METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto19 – Príklady 4

VZOROVÉ RIEŠENIA

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Príklad 1

$$\left.\begin{aligned} \varphi_2 &= E_0 r\cos\theta + \frac{B}{r^2}\cos\theta \\ \varphi_1 &= C r\cos\theta \end{aligned}\right\} \quad \begin{aligned} \varphi_1(r, \pm\pi/2) &= 0 \\ \varphi_2(r, \pm\pi/2) &= 0 \end{aligned}$$

$$\varphi_1(R,\theta) = \varphi_2(R,\theta) \implies E_0 R + \frac{B}{R^2} = CR$$

$$\epsilon \left.\frac{\partial\varphi_1}{\partial r}\right|_R = \epsilon_0 \left.\frac{\partial\varphi_2}{\partial r}\right|_R \implies \left[E_0 - \frac{2B}{R^3}\right] = \frac{\epsilon}{\epsilon_0} C$$

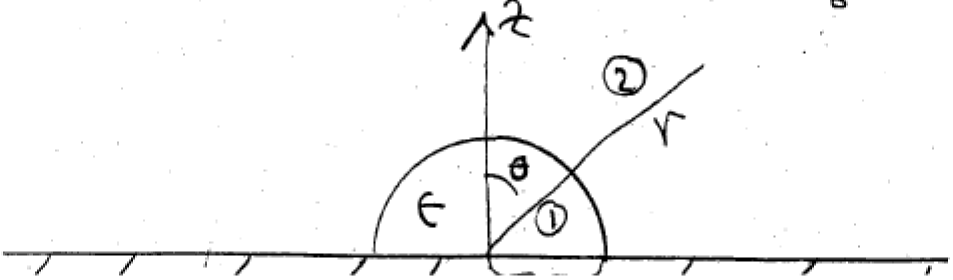
$$\therefore \quad 3E_0 = C\left[2 + \epsilon/\epsilon_0\right], \qquad B = R^3\left[C - E_0\right]$$

$$C = \frac{3E_0}{2+\epsilon/\epsilon_0} \qquad \qquad B = R^3 \frac{\left[1 - \epsilon/\epsilon_0\right]E_0}{\left[2 + \epsilon/\epsilon_0\right]}$$

$$\downarrow \underset{\sim}{E_0} \qquad \qquad \qquad \downarrow \underset{\sim}{E_0}$$

$$\varphi_1(r,\theta) = \frac{3E_0}{2+\epsilon/\epsilon_0} r\cos\theta$$

$$\varphi_2(r,\theta) = E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\frac{(1-\epsilon/\epsilon_0)}{(2+\epsilon/\epsilon_0)}\cos\theta$$



Príklad 2

1. The state vector is

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$$

and the Schrödinger equation is

$$i\hbar\dot{\psi}_0 = ig\psi_1$$
$$i\hbar\dot{\psi}_1 = -ig\psi_0 + \Delta\,\psi_1$$

from which it follows that

$$-\hbar^2\ddot{\psi}_1 = g^2\psi_1 + i\hbar\Delta\,\psi_1$$

and therefore, since $\psi_0(0) = 1$ and $\psi_1(0) = 0$,

$$\psi_1(t) = A\left(e^{-i\omega_1 t} - e^{-i\omega_2 t}\right)$$

for some constant A and

$$\omega_{1,2} = \frac{\Delta}{2\hbar} \pm \sqrt{\frac{\Delta^2}{4\hbar^2} + \frac{g^2}{\hbar^2}}.$$

From the Schrödinger equation for ψ_0 we can now deduce that

$$\psi_0(t) = \frac{iAg}{\hbar}\left(\frac{1}{\omega_1}e^{-i\omega_1 t} - \frac{1}{\omega_2}e^{-i\omega_2 t}\right)$$

and therefore

$$\frac{iAg}{\hbar}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = 1$$

or

$$A = \frac{-i/2}{\sqrt{1 + \frac{\Delta^2}{4g^2}}}.$$

The desired probability is

$$|\psi_1(t)|^2 = |A|^2\left[2 - 2\cos(\omega_1 - \omega_2)t\right] = \frac{\sin^2 \frac{gt}{2\hbar}\sqrt{1 + \frac{\Delta^2}{4g^2}}}{1 + \frac{\Delta^2}{4g^2}}.$$

2. Continuous measurement of D forces the system to stay in one eigenstate of D. From the initial condition, this eigenstate is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If you have the solution to the first part in hand, then you can consider a sequence of measurements separated in time by ϵ; we eventually take the limit $\epsilon \to 0$. For sufficiently small ϵ, the probability that the first measurement yields one is

$$p_1 = \frac{g^2\epsilon^2}{4\hbar^2},$$

so the probability of remaining in the initial state is

$$p_0 = 1 - p_1 = 1 - \frac{g^2\epsilon^2}{4\hbar^2}.$$

All we really need is that $p_1 \propto \epsilon^2$, which is a consequence of the Schrödinger equation; the detailed solution is not necessary. After n measurements, the probability of still being in the initial state is at least p_0^n. For fixed time $t > 0$, choose $\epsilon = t/n$. Then the probability of being in initial state at time t is

$$P_0(t) \geq \left(1 - \frac{g^2 t^2}{4\hbar^2 n^2}\right)^n$$

and, for continuous measurement,

$$\lim_{n\to\infty} \log P_0(t) \geq n \log \left(1 - \frac{g^2 t^2}{4\hbar^2 n^2}\right) = 0.$$

We have $\lim_{n\to\infty} P_0(t) = 1$, and the system never leaves the initial state.

Príklad 3

SOLUTION: The surface consists of N horizontal steps, $N_\uparrow$ upward steps, and $N_\downarrow$ downward steps. The degrees of freedom the system possesses are whether after each horizontal step the surface goes upward, downward, or remains at the same level. Let us represent these three possibilities by a scalar variable $\sigma = +1$, 0, or -1, respectively. We further label each step by a subscript $i \in \{1, \ldots, N\}$.

(a) With $H = \varepsilon \sum_i (1 + \sigma_i^2)$, the energy is written as a sum over the N columns. The contribution from each column is ε if $\sigma = 0$, *i.e.* if there is no step, and 2ε if $\sigma = \pm 1$, *i.e.* if there is a step in either direction. Since each step adds an extra lattice length to the length of the surface, this Hamiltonian properly accounts for the surface energy of ε per lattice length.

(b) The partition function is a sum over all configurations. This may be represented as a product over the steps, *viz.*

$$Z = \mathrm{Tr}\, e^{-H/k_B T} = \prod_{i=1}^{N} \sum_{\sigma_i=-1}^{1} e^{-(1+\sigma_i^2)/k_B T}$$

$$= e^{-N\varepsilon/k_B T} \left(1 + 2\, e^{-\varepsilon/k_B T}\right)^N .$$

(c) The free energy is

$$
\begin{aligned}
F &= k_{\mathrm{B}} T \ln Z \\
&= N\varepsilon - N k_{\mathrm{B}} T \ln\left(1 + 2\, e^{-\varepsilon/k_{\mathrm{B}}T}\right) .
\end{aligned}
$$

In the low temperature regime $k_{\mathrm{B}}T \ll \varepsilon$, we have $F \approx N\varepsilon$, which is the energy of a flat surface, whose length is the minimum value possible, N. In the high temperature regime $k_{\mathrm{B}}T \gg \varepsilon$, we have $-Nk_{\mathrm{B}}T \ln 3$, which reflects the fact that the surface is completely randomized, with 3^N equally probable configurations yielding an entropy $S = Nk_{\mathrm{B}} \ln 3$, as $T \to \infty$. The entropy term $-TS$ dominates the average energy E at these high temperatures.

(d) The total surface length is $L = N + N_{\uparrow} + N_{\downarrow} = N \cdot (1 + 2p)$, where p is the probability for an upward or downward step:

$$
p = \frac{e^{-\varepsilon/k_{\mathrm{B}}T}}{1 + 2\, e^{-\varepsilon/k_{\mathrm{B}}T}} .
$$

Thus, $\langle L \rangle_{T\to 0} = N$, while $\langle L \rangle_{T\to\infty} = \frac{5}{3}N$.