METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto19 – Príklady 5

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Príklad 1

5. Nech veľkosť rýchlosti guličky pri dopade na plôšku je v a hľadaná výška je h. Podľa zákona zachovania energie pre guličku s hmotnosťou m dostaneme $mg(H-h) = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2g(H-h)}$. Odraz je pružný, preto uhol dopadu je rovný uhlu odrazu, po ktorom bude mať rýchlosť vodorovný smer. Pôjde teda o pohyb zložený z voľného pádu z výšky h a rovnomerného pohybu s rýchlosťou v vo vodorovnom smere. Tento pohyb teda bude trvať čas $t = \sqrt{\frac{2h}{g}}$ (lebo $h = \frac{1}{2}gt^2$). Vo vodorovnom smere zatiaľ preletí gulička vzdialenosť $vt = v\sqrt{\frac{2h}{g}} = \sqrt{4h(H-h)}$, čo je však podľa zadania rovné $\frac{H}{2}$. Platí teda rovnosť

$$\frac{H}{2} = \sqrt{4h(H-h)}.$$

Úpravou dostaneme kvadratickú rovnicu s koreňmi $\frac{16\pm\sqrt{256-64}}{32}H$. Podmienku $h < H$ spĺňa riešenie $\frac{2-\sqrt{3}}{4}H \approx 0,067\ H$.

Príklad 2

We need to find force acting on liquid in one quarter of the pipe. For a small segment of the pipe,

$$dF = \frac{dm\, v^2}{R} = \rho S d\phi\, v^2$$

and it is pointing towards the center, providing centripetal acceleration to the liquid. Projection of the total force on the horizontal axis is $F_x = \int_0^{\pi/2} dF \cos\phi$, and the answer is $T = F_x$,

$$\boxed{T = \rho S v^2}$$

This we can get from more sophisticated hydrodynamics as well, by considering momentum flux equation

$$\frac{\partial}{\partial t}\rho v_i = -\nabla_j \Pi_{ij} + f_i$$

where $\Pi_{ij} = \delta_{ij}p + \rho v_i v_j$ is momentum density tensor, and f_i is external force density. Considering left quarter of the pipe, for steady flow there is no momentum change in this volume, and thus we must have for the external force acting on the volume

$$F_i = \int dV f_i = \int dV\, \nabla_j \Pi_{ij} = \oint dS_j \Pi_{ij} = \oint \rho v_i(\mathbf{v}d\mathbf{S})$$

where we used Gauss theorem and the fact that the pressure is uniform. The surface integral is not zero only through opening parts of the pipe, since on the sides $\mathbf{v} \perp d\mathbf{S}$. For x-component of the force we integrate only over the vertical cross-section at the top of the semi-circle, to immediately get

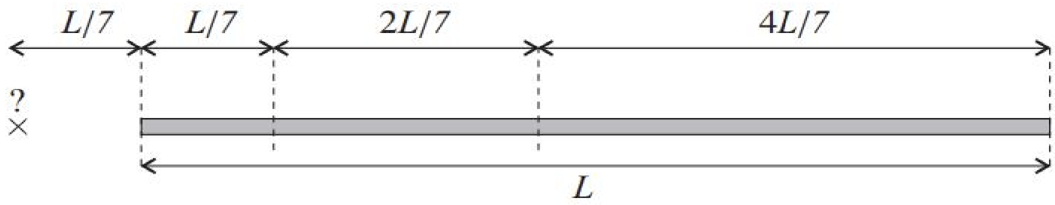
$$T = F_x = \rho v(vS) = \rho S v^2$$

Príklad 3

32. Označme si dĺžkovú hustotu náboja na našej tyči ako λ a situáciu si trochu zovšeobecnime. Predstavme si, že máme homogénne nabitú tyč s rovnakou hustotou náboja λ a dĺžkou αL. Intenzitu elektrického poľa vo vzdialenosti αL od jedného jej konca označme ako E_α. Zadanie nám hovorí, že $E_1 = E$.

Situácia s E_α je však len α-krát zväčšená (teda vlastne zmenšená, ak $\alpha < 1$) situácia s E_1. Ak by sme zobrali tyč dĺžky L vo vzdialenosti L od nás, nasekali ju na kúsky a každý jej kúsok α-krát vzdialili a α-krát zväčšili, dostali by sme tyč dĺžky αL vo vzdialenosti αL od nás. Každý spomínaný kúsok by pritom zväčšil svoj náboj α-krát, máme totiž vždy rovnakú hustotu náboja. Príspevok elektrickej intenzity od kúsku s nábojom q vo vzdialenosti r by sa teda zmenil z $k\frac{q}{r^2}$ na $k\frac{\alpha q}{\alpha^2 r^2} = \frac{1}{\alpha} k \frac{q}{r^2}$. Inými slovami, elektrická intenzita každého kúsku sa zväčší $\frac{1}{\alpha}$-krát, a preto $E_\alpha = \frac{1}{\alpha} E_1 = \frac{1}{\alpha} E$.

Teraz si už len stačí všimnúť, že ak zoberieme tri tyče s dĺžkami $\frac{1}{7}L$, $\frac{2}{7}L$ a $\frac{4}{7}L$ a umiestnime ich do vzdialeností $\frac{1}{7}L$, $\frac{2}{7}L$ a $\frac{4}{7}L$ od nás, spoja sa do jednej tyče dĺžky L, ktorá bude od nás vzdialená $\frac{1}{7}L$.



Inými slovami, dostaneme presne tú situáciu, na ktorú sa pýta zadanie. Intenzita elektrického poľa teda bude

$$E_{1/7} + E_{2/7} + E_{4/7} = 7E + \frac{7}{2}E + \frac{7}{4}E = \frac{49}{4}E.$$

Príklad 4

SOLUTION: We conserve 4-momentum in the lab frame:

$$p_\pi^\mu + p_n^\mu = p_K^\mu + p_\Lambda^\mu \ ,$$

where $p = (p^0, p^1, p^2, p^3) = (E/c, \boldsymbol{p})$ is the 4−momentum. We use a $(-,+,+,+)$ metric, in which case the scalar product of two 4-vectors is $a \cdot b \equiv a_\mu \, b^\mu = -a^0 \, b^0 + \boldsymbol{a} \cdot \boldsymbol{p}$, and is (inertial) frame-independent. For a particle of mass m, then, $p \cdot p = m^2 c^2$ (evaluate in rest frame). Thus,

$$\begin{aligned} p_\Lambda \cdot p_\Lambda &= (p_\pi + p_n - p_K) \cdot (p_\pi + p_n - p_K) \\ &= -m_\pi^2 c^2 - m_n^2 c^2 - m_K^2 c^2 + 2\, p_\pi \cdot p_n - 2\, p_\pi \cdot p_K - 2\, p_n \cdot p_K \\ &= -m_\Lambda^2 c^2 \ . \end{aligned}$$

In the lab frame,

$$p_\pi = (E_\pi/c, \boldsymbol{p}_\pi) \quad , \quad p_n = (m_n c, 0) \quad , \quad p_K = (E_K/c, \boldsymbol{p}_K) \ ,$$

so $p_\pi \cdot p_n = -E_\pi m_n$, $p_\pi \cdot p_K = -E_\pi E_K/c^2 + \boldsymbol{p}_\pi \cdot \boldsymbol{p}_K$, and $p_n \cdot p_K = -E_K m_n$. Substituting these dot products into our earlier formula, we obtain

$$-m_\Lambda^2 c^2 = -m_\pi^2 c^2 - m_n^2 c^2 - m_K^2 c^2 - 2m_n E_\pi + 2m_n E_K + 2E_\pi E_K/c^2 - 2\boldsymbol{p}_\pi \cdot \boldsymbol{p}_K \ .$$

We're told the lab frame angle between pion and kaon is 90°, so we set $\boldsymbol{p}_\pi \cdot \boldsymbol{p}_K = 0$ and obtain

$$E_\pi = \frac{m_\Lambda^2 c^4 - m_\pi^2 c^4 - m_n^2 c^4 - m_K^2 c^4 + 2E_K m_n c^2}{2(m_n c^2 - E_K)}$$

The minimum of E_π is achieved when E_K takes the smallest possible value, which is $E_K = m_K c^2$. Finally, then,

$$E_\pi = \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n m_K}{2(m_n - m_K)}\, c^2 = 1149\,\text{MeV} \ ,$$

and so the threshold kinetic energy of the pion is

$$T_\pi = E_\pi - m_\pi c^2 = 1009\,\text{MeV} \ .$$

Príklad 5

`SOLUTION:` This problem is conveniently solved using the method of images.

(a) An equipotential $\phi = 0$ is achieved over the entire sphere by placing an image charge of strength $\tilde{Q} = -(a/b)\,Q$ a distance a^2/b from the center, also at $\theta = 0$. Even if we did not remember these values, they could easily be determined by supposing the image charge lies a distance d from the center, and then demanding that the potential vanish anywhere on the surface of the sphere:

$$\phi(R, \theta, \phi) = \frac{Q}{\sqrt{a^2 + b^2 - 2ab\,\cos\theta}} + \frac{\tilde{Q}^2}{\sqrt{a^2 + d^2 - 2ad\,\cos\theta}} = 0 \quad \forall\, \theta \ .$$

After pushing one radical over to the other side of the equation, inverting both sides, and squaring, one then separately equates the constant terms on both sides as well as the coefficients of $\cos\theta$. This yields two equations:

$$b\,\tilde{Q}^2 = d\,Q^2$$
$$(a^2 + b^2)\,\tilde{Q}^2 = (a^2 + d^2)\,Q^2 \ ,$$

which yield the familiar results $\tilde{Q} = -aQ/b$ and $d = a^2/b$. The potential everywhere is then

$$\phi(r,\theta,\varphi) = \frac{Q}{\sqrt{r^2+b^2-2br\,\cos\theta}} - \frac{Q}{\sqrt{\left(\frac{br}{a}\right)^2 + a^2 - 2br\cos\theta}} \; .$$

(b) It is tempting to compute the potential due to the image charge at Q,

$$\phi_{\rm image}(r)\big|_{\theta=0} = -\frac{Qa}{br-a^2} \qquad \Longrightarrow \qquad \phi_{\rm image}(b) = -\frac{Qa}{b^2-a^2} \; ,$$

multiply by Q, and conclude that $W = aQ^2/(b^2-a^2)$ is the work required. This is wrong! The reason is that *the image charge moves with* Q. To get the right answer, integrate $F\,dr$, where F is the radial component of the force, $\boldsymbol{F} = Q\boldsymbol{E}$. The electric field due to the image at Q is

$$E(r) = -\frac{\partial\phi_{\rm image}(r)}{\partial r}\bigg|_{b=r} = -\frac{Qar}{(r^2-a^2)^2} \; .$$

Next we multiply by Q and then integrate to get the work done *on* the charge:

$$W = -Q\int\limits_b^\infty dr\,E(r) = a\,Q^2\int\limits_b^\infty \frac{r\,dr}{(r^2-a^2)^2} = \frac{a\,Q^2}{2(b^2-a^2)} \; .$$

The wrong answer we obtained by the simplistic analysis is a factor of two too large.