VZOROVÉ RIEŠENIA

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Príklad 1

(a) Since A and B are Hermitian, we have that $(A+B)^2$ is also Hermitian. But $2\,AB$ is Hermitian if and only if A commutes with B. Therefore we have that $(A + B)^2 = A^2 + B^2 + 2\,AB = 2\,AB$ (where the first equality uses $[A, B] = \hat{0}$), and hence $A^2 + B^2 = \hat{0}$. Taking the expectation of both sides of this last equation in an arbitrary state $|\psi\rangle$ yields $\langle A\psi|A\psi\rangle + \langle B\psi|B\psi\rangle = 0$, where we have used the Hermiticity of A and B. Since both terms on the LHS of this equation are ≥ 0 for any A and $|\psi\rangle$, we see that $\langle A\psi|A\psi\rangle = \langle B\psi|B\psi\rangle = 0$. But $\langle \phi|\phi\rangle = 0$ if and only if the vector $|\phi\rangle$ is the zero vector $|0\rangle$. Hence $A\,|\psi\rangle = B\,|\psi\rangle = |0\rangle$ for all states $|\psi\rangle$, which shows that $A = B = \hat{0}$.

(b) Let $|\phi\rangle$ be a unit vector in $\mathcal{H}$ which is orthogonal to $|\psi\rangle$. (The vector $|\phi\rangle$ is unique up to an overall phase.) Then $\{|\psi\rangle , |\phi\rangle\}$ is an orthonormal basis for $\mathcal{H}$. We can now (for example) choose $A = \alpha\,(|\psi\rangle \langle\phi| + |\phi\rangle \langle\psi|)$ and $B = \beta\,(|\psi\rangle \langle\psi| - |\phi\rangle \langle\phi|)$, where α and β are non-zero real numbers. The eigenvalues of A are $\pm\alpha$, and the state of the system after the first measurement is $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \pm |\phi\rangle)$ if the value $\pm\alpha$ is obtained (each occurs with probability $1/2$). We want to show that if we now measure B (whose eigenvalues are $\pm\beta$), there will be a non-zero probability of obtaining the value $-\beta$, in which case the state of the system after the measurement will be $|\phi\rangle$ (which is orthogonal to $|\psi\rangle$). But it is immediate from the form of the states $|\chi_\pm\rangle$ that the probability of obtaining $-\beta$ in the measurement of B is $1/2$, regardless of the outcome of the first measurement.

Príklad 2

First, determine the mass M in terms of ρ and R using displaced liquid volume V_d ;

Bouyancy $\Rightarrow M = \rho V_d$

$\theta_{max} = \cos^{-1}\left(\dfrac{R}{2R}\right) = \dfrac{\pi}{3}$

$d^3_r = r\,dr\,d\phi\,dz$

$\quad\quad = 2\pi r\,dr\,dz$

$r = R\sin\theta \Rightarrow dr = R\cos\theta\,d\theta$

$z = R\cos\theta \Rightarrow dz = +R\sin\theta\,d\theta$

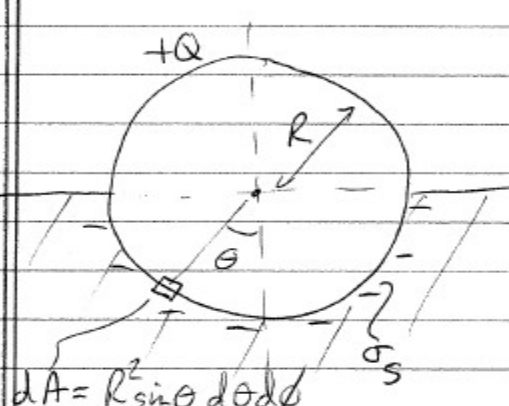$\Rightarrow V_d = \displaystyle\int_0^{\pi/3} \pi\,(R\sin\theta)^2\,(+R\sin\theta\,d\theta) = +\pi R^3 \displaystyle\int_0^{\pi/3} d\theta\,\sin^3\theta$

$$= +\pi R^3 \left\{ \left[-\cos\theta \right]_0^{\pi/3} + \left[\frac{1}{3}\cos^3\theta \right]_0^{\pi/3} \right\}$$

$$= +\pi R^3 \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3} \right) = \frac{5}{24}\pi R^3$$

$$\therefore \quad M = \frac{5}{24}\pi \rho R^3 \qquad \cdots \text{(A)}$$

When the sphere is charged, there is an additional electric force between the charge Q and the induced surface charge in the dielectric liquid;

$$dq = \sigma_s R^2 \sin\theta\, d\theta\, d\phi$$

$$\Rightarrow F_e = \frac{Q}{4\pi\epsilon_0} \int_0^{\pi/2} \frac{2\pi \sigma_s R^2 \sin\theta \cos\theta\, d\theta}{R^2}$$

$$dA = R^2 \sin\theta\, d\theta\, d\phi$$

... downward component

$$\Rightarrow F_e = \frac{\sigma_s Q}{2\epsilon_0} \int_0^{\pi/2} d\theta \cdot \frac{1}{2}\sin 2\theta = \frac{\sigma_s Q}{4\epsilon_0}\left[-\frac{1}{2}\cos 2\theta \right]_0^{\pi/2}$$

$$= \frac{\sigma_s Q}{4\epsilon_0}\left(\frac{1}{2} - -\frac{1}{2} \right) = \frac{\sigma_s Q}{4\epsilon_0}$$

All that remains is to determine σ_s since we know that

$$F_e + Mg = \frac{1}{2}\left(\frac{4}{3}\pi R^3 \right)\rho g \qquad \text{... half sphere submerged}$$

$$\Rightarrow \frac{\sigma_s Q}{4\epsilon_0} = \frac{11}{24}\pi R^3 \rho \cdot g \qquad \text{... using (A)}$$

Now, $D = \epsilon_0 \epsilon_r E = \epsilon_0 E + P$, where $P = \sigma_s$
and $\epsilon_0 E = \sigma = \dfrac{Q}{4\pi R^2}$, hence

$$\epsilon_r \sigma = \sigma + \sigma_s \implies \sigma_s = \dfrac{Q(\epsilon_r - 1)}{4\pi R^2 \epsilon_r}$$

Thus,

$$\dfrac{Q^2(\epsilon_r - 1)}{4\pi R^2 \epsilon_0 \epsilon_r} = \dfrac{11}{6}\pi R^3 \varrho \cdot g$$

$$\therefore Q = \sqrt{\dfrac{22\pi^2 R^5 \epsilon_0 \epsilon_r \varrho g}{3(\epsilon_r - 1)}}$$

Príklad 3

a)

$$P(\vec{v}) = \left(\dfrac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\dfrac{m\vec{v}^2}{2k_B T}\right)$$

b) Due to the equipartition theorem

$$\langle \varepsilon \rangle = \dfrac{m}{2}\langle \vec{v}^2 \rangle = \dfrac{3}{2}k_B T$$

and hence

$$E_0 = \langle E \rangle = \dfrac{3}{2}N k_B T$$

c) We instantaneously removed all particles with a kinetic energy

$$\varepsilon_{kin} = \dfrac{1}{2}mv^2 \geq n k_B T$$

The number of remaining particle, N_{new}, is given by

$$N_{new} = N_0 \left(\dfrac{m}{2\pi k_B T}\right)^{3/2} \int' d^3 v \exp\left[-\dfrac{mv^2}{2k_B T}\right]$$

$$= N_0 \left(\dfrac{m}{2\pi k_B T}\right)^{3/2} 4\pi \int_0^{v_c} dv\, v^2 \exp\left[-\dfrac{mv^2}{2k_B T}\right]$$

where

$$v_c = \sqrt{\dfrac{2n k_B T}{m}}$$

I next perform the variable transformation

$$x = \frac{mv^2}{2k_BT} \implies dx = \frac{mv}{k_BT}dv = \frac{m}{k_BT}\sqrt{\frac{2k_BTx}{m}}\,dv = \sqrt{\frac{2mx}{k_BT}}\,dv$$

and thus

$$N_{new} = N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}4\pi\sqrt{\frac{k_BT}{2m}}\int_0^n dx\,\frac{2k_BT}{m}x^{1/2}\exp[-x]$$

$$= N_0\frac{2}{\sqrt{\pi}}\int_0^n dx\,x^{1/2}\exp[-x]$$

$$= N_0\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)\right]$$

Next, we compute the remaining energy that is contained in the system after the particles are removed.

$$E_{new} = N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int^t d^3v\left(\frac{1}{2}mv^2\right)\exp\left[-\frac{mv^2}{2k_BT}\right]$$

$$= N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{m}{2}4\pi\int_0^{v_c}dv\,v^4\exp\left[-\frac{mv^2}{2k_BT}\right]$$

I next perform the variable transformation

$$x = \frac{mv^2}{2k_BT} \implies dx = \frac{mv}{k_BT}dv = \frac{m}{k_BT}\sqrt{\frac{2k_BTx}{m}}\,dv = \sqrt{\frac{2mx}{k_BT}}\,dv$$

and thus

$$E_{new} = N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{m}{2}4\pi\sqrt{\frac{k_BT}{2m}}\int_0^n dx\left(\frac{2k_BT}{m}\right)^2 x^{3/2}\exp[-x]$$

$$= N_0\frac{2k_BT}{\sqrt{\pi}}\int_0^n dx\,x^{3/2}\exp[-x]$$

$$= N_0\frac{k_BT}{2\sqrt{\pi}}\left[3\sqrt{\pi}\,\mathrm{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)\right]$$

After equilibration, the new temperature is given by

$$E_{new} = \frac{3}{2}N_{new}k_BT_{new}$$

or

$$T_{new}(n) = \frac{2}{3}\frac{E_{new}}{k_BN_{new}} = \frac{2}{3}\frac{N_0\frac{k_BT}{2\sqrt{\pi}}[3\sqrt{\pi}\,\mathrm{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)]}{k_BN_0\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)\right]}$$

$$= \frac{1}{6}\frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)}{\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)}T$$

and thus

$$T_{new} = T/2$$

requires

$$\frac{1}{6}\frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)}{\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)} = \frac{1}{2}$$

and thus

$$n = 1.527$$

Príklad 4

(a) The total energy is $E = (M + m\gamma)c^2$ where $\gamma \equiv 1/\sqrt{1 - (v/c)^2}$, and the total momentum is $P = mv\gamma$. Since $E^2 - P^2c^2$ is Lorentz invariant, the energy in the center of mass frame E' is given by

$$E' = \sqrt{E^2 - P^2c^2} = c\sqrt{(M + m\gamma)^2c^2 - m^2v^2\gamma^2} = c^2\sqrt{M^2 + 2mM\gamma + m^2} .$$

After the fission, half of E' is carried by one of the daughter nuclei with momentum p' whose energy is $c\sqrt{(p')^2 + (M')^2c^2}$. Equating the two gives

$$p' = \frac{c}{2}\sqrt{M^2 + 2mM\gamma + m^2 - 4(M')^2} .$$

(b) Let the magnitude of the momentum of either e^- or $\bar{\nu}_e$ be p_2. From momentum conservation, we have $2p_2 \cos\theta = p_1$ and energy conservation gives $2p_2c + c\sqrt{p_1^2 + m_p^2c^2} = m_nc^2$. Eliminating p_2 from the first equation, and using the second, we get

$$\cos\theta = \frac{p_1}{m_nc - \sqrt{(p_1)^2 + m_p^2c^2}} .$$

The expression on the right-hand side is a monotonically increasing function of p_1. Since $0 \le \cos\theta \le 1$, we have

$$0 \le p_1 \le \frac{(m_n^2 - m_p^2)c}{2m_N} .$$

When $p_1 = 0$, we have $\theta = \pi/2$ and e^- and $\bar{\nu}_e$ are back to back. In the other limit when $p_1 = (m_n^2 - m_p^2)c/(2m_N)$ we have $\theta = 0$ and $(e^-, \bar{\nu}_e)$ pair and the proton are back to back.