \right). \quad (21)$$

where

$$D_\alpha\phi = i[P_\alpha + gA_\alpha, \phi],$$

$$F_{12} = ig[P_1, A_2] - ig[P_2, A_1] + ig^2[A_1, A_2],$$

$$[X, Y] = i\epsilon(1 - Z), \epsilon P_1 = Y, \epsilon P_2 = -X$$



- A rather complicated three matrix model.
- Standard analysis of this model suggests that it is not renormalizable even with the GW trick [Buric, Nenadovic, Prekrat '16].
- Can we see that in the phase structure of the corresponding matrix model – is there a striped phase? [work in progress]

