METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto20 – Príklady 5

VZOROVÉ RIEŠENIA

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Príklad 1

Príklad 2

The force $\mathbf{F}$ on a particle with rest mass m is the rate of change its momentum $\mathbf{p}$ as given by text Eq. (1.36):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{34}$$

where as given by text Eq. (1.35):

$$\mathbf{p} = \gamma m \, \mathbf{v} \tag{35}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{36}$$

where $\mathbf{v}$ is the velocity of the particle. Eq. (34) with $\mathbf{p}$ given by Eq. (35) is the relativistic generalization of Newton's Second Law.

Now

$$\mathbf{v} = v\mathbf{u}_t \tag{37}$$

where $\mathbf{u}_t$ is a unit vector along the tangent of the particle's trajectory, and the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n \tag{38}$$

where $\mathbf{u}_n$ is a unit vector orthogonal to $\mathbf{u}_t$ directed towards the centre of curvature of the trajectory and ρ is the radius of curvature of the trajectory, so

$$\frac{d\mathbf{p}}{dt} = \gamma m \left(\gamma^2 \frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n\right) \tag{39}$$

When

$$\mathbf{F} = F\mathbf{u}_t \tag{40}$$

that is, when $\mathbf{F}$ is parallel to $\mathbf{v}$, if follows that

$$\rho = \infty \tag{41}$$

That is, the particle moves in a straight line, and

$$a = \tilde{a}/\gamma^3 \tag{42}$$

where

$$a = \frac{dv}{dt} \tag{43}$$

and

$$\tilde{a} = \frac{F}{m} \tag{44}$$

When a particle of charge q moves in an electric field $\mathbf{E}$, the force $\mathbf{F}$ on the particle is $\mathbf{F} = q\mathbf{E}$. If the particle moves in the direction of $\mathbf{E}$, then $\mathbf{F}$ and $\mathbf{v}$ are parallel. Accordingly, Eq. (42) holds with $\tilde{a} = qE/m$.

It follows from Eq. (42) that $a \to 0$ as $v \to c$ and also that

$$F \simeq ma \quad \text{when } v \ll c \tag{46}$$

which is the nonrelativistic result.

When F is constant and the particle starts from rest at $t = 0$, its speed $v(t)$ is found by integrating Eq. (42):

$$\int_0^{v(t)} \frac{dv'}{(1 - v'^2/c^2)^{3/2}} = \tilde{a}t \tag{47}$$

to be

$$v(t) = \frac{\tilde{a}t}{\sqrt{1 + (\tilde{a}t/c)^2}}. \tag{48}$$

It follows that $v(t) \to c$ as $t \to \infty$ and also that

$$v(t) \simeq \tilde{a}t \quad \text{when } \tilde{a}t \ll c \tag{49}$$

which is the non-relativistic result.

The position $x(t)$ of the particle is found by integrating $v = dx/dt$:

$$x(t) = \int_0^t v(t')dt' = \left(\sqrt{1 + (\tilde{a}t/c)^2} - 1\right) c^2/\tilde{a}. \tag{50}$$

It follows that $x \to ct$ as $t \to \infty$ and also that

$$x(t) \simeq \frac{1}{2}\tilde{a}t^2 \quad \text{when } \tilde{a}t \ll c \tag{51}$$

which is the nonrelativistic result.

The position $x(v)$ is found by integrating $a(v) = dv/dt = vdv/dx$:

$$x(v) - x(0) = \int_{x(0)}^{v} \frac{v'dv'}{a(v')} = (\gamma - 1)c^2/\tilde{a}. \tag{52}$$

It follows that $x(v) - x(0) \to \infty$ as $v \to c$ and also that

$$v^2 \simeq 2\tilde{a}[x(v) - x(0)] \quad \text{when } v \ll c \tag{53}$$

which is the nonrelativistic result.

When a particle moves with constant speed, that is, when

$$\frac{dv}{dt} = 0 \tag{56}$$

it follows from Eqs. (34) and (39) that

$$\tilde{\omega}\rho = \gamma v \tag{57}$$

where

$$\tilde{\omega} = \frac{qB\sin\theta}{m} \tag{58}$$

where θ is the angle that $\mathbf{v}$ makes with $\mathbf{B}$. For a proton moving perpendicular to a 1.00 T magnetic field, $\tilde{\omega} = 95.8$ MHz.

The right side of Eq. (57) is constant. Accordingly, the radius of curvature ρ of the particle's trajectory changes to accommodate changes in the magnetic field $\mathbf{B}$.

When **B** is constant (that is, time-independent and homogeneous), it follows from Eq. (57) that the particle moves in a circle with radius

$$r = \gamma v/\bar{\omega} \tag{59}$$

and speed

$$v = \bar{\omega} r/\gamma = \frac{\bar{\omega} r}{\sqrt{1 + (\bar{\omega} r/c)^2}}. \tag{60}$$

The angular frequency $\omega = v/r$ of the circular motion is

$$\omega = \bar{\omega}/\gamma = \frac{\bar{\omega}}{\sqrt{1 + (\bar{\omega} r/c)^2}} \tag{61}$$

as above.

It follows that $r \to \infty$ as $v \to c$ and and also that

$$v \simeq \bar{\omega} r \quad \text{when } v \ll c \tag{62}$$

which is the nonrelativistic result.

It follows also that

$$\omega \simeq \bar{\omega} \quad \text{when } \bar{\omega} r \ll c \tag{63}$$

For a proton moving perpendicular to a 1.00 T magnetic field, this requires that $r \ll 3.13$ m.

The above results limit the range of speeds attainable in a conventional particle-accelerating cyclotron which relies, as with Eq. (63), on a constant-frequency accelerating potential to increase particle speeds and a time-independent homogeneous magnetic field to make particles move in circles.

This limitation is overcome at the TRIUMF cyclotron on the UBC campus which accelerates protons to 520 MeV (0.75c), and has a diameter of 17.1 m. This is accomplished by increasing the magnetic field with radius to accommodate the Lorentz factor γ. For more information on TRIUMF, see http://www.triumf.ca.

Príklad 3

__Solution__: Let $\omega = 2\pi f$ and use $c \equiv 1$. Then, $(\omega, \vec{k})$ transforms as a __4-vector__ under Lorentz transformations, with $|\vec{k}| = \omega$.

Let x'-y' axes be fixed with respect to the mirror (moving at $v_x = +v$ with respect to the lab axes x-y). $\left(\gamma \equiv \frac{1}{\sqrt{1-v^2}}\right)$

Incident beam: (mirror frame) $\omega' = \gamma(\omega - v k_x)$ and $k_x' = \gamma(k_x - v\omega)$ and $k_y' = k_y$ as measured in the mirror frame of reference

In the __mirror frame__, the incident + reflected beams have the same frequency and the angle of incidence equals the angle of reflection.

Reflected beam: (mirror frame) $\omega_R' = \omega'$, $k_{Rx}' = -k_x'$, $k_{Ry}' = k_y'$.

We now Lorentz transform the reflected beam's (ω_R', k_R') back to the laboratory frame (which moves at $v_x = -v$ with respect to the mirror):

$$\omega_R = \gamma(\underbrace{\omega_R'}_{\omega'} + v\underbrace{k_{Rx}'}_{-k_x'}) \text{ and } k_{Rx} = \gamma(\underbrace{k_{Rx}'}_{-k_x'} + v\underbrace{\omega_R'}_{\omega'}) \text{ and } k_{Ry} = \underbrace{k_{Ry}'}_{k_y'}.$$

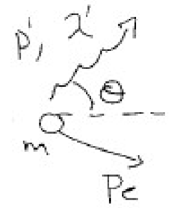
$$\therefore \omega_R = \gamma(\underbrace{\omega'}_{\gamma(\omega - vk_x)} - v\underbrace{k_x'}_{\gamma(k_x - v\omega)}) \text{ and } k_{Rx} = \gamma(-\underbrace{k_x'}_{\gamma(k_x - v\omega)} + v\underbrace{\omega'}_{\gamma(\omega - vk_x)}) \text{ and } k_{Ry} = \underbrace{k_y'}_{k_y}$$

$$\therefore \underbrace{\omega_R}_{2\pi f_R} = \gamma^2[\underbrace{\omega}_{2\pi f}(1+v^2) - 2v\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Rx}}_{(\omega_R\cos\theta_R)} = \gamma^2[2v\omega - (1+v^2)\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Ry}}_{(\omega_R\sin\theta_R)} = \underbrace{k_y}_{(\omega\sin\theta)}.$$

$$\therefore \boxed{\omega_R = \frac{\omega}{(1-v^2)}\left[1 + 2v\cos\theta + v^2\right]} \text{ and } \boxed{\tan\theta_R = \frac{(1-v^2)\sin\theta}{[2v + (1+v^2)\cos\theta]}}$$

Príklad 4

Relativity problem solution



From conservation of energy

$$pc + mc^2 = p'c + \sqrt{p_e^2c^2 + m^2c^4}$$

$$(pc + mc^2 - p'c)^2 = p_e^2c^2 + m^2c^4$$

and conservation of momentum

$$\vec{p} - \vec{p}' = \vec{p}_e \Rightarrow (\vec{p} - \vec{p}')^2 = p_e^2$$

$$\therefore \cancel{p^2} + \cancel{p'^2} + \cancel{m^2c^2} - \cancel{2}pp' - \cancel{2}p'mc + \cancel{2}pmc - \cancel{p^2} - \cancel{p'^2} + \cancel{2}\vec{p}\cdot\vec{p}' = \cancel{m^2c^2}$$

$$pp' - \vec{p}\cdot\vec{p}' = (p - p')mc$$

$$pp'(1 - \cos\theta) = (p - p')mc$$

$$1 - \cos\theta = \left(\frac{1}{p'} - \frac{1}{p}\right)mc$$

where $p = \frac{h}{\lambda}$

$$\therefore \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)}$$

b. from above $\frac{1}{p'c} = \frac{1}{pc} + \frac{1}{mc^2}(1-\cos\theta)$

$$\therefore p'c = \frac{mc^2}{1-\cos\theta + \frac{mc}{p}}$$

and $K = \sqrt{p_e^2c^2 + m^2c^4} - mc^2 = pc - p'c$

but from energy conservation that equals $pc - p'c$ (from 1st equation in a)

$$\therefore \quad K = pc - \frac{mc^2}{1-\cos\theta + \frac{mc}{p}}$$

$$K = \frac{pc(1-\cos\theta + \frac{mc}{p}) - mc^2}{1-\cos\theta + \frac{mc}{p}}$$

$$K = \quad pc(1-\cos\theta)$$

$$K = \frac{pc(1-\cos\theta)}{1-\cos\theta + \frac{mc}{p}}$$

$$\boxed{K = \frac{(1-\cos\theta)\frac{hc}{\lambda}}{1-\cos\theta + \frac{mc\lambda}{h}}}$$