METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto19 – Príklady 2

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Príklad 1

SOLUTION:

(a) From Maxwell, we have

$$\begin{aligned}\nabla \times \boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \\ &= \frac{4\pi}{c}(\boldsymbol{j} + \boldsymbol{j}_{\rm d}) \,,\end{aligned}$$

and since $\nabla \cdot (\nabla \times \boldsymbol{B}) = 0$, it follows that $\nabla \cdot \boldsymbol{J} = 0$.

(b) From $\nabla \cdot \boldsymbol{E} = 4\pi\rho$, we have $\boldsymbol{E} = q\,\hat{\boldsymbol{r}}/r^2$ for a spherical distribution of charges. Thus,

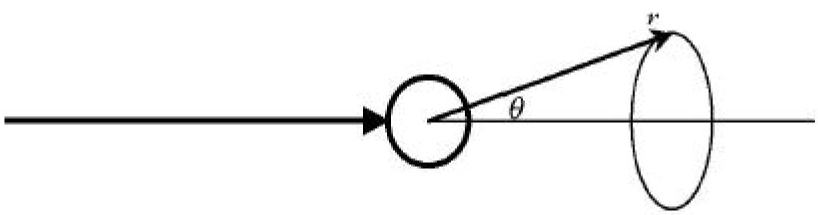
$$\boldsymbol{j}_{\rm d} = \frac{I}{4\pi r^2}\,\hat{\boldsymbol{r}} \,.$$

Note that $\nabla \cdot \boldsymbol{j}_{\rm d} = I\,\delta(\boldsymbol{r})$, which vanishes outside the sphere. Since $\nabla \cdot \boldsymbol{j} = 0$ outside the sphere as well, we have that $\nabla \cdot \boldsymbol{J} = 0$.

(c) From axial symmetry, we expect circular magnetic field lines. So use the integral form of Ampère's law,

$$\oint\limits_{\partial\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{4\pi}{c}\int\limits_{\Sigma} dA\;\hat{\boldsymbol{n}} \cdot \boldsymbol{J} \,,$$

where Σ is any two-dimensional surface, and $\hat{\boldsymbol{n}}$ is the local surface normal. Consider the $\boldsymbol{B}$ field along a circular loop a distance r from the center of the sphere, at an angle θ with respect to the wire's axis: Since there is no physical charge flowing through the loop, the total



current is just the displacement current from part (b). Let Σ be the

cap of a sphere of radius r, subtending a solid angle Ω. We therefore have

$$2\pi B r \sin\theta = \frac{4\pi}{c} \cdot \frac{\Omega}{4\pi} \cdot I = \frac{\Omega\, I}{c} \,,$$

where $r\sin\theta$ is the radius of the loop, and $\Omega = 2\pi(1-\cos\theta)$ is the solid angle subtended by the loop. We therefore have

$$B(r,\theta) = \frac{(1-\cos\theta)\,I}{cr\,\sin\theta} = \frac{I}{cr}\,\tan(\tfrac{1}{2}\theta) \,.$$

Note that there are two choices we could make for our cap. The complementary region Σ' would subtend solid angle $4\pi - \Omega$, and is pierced by the wire. In this case, both $\boldsymbol{j}$ and $\boldsymbol{j}_{\rm d}$ contribute to $\boldsymbol{J}$, and after considering the opposite orientation of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{r}}$ on Σ', we obtain

$$2\pi B r \sin\theta = \frac{4\pi}{c}\left\{ -\frac{4\pi - \Omega}{4\pi} \cdot I + I \right\} = \frac{\Omega\, I}{c} \,,$$

as before.

(d) Near the wire, we have $\theta \to \pi$, and $\cos\theta \to 1$, and we recover the familiar expression

$$B(r,\theta) \approx \frac{2I}{cr\sin\theta} = \frac{2I}{cR} \,,$$

where $R = r\sin\theta$ is the perpendicular distance from the wire.

Príklad 2

SOLUTION: We use Fermi's Golden Rule for the transition rate,

$$\Gamma_{\mathrm{i}\to\mathrm{f}} = \frac{2\pi}{\hbar} \left|\langle \mathrm{f}|\hat{V}(\omega)|\mathrm{i}\rangle\right|^2 \delta(E_{\mathrm{f}} - E_{\mathrm{i}} - \hbar\omega) \,,$$

valid for harmonic perturbations of the form $V(t) = \hat{V}(\omega)\, e^{-i\omega t}$. (For a real harmonic potential, sum over positive and negative frequency components.) Our potential is

$$V(t) = -\,e\boldsymbol{E}_0 \cdot \boldsymbol{r}\, \cos(\omega t) \,,$$

so $\hat{V}(\omega) = \hat{V}(-\omega) = -e\boldsymbol{E}_0 \cdot \boldsymbol{r}$. The matrix element we seek is then

$$\begin{aligned}\mathcal{M} &= -e\langle \psi_{\boldsymbol{k}}|\boldsymbol{E}_0 \cdot \boldsymbol{r}|\psi_0\rangle \\ &= -eE_0\,(\pi a_{\mathrm{B}}^3)^{-1/2}\, V^{-1/2} \int d^3\!r\, e^{-r/a_{\mathrm{B}}}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \,,\end{aligned}$$

where we take $\boldsymbol{E}_0 = E_0\,\hat{\boldsymbol{e}}$. We may, without loss of generality, take $\boldsymbol{k}$ to lie along $\hat{\boldsymbol{z}}$. Writing $\boldsymbol{r}$ and $\hat{\boldsymbol{e}}$ in polar coordinates, we then have

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{r} = r\cos\theta\cos\vartheta + r\sin\theta\sin\vartheta\cos(\phi - \varphi) \,,$$

where (θ,ϕ) and (ϑ,φ) are the polar and azimuthal angles for $\boldsymbol{r}$ and $\hat{\boldsymbol{e}}$, respectively. The last term integrates to zero. The matrix element is then

$$\mathcal{M} = -2\pi eE_0\,(\pi a_{\mathrm{B}}^3 V)^{-1/2}\, \cos\vartheta \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin\theta\, r\cos\theta\, e^{-r/a_{\mathrm{B}}}\, e^{-ikr\cos\theta} \,.$$

The double integral is straightforward:

$$\begin{aligned}\int_{-1}^{1} d\mu\, \mu \int_0^\infty dr\, r^3\, e^{-(a_{\mathrm{B}}^{-1} + ik\mu)r} &= 6\int_{-1}^{1} d\mu\, \frac{\mu}{(a_{\mathrm{B}}^{-1} + ik\mu)^4} \\ &= -\frac{6}{k^2} \int_{a_{\mathrm{B}}^{-1} - ik}^{a_{\mathrm{B}}^{-1} + ik} \frac{s - a_{\mathrm{B}}^{-1}}{s^4} \\ &= -\frac{16i\,k\,a_{\mathrm{B}}^{-1}}{(a_{\mathrm{B}}^{-2} + k^2)^3} \,.\end{aligned}$$

The matrix element must be squared, then summed over all final $\boldsymbol{k}$ states. Recalling the relation

$$\sum_{\boldsymbol{k}} A(\boldsymbol{k}) = V \int \frac{d^3k}{(2\pi)^3} A(\boldsymbol{k}) \, ,$$

we obtain for the total transmission rate

$$\begin{aligned}
\Gamma &= \frac{2\pi}{\hbar} \int_0^\infty \frac{dk\,k^2}{4\pi^2} \int_0^\pi d\vartheta\, \sin\vartheta \cdot \frac{4\pi^2 e^2 E_0^2 \cos^2\vartheta}{\pi a_{\mathrm{B}}^3} \cdot \frac{256\, k^2 a_{\mathrm{B}}^{-2}}{(a_{\mathrm{B}}^{-2} + k^2)^6} \cdot \delta\!\left(\frac{\hbar k^2}{2m} - \hbar\omega + \frac{\hbar^2}{2m a_{\mathrm{B}}^2}\right) \\
&= \frac{256}{3\hbar} e^2 E_0^2 a_{\mathrm{B}}^3 \left(\frac{\omega_0}{\omega}\right)^3 \left(\frac{\omega}{\omega_0} - 1\right)^{3/2} ,
\end{aligned}$$

where $\omega_0 \equiv \hbar/2ma_{\mathrm{B}}^2 = me^4/2\hbar^3$ is the lowest ionization frequency. Note that $\Gamma \to 0$ at the ionization edge, $\omega = \omega_0$. The approximation of ionized states by plane waves is accurate only for $\omega \gg \omega_0$.

Príklad 3

the relevant part of the ideal gas partition function for the unmixed and mixed cases are

unmixed: $Z = Z_1 Z_2$

$$Z_1 \sim \frac{V^{N_1}}{N_1!} \frac{V^{M_1}}{M_1!} \qquad Z_2 = \frac{V^{N_2}}{N_2!} \frac{V^{M_2}}{M_2!}$$

$$S \sim -\frac{\partial F}{\partial T} \simeq k N_1 \ln \frac{V}{(N_1 \lambda_\alpha^3)} + k M_1 \ln \frac{V}{(M_1 \lambda_\beta^3)}$$

$$+ k N_2 \ln \frac{V}{(N_2 \lambda_\alpha^3)} + k M_2 \ln \frac{V}{(M_2 \lambda_\beta^3)}$$

after mixing (volume is now $2V$), where λ_α and λ_β are the thermal wavelengths for the gases α and β, $\lambda_\alpha = \text{const} \cdot \frac{\hbar}{\sqrt{2m_\alpha k_B T}}$

$$S = k (N_1 + N_2) \ln \frac{2V}{(N_1 + N_2)}$$

$$+ k (M_1 + M_2) \ln \frac{2V}{(M_1 + M_2)}$$

So

$$\Delta S = S' - S = K\left[N_1 \ln \frac{2N_1}{(N_1+N_2)} + N_2 \ln \frac{2N_2}{(N_1+N_2)} + M_1 \ln \frac{2M_1}{(M_1+M_2)} + M_2 \ln \frac{2M_2}{(M_1+M_2)} \right]$$

if $N_1 = N_2 = \frac{1}{2}(N_1+N_2)$ $\Delta S = 0$: if $M_1 = N_2 = 0$ $\Delta S = 2N_1 K \ln 2$