METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto19 – Príklady 4

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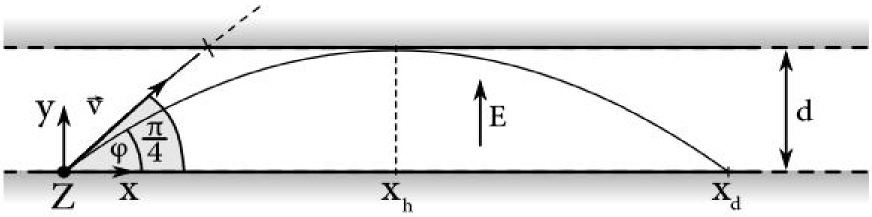
Príklad 1

22 Keďže uvažované elektrické pole je homogénne, na všetky vylietavajúce elektróny bude smerom nadol pôsobiť konštantná sila veľkosti Ee. Tá bude udeľovať elektrónom zrýchlenie $a = \frac{Ee}{m_e}$. Môžme si tiež uvedomiť, že gravitačná sila pôsobiaca na elektróny je oproti elektrickej sile zanedbateľná. Celá sústava je rotačne súmerná, takže vo výpočte sa stačí obmedziť na jeden rez obsahujúci zdroj. Analogicky k šikmému vrhu v homogénnom gravitačnom poli môžeme popísať súradnice elektrónu uniknuvšieho zo zdroja pod uhlom φ

$$x = v_x t = v\cos\varphi \cdot t,$$

$$y = v_y t - \frac{1}{2}at^2 = v\sin\varphi \cdot t - \frac{1}{2}at^2.$$



V závislosti na počiatočnej rýchlosti a elevačnom uhle elektrónu môžu vo všeobecnosti nastať tri prípady:

- elektrón narazí do hornej dosky kondenzátora;
- elektrón sa obtrie o hornú dosku kondenzátora;
- elektrón nedosiahne dostatočnú výšku na to, aby sa dotkol hornej dosky kondenzátora.

Pozrime sa na elektrón, ktorý sa práve obtrie o hornú dosku. Letiaci elektrón dosiahne svoju maximálnu výšku po čase

$$T = \frac{v_y}{a} = \frac{v\sin\varphi}{a}.$$

Ak tento čas dosadíme do vzorca pre y-súradnicu a položíme ju rovnú d, zistíme pod akým uhlom musí byť vypustený elektrón, aby svoju maximálnu výšku dosiahol práve vo výške dosky, čo teda znamená, že sa o ňu obtrie.

$$d = \frac{v^2\sin^2\varphi}{a} - \frac{v^2\sin^2\varphi}{2a} = \frac{v^2\sin^2\varphi}{2a},$$

$$\varphi = \arcsin\frac{\sqrt{2ad}}{v}.$$

Ak do tohoto vzťahu dosadíme zadané hodnoty, vyjde uhol $\varphi \doteq 36{,}37°$. Tento uhol je menší ako 45°, čo znamená, že všetky elektróny, ktoré by mohli doletieť ďalej, narazia do hornej dosky. Ak dosadíme tento uhol spolu s časom T do rovnice pre x-súradnicu, zistíme v akej vzdialenosti od stredu sa budú elektróny obtierať o hornú dosku

$$x_h = \frac{v^2\cos\varphi\sin\varphi}{a}.$$

Vieme teda, že vrchol trajektórie takéhoto elektrónu je vo vodorovnej vzdialenosti x_h od zdroja. Ak si uvedomíme, že elektrón dosiahne vrchol trajektórie presne v polovici prejdenej vodorovnej vzdialenosti, zistíme, že maximálny dolet na spodnej doske je $x_d = 2x_h$.

Po dosadení hodnôt zo zadania dostávame $x_h = 2{,}715$ m. Ak vezmeme do úvahy rotačnú súmernosť úlohy, zistíme, že elektróny budú na hornú dosku dopadať do kruhu s polomerom 2,715 m a na dolnú dosku do kruhu s polomerom 5,43 m. Pre celkovú plochu dopadu teda dostávame výsledok

$$S = \pi\left(x_h^2 + x_d^2\right) = 5\pi x_h^2 \doteq 116\ \mathrm{m}^2.$$

Príklad 2

Let y be the distance above the horizontal. The potential energy of the cylinder is just mgy and of the mass point is $mgy + mga\cos\theta/2$ and $y = a\theta\sin\alpha$, so the total potential energy is

$$V(\theta) = mg\left(2a\theta\sin\alpha + a\cos\theta/2\right).$$

The cylinder will roll freely when there are no stable points:

$$\begin{aligned} \frac{dV}{d\theta} &= mg\left(2\sin\alpha - \frac{1}{2}\sin\theta\right) = 0 \rightarrow \\ 4\sin\alpha &= \sin\theta \end{aligned}$$

so when $\sin\theta > 1/4$, the cylinder begins to roll.

Alternatively, one may take torques around the contact point.

Príklad 3

Solution 1.10. In plane-polar coordinates, the Lagrangian for a particle moving in a central potential $V(r)$ is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{10.89}$$

where m is the mass of the particle. The potential is given in the question as

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2. \tag{10.90}$$

The θ-component of Lagrange's equation is

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \equiv l. \tag{10.91}$$

The hamiltonian of our system is then

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{eff}(r), \tag{10.92}$$

with $p_r = m\dot{r}$ and

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r). \tag{10.93}$$

The term $l^2/2mr^2$ is referred to as an "angular momentum barrier." Solving the equations of motion for this hamiltonian is equivalent to solving Lagrange's equations for the Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 - V_{eff}(r). \tag{10.94}$$

This is a completely general result for the motion of a particle in a central potential and could easily have been our starting point in this problem (e.g., Goldstein, Chapter 3).

It may seem unnecessarily long-winded to go through this procedure, but note that the sign of the angular momentum barrier in (10.94) is *opposite* to what we would have gotten if we had naively replaced $\dot{\theta}$ with l/mr^2 in the Lagrangian (10.89). This is due to the fact that the Lagrangian is a function of the time derivative of the position, and not of the canonical momentum.

The equation of motion from (10.94) is

$$m\ddot{r} = -\frac{d}{dr}V_{eff}(r). \tag{10.95}$$

If the particle is in a circular orbit at $r = r_0$ we require that the force on it at that radius should vanish,

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} = 0. \tag{10.96}$$

Using our expression for V_{eff} (10.93), we derive an expression relating the angular momentum l to the radius of the orbit r_0:

$$\frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - br_0 = 0. \tag{10.97}$$

We are interested in perturbations about this circular orbit. Provided the perturbation remains small, we can expand $V_{eff}(r)$ about r_0,

$$V_{eff}(r) = V_{eff}(r_0) + (r - r_0)V'_{eff}(r_0) + \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0) + \cdots. \tag{10.98}$$

If we use this expansion in the Lagrangian (10.94) together with the condition (10.96), we find

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0), \tag{10.99}$$

where we have dropped a constant term. This is just the Lagrangian for a simple harmonic oscillator, describing a particle undergoing radial oscillations with frequency

$$\omega^2 = \frac{1}{m}V''_{eff}(r_0). \tag{10.100}$$

Differentiating $V_{eff}(r)$ twice gives us

$$\frac{3l^2}{mr_0^4} - \frac{2k}{r_0^3} + b = m\omega^2. \tag{10.101}$$

We can eliminate l between equations (10.101) and (10.97) to give the frequency of radial oscillations:

$$\omega = \left(\frac{k}{mr_0^3} + \frac{4b}{m}\right)^{1/2}. \tag{10.102}$$

To find the rate of precession of the perihelion, we need to know the period of the orbit. From the definition of angular momentum l equation (10.91), we have an equation for the orbital angular velocity ω_1,

$$\omega_1 \equiv \frac{d\theta}{dt} = \frac{l}{mr^2}. \tag{10.103}$$

Let us write $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is sinusoidal with frequency ω and average value zero. We substitute $r(t)$ into equation (10.103) and expand in $\epsilon(t)$:

$$\frac{d\theta}{dt} = \frac{l}{mr_0^2}\left(1 - \frac{2\epsilon}{r_0} + \mathcal{O}(\epsilon^2)\right). \tag{10.104}$$

To zeroth order in the small quantities br_0^3/k and ϵ/r_0, the period of the orbit T_1 is the same as the period of oscillations $T_2 = 2\pi/\omega$. Therefore we can average ϵ over T_1 rather than T_2 and still get zero, to within terms of second order, which we are neglecting. The average angular velocity is therefore

$$\bar{\omega}_1 = \frac{2\pi}{T_1} \approx \frac{l}{mr_0^2} = \sqrt{\frac{k}{mr_0^3} + \frac{b}{m}}, \tag{10.105}$$

where we have made use of (10.97).

Now consider one complete period of the radial oscillation. This takes place in time $T_2 = 2\pi/\omega$. In this time the particle travels along its orbit through an angle of

$$\theta = 2\pi\frac{\bar{\omega}_1}{\omega} = 2\pi\frac{\sqrt{k/mr_0^3 + b/m}}{\sqrt{k/mr_0^3 + 4b/m}}$$

$$\approx 2\pi\left(1 - \frac{3br_0^3}{2k}\right). \tag{10.106}$$

In other words, the particle does not quite orbit through 2π before the radial oscillation is completed. Each time around the perihelion precesses backwards through an angle

$$\delta\theta = 3\pi\frac{br_0^3}{k}, \tag{10.107}$$

and it gets around in time T_2, so the precession rate is

$$\alpha = \frac{\delta\theta}{T_2} = \frac{3\pi br_0^3}{k}\frac{\sqrt{k/mr_0^3 + 4b/m}}{2\pi}$$

$$\approx \frac{3b}{2}\sqrt{\frac{r_0^3}{mk}}. \tag{10.108}$$

Príklad 4

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \& \qquad \frac{\partial \rho}{\partial t} = 0$$

$$\hookrightarrow \quad \vec{\nabla}\cdot\vec{J} = 0 \quad \hookrightarrow \quad \vec{J} = J\,\hat{x}$$

↑ constant!

$$\vec{J} = \sigma \vec{E} \quad \hookrightarrow \quad \vec{E} = \frac{Jx}{a}\,\hat{x}$$

$$V_0 = -\int_0^L \vec{E}\cdot dx = -\frac{JL^2}{2a} \longrightarrow J = -\frac{2aV_0}{L^2}$$

so

$$\boxed{\vec{J} = -\frac{2aV_0}{L^2}\,\hat{x}} \qquad \boxed{\vec{E} = -\frac{2x}{L^2}V_0\,\hat{x}}$$

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} \Rightarrow \boxed{\rho = -\frac{2\varepsilon_0 V_0}{L^2}}$$

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Príklad 5

(a) The electric potential V satisfies the Laplace equation, $\nabla^2 V = 0$. Given the boundary conditions

$$V(x, y=0) = 0 = V(x, y=a), \qquad \text{and} \qquad V(x=0, y) = V_0,$$

the solution is of the form

$$V(x,y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{ikx}.$$

Inserting this solution into the Laplace equation, we have

$$-\left(\frac{\pi}{a}\right)^2 - k^2 = 0,$$

or $k = \pm i\pi/a$. Thus, the solution (for $x \geq 0$) is

$$V(x,y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{-\pi x/a}.$$

(We can ignore $x \geq L/2$ since $e^{-piL/2a} \ll 1$ for $L/a \gg 1$.)

(b) To find the charge density σ at the surface of the conductors, we need the electric field $\vec{E}$ at the surface. The latter can be obtained from the potential $V(x, y)$ as

$$\vec{E} = -\vec{\nabla} V = \frac{\pi V_0}{a} \left[\sin\left(\frac{\pi y}{a}\right) \hat{\boldsymbol{x}} - \cos\left(\frac{\pi y}{a}\right) \hat{\boldsymbol{y}} \right] e^{-\pi x/a}.$$

At the surfaces of the conducting plates at $y = 0$ and $y = a$, the induced charge densities are the same, with

$$\sigma(x, y = 0) = \sigma(x, y = a) = \epsilon_0 \vec{E} \cdot \hat{\boldsymbol{n}} = -\frac{\epsilon_0 \pi V_0}{a} e^{-\pi x/a}, \qquad x \geq 0$$

for both plates.

(c) Force exerted on a conductor is given by

$$\vec{F} = \int \sigma \, \vec{E}_{\text{ext}} \, dA,$$

integrated over the surface area of the conductor, with $E_{\text{ext}} = E_{\text{self}} = E/2$.

On the upper plate (and $x \geq 0$),

$$\begin{aligned} \vec{F} &= L \int_0^{L/2 \to \infty} dx \, \sigma(x, y = a) \cdot \frac{1}{2} \vec{E}(x, y = a) \\ &= -\frac{\epsilon_0 \pi^2 V_0^2 L}{2a^2} \left[\int_0^\infty e^{-2\pi x/a} dx \right] \hat{\boldsymbol{y}} \\ &= -\frac{\pi}{4} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}} \end{aligned}$$

Including also the part from $x \leq 0$, the total force exerted on the top plate is

$$\vec{F}_{\text{upper}} = -\frac{\pi}{2} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}},$$

i.e., the top plate is attracted towards the lower plate.

By symmetry, the lower plate is attracted towards the upper plate with force of the same magnitude, i.e.,

$$\vec{F}_{\text{lower}} = +\frac{\pi}{2} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}}.$$