METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto19 – Príklady 1

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Príklad 1

Solution 1.10. In plane-polar coordinates, the Lagrangian for a particle moving in a central potential $V(r)$ is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{10.89}$$

where m is the mass of the particle. The potential is given in the question as

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2. \tag{10.90}$$

The θ-component of Lagrange's equation is

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \equiv l. \tag{10.91}$$

The hamiltonian of our system is then

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{eff}(r), \tag{10.92}$$

with $p_r = m\dot{r}$ and

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r). \tag{10.93}$$

The term $l^2/2mr^2$ is referred to as an "angular momentum barrier." Solving the equations of motion for this hamiltonian is equivalent to solving Lagrange's equations for the Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 - V_{eff}(r). \tag{10.94}$$

This is a completely general result for the motion of a particle in a central potential and could easily have been our starting point in this problem (e.g., Goldstein, Chapter 3).

It may seem unnecessarily long-winded to go through this procedure, but note that the sign of the angular momentum barrier in (10.94) is *opposite* to what we would have gotten if we had naively replaced $\dot{\theta}$ with l/mr^2 in the Lagrangian (10.89). This is due to the fact that the Lagrangian is a function of the time derivative of the position, and not of the canonical momentum.

The equation of motion from (10.94) is

$$m\ddot{r} = -\frac{d}{dr}V_{\mathit{eff}}(r). \tag{10.95}$$

If the particle is in a circular orbit at $r = r_0$ we require that the force on it at that radius should vanish,

$$\left.\frac{dV_{\mathit{eff}}}{dr}\right|_{r=r_0} = 0. \tag{10.96}$$

Using our expression for V_{eff} (10.93), we derive an expression relating the angular momentum l to the radius of the orbit r_0:

$$\frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - br_0 = 0. \tag{10.97}$$

We are interested in perturbations about this circular orbit. Provided the perturbation remains small, we can expand $V_{\mathit{eff}}(r)$ about r_0,

$$V_{\mathit{eff}}(r) = V_{\mathit{eff}}(r_0) + (r - r_0)V'_{\mathit{eff}}(r_0) + \frac{1}{2}(r - r_0)^2 V''_{\mathit{eff}}(r_0) + \cdots. \tag{10.98}$$

If we use this expansion in the Lagrangian (10.94) together with the condition (10.96), we find

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(r - r_0)^2 V''_{\mathit{eff}}(r_0), \tag{10.99}$$

where we have dropped a constant term. This is just the Lagrangian for a simple harmonic oscillator, describing a particle undergoing radial oscillations with frequency

$$\omega^2 = \frac{1}{m}V''_{\mathit{eff}}(r_0). \tag{10.100}$$

Differentiating $V_{\mathit{eff}}(r)$ twice gives us

$$\frac{3l^2}{mr_0^4} - \frac{2k}{r_0^3} + b = m\omega^2. \tag{10.101}$$

We can eliminate l between equations (10.101) and (10.97) to give the frequency of radial oscillations:

$$\omega = \left(\frac{k}{mr_0^3} + \frac{4b}{m}\right)^{1/2}. \tag{10.102}$$

To find the rate of precession of the perihelion, we need to know the period of the orbit. From the definition of angular momentum l equation (10.91), we have an equation for the orbital angular velocity ω_1,

$$\omega_1 \equiv \frac{d\theta}{dt} = \frac{l}{mr^2}. \tag{10.103}$$

Let us write $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is sinusoidal with frequency ω and average value zero. We substitute $r(t)$ into equation (10.103) and expand in $\epsilon(t)$:

$$\frac{d\theta}{dt} = \frac{l}{mr_0^2}\left(1 - \frac{2\epsilon}{r_0} + \mathcal{O}(\epsilon^2)\right). \tag{10.104}$$

To zeroth order in the small quantities br_0^3/k and ϵ/r_0, the period of the orbit T_1 is the same as the period of oscillations $T_2 = 2\pi/\omega$. Therefore we can average ϵ over T_1 rather than T_2 and still get zero, to within terms of second order, which we are neglecting. The average angular velocity is therefore

$$\bar{\omega_1} = \frac{2\pi}{T_1} \approx \frac{l}{mr_0^2} = \sqrt{\frac{k}{mr_0^3} + \frac{b}{m}}, \tag{10.105}$$

where we have made use of (10.97).

Now consider one complete period of the radial oscillation. This takes place in time $T_2 = 2\pi/\omega$. In this time the particle travels along its orbit through an angle of

$$\theta = 2\pi\frac{\bar{\omega_1}}{\omega} = 2\pi\frac{\sqrt{k/mr_0^3 + b/m}}{\sqrt{k/mr_0^3 + 4b/m}}$$

$$\approx 2\pi\left(1 - \frac{3br_0^3}{2k}\right). \tag{10.106}$$

In other words, the particle does not quite orbit through 2π before the radial oscillation is completed. Each time around the perihelion precesses backwards through an angle

$$\delta\theta = 3\pi\frac{br_0^3}{k}, \tag{10.107}$$

and it gets around in time T_2, so the precession rate is

$$\alpha = \frac{\delta\theta}{T_2} = \frac{3\pi br_0^3}{k}\frac{\sqrt{k/mr_0^3 + 4b/m}}{2\pi}$$

$$\approx \frac{3b}{2}\sqrt{\frac{r_0^3}{mk}}. \tag{10.108}$$

Príklad 2

(a) The electric potential V satisfies the Laplace equation, $\nabla^2 V = 0$. Given the boundary conditions

$$V(x, y=0) = 0 = V(x, y=a), \qquad \text{and} \qquad V(x=0, y) = V_0,$$

the solution is of the form

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{ikx}.$$

Inserting this solution into the Laplace equation, we have

$$-\left(\frac{\pi}{a}\right)^2 - k^2 = 0,$$

or $k = \pm i\pi/a$. Thus, the solution (for $x \geq 0$) is

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{-\pi x/a}.$$

(We can ignore $x \geq L/2$ since $e^{-piL/2a} \ll 1$ for $L/a \gg 1$.)

(b) To find the charge density σ at the surface of the conductors, we need the electric field $\vec{E}$ at the surface. The latter can be obtained from the potential $V(x, y)$ as

$$\vec{E} = -\vec{\nabla} V = \frac{\pi V_0}{a} \left[\sin\left(\frac{\pi y}{a}\right) \hat{x} - \cos\left(\frac{\pi y}{a}\right) \hat{y}\right] e^{-\pi x/a}.$$

At the surfaces of the conducting plates at $y = 0$ and $y = a$, the induced charge densities are the same, with

$$\sigma(x, y=0) = \sigma(x, y=a) = \epsilon_0 \vec{E} \cdot \hat{n} = -\frac{\epsilon_0 \pi V_0}{a} e^{-\pi x/a}, \qquad x \geq 0$$

for both plates.

(c) Force exerted on a conductor is given by

$$\vec{F} = \int \sigma \, \vec{E}_{\text{ext}} \, dA,$$

integrated over the surface area of the conductor, with $E_{\text{ext}} = E_{\text{self}} = E/2$.

On the upper plate (and $x \geq 0$),

$$\begin{aligned}
\vec{F} &= L \int_0^{L/2 \to \infty} dx \, \sigma(x, y=a) \cdot \frac{1}{2} \vec{E}(x, y=a) \\
&= -\frac{\epsilon_0 \pi^2 V_0^2 L}{2a^2} \left[\int_0^{\infty} e^{-2\pi x/a} dx\right] \hat{y} \\
&= -\frac{\pi}{4} \epsilon_0 V_0^2 L \, \hat{y}
\end{aligned}$$

Including also the part from $x \leq 0$, the total force exerted on the top plate is

$$\vec{F}_{\text{upper}} = -\frac{\pi}{2}\epsilon_0 V_0^2 L\,\hat{y},$$

i.e., the top plate is attracted towards the lower plate.

By symmetry, the lower plate is attracted towards the upper plate with force of the same magnitude, i.e.,

$$\vec{F}_{\text{lower}} = +\frac{\pi}{2}\epsilon_0 V_0^2 L\,\hat{y}.$$

Príklad 3

The potential due to a uniformal spherical volume with net charge e and radius r_0 is

$$\begin{aligned} U &= -\frac{er^2}{2r_0^3} + \frac{3e}{2r_0}, \qquad (r < r_0) \\ &= \frac{e}{r}, \qquad (r > r_0) \end{aligned}$$

where the constant of integration in the first expression has been chosen to make U continuous at r_0.

The perturbation V in the Hydrogen atom potential is

$$\begin{aligned} \Delta V &= \frac{e^2 r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}, \qquad (r < r_0) \\ &= 0, \qquad (r > r_0) \end{aligned}$$

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The energy shift is

$$\Delta E = \langle \Delta V \rangle$$

to first order in perturbation theory. For the $1s$ state,

$$\Delta E = \int |\psi|^2 \, \Delta V = \int_0^{r_0} 4\pi r^2 \mathrm{d}r \; |\psi|^2 \, \Delta V$$

Since $r_0 \ll a_0$, the typical scale of variation of the wavefunction, $\psi \approx \psi(0)$, and

$$\Delta E \approx |\psi(0)|^2 \int_0^{r_0} 4\pi r^2 \mathrm{d}r \left[\frac{e^2 r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}\right] = \frac{2}{5} e^2 \pi r_0^2 \, |\psi(0)|^2$$

For the 1s state,

$$\psi = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0}$$

so

$$\Delta E \approx \frac{2}{5}\frac{e^2 r_0^2}{a_0^3} = \frac{2}{5}\frac{e^2}{a_0}\frac{r_0^2}{a_0^2}$$

The ground state energy of H is $-e^2/(2a_0) = -13.6$ eV, and $a_0 = 0.529 \times 10^{-10}$ m, so

$$\Delta E = \frac{2}{5}(2 \times 13.6\,\mathrm{eV})\frac{r_0^2}{a_0^2} = 3.9 \times 10^{-9}\,\mathrm{eV}$$

The wavefunction of the $2p$ state vanishes at the orgin. This suppresses ΔE by an additional factor of $r^2/a_0^2 \sim 10^{-10}$

Príklad 4

SOLUTION: For a non-interacting ideal gas,

$$E = -\frac{\partial}{\partial \beta} N \ln \zeta \, ,$$

where ζ is the single-molecule partition function

$$\zeta = \sum_{n=0}^{\infty} (n+1) \exp(-\beta n \varepsilon) \, .$$

This partition function can be evaluated as follows ($x \equiv \beta\varepsilon$):

$$\zeta = -e^x \frac{d}{dx} \sum_{n=0}^{\infty} \exp\left(-(n+1)x \right) = -e^x \frac{d}{dx} \frac{e^{-x}}{1-e^{-x}} = [1 - \exp(-\beta\varepsilon)]^{-2}.$$

Hence, the sought contribution to the energy is

$$E = \frac{2N\varepsilon}{\exp(\varepsilon/kT) - 1} \, .$$

Alternatively, one can reproduce this result as follows. One can imagine that every molecule has two independent internal degrees of freedom of harmonic oscilator type, with energy spacing ε each. It is easy to see that this model gives the same spectrum and degeneracies if the energy is counted from the ground state. With this convention, the average energy of a single harmonic oscillator is $\varepsilon n_B(\varepsilon)$, where $n_B(\varepsilon)$ is the Bose-Einstein occupation number. Therefore, for the entire gas we get $E = 2N\varepsilon n_B(\varepsilon)$, in agreement with the first derivation.