METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima20 – Príklady 4

VZOROVÉ RIEŠENIA

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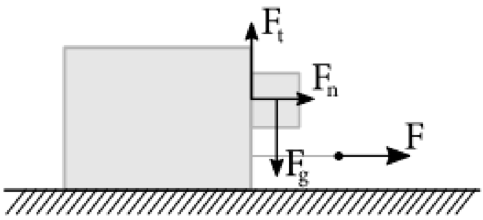
Príklad 1

Prvoradé je vedieť, aké sily pôsobia v našej sústave, špeciálne, aké sily pôsobia na „levitujúcu" kocku.



V horizontálnom smere pôsobí iba normálová sila F_n od veľkej kocky. Pohybová rovnica[1] v horizontálnom smere pre „levitujúcu" kocku má teda tvar

$$ma = F_n.$$

Vo vertikálnom smere pôsobia dve sily, gravitačná $F_g = mg$, a trecia $F_t \leq fF_n$. Keďže nás zaujíma prípad, kde sa „levitujúca" kocka nebude pohybovať vo vertikálnom smere, s určitosťou musí platiť rovnosť síl

$$F_g = F_t \Rightarrow mg \leq fF_n.$$

Posledný dielik puzzle je pohybová rovnica celého systému. Obe kocky sa hýbu so zrýchlením a v horizontálnom smere a jediná vonkajšia sila, ktorá na nich pôsobí je sila F, ktorou pôsobia Žaba s Janom na lano, takže

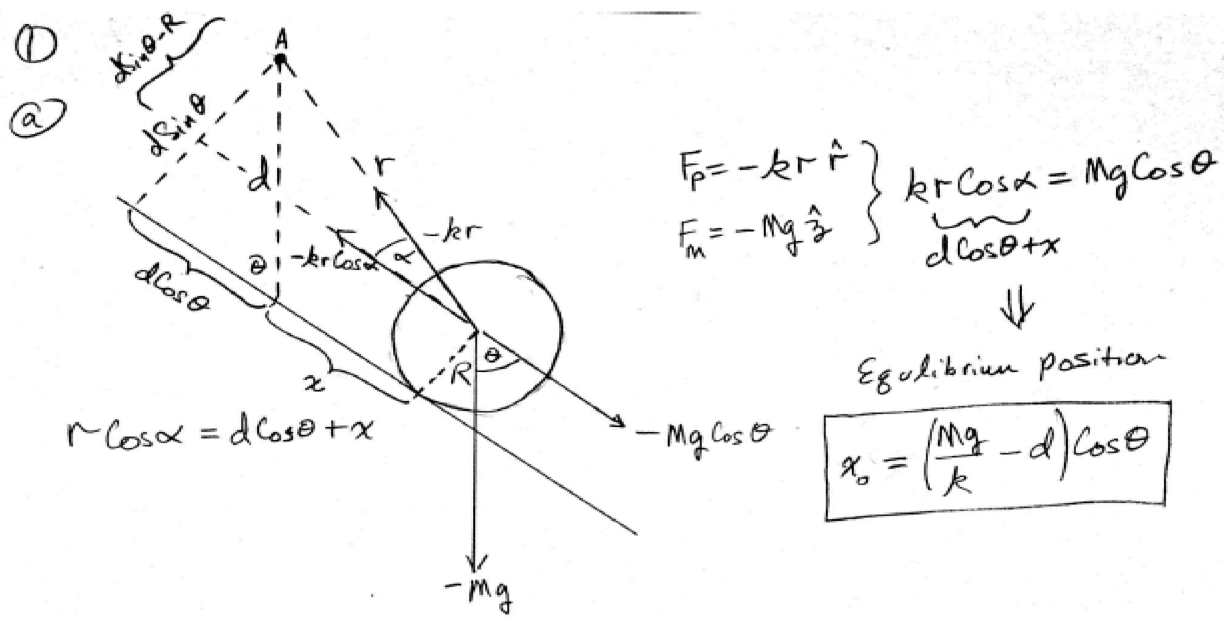
$$(m + M)\,a = F.$$

Spojením všetkých troch pohybových rovníc dostávame podmienku pre silu F

$$F \geq (m + M)\,\frac{g}{f},$$

a teda najmenšiu hodnotu má F vtedy, ak nastane rovnosť.

Príklad 2



(b) (i) $F = -kr \Rightarrow V_p = -\int F\,dr = \frac{1}{2}kr^2$ $\quad V_m = +Mgx\cos\theta$

(ii) $r^2 = (x + d\cos\theta)^2 + (d\sin\theta - R)^2$

(iii) $y = s = R\theta \Rightarrow \dot{y} = \dot{s} = R\dot{\theta}$

(iv) $I = \frac{1}{2}MR^2$

(v) $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2 + \frac{1}{2}M\dot{x} = \frac{3}{4}M\dot{x}^2$

Define $y = x - x_0$

Then $y = x - \left(\frac{Mg}{k} - d\right)\cos\theta$

And $\dot{y} = \dot{x}\ ;\ \ddot{y} = \ddot{x}$

Thus $\mathcal{L} = T - V = \frac{3}{4}M\dot{x}^2 - \frac{1}{2}k\left[(x+d\cos\theta)^2 + (d\sin\theta - R)^2\right] + Mgx\cos\theta$

Using the ELEs $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{3}{2}M\ddot{x} = \frac{\partial \mathcal{L}}{\partial x} = -k(x + d\cos\theta) + Mg\cos\theta$

Hence $\ddot{x} + \frac{2k}{3M}\underbrace{\left(x - \left[\frac{Mg}{k} - d\right]\cos\theta\right)}_{y} = 0 \Rightarrow \ddot{y} + \frac{2k}{3M}y = 0$

$\Rightarrow \boxed{\omega = \sqrt{\frac{2k}{3M}}}$

Príklad 3

a) Since the problem has an aziumuthal symmetry, we have

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)$$

Using that

$$P_0(\cos\theta) = 1$$
$$P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]$$

we obtain

$$\phi(\theta) = \phi_0\cos^2\theta = \frac{\phi_0}{3}[2P_2(\cos\theta) + 1] = \frac{\phi_0}{3}[2P_2(\cos\theta) + P_0(\cos\theta)]$$

And thus

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\,\frac{\phi_0}{3}[2P_2(\cos\theta) + P_0(\cos\theta)]P_m(x)\\ &= \frac{\phi_0}{3}\left[2\frac{2}{2m+1}\delta_{m,2} + \frac{2}{2m+1}\delta_{m,0}\right]\\ &= \frac{\phi_0}{3}\left[\frac{4}{5}\delta_{m,2} + 2\delta_{m,0}\right]\end{aligned}$$

If we want to evaluate the potential inside of the sphere, we need to set $B_l = 0$ and obtain

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\left[\sum_l A_l R^l P_l(x)\right]P_m(x)\\ &= A_m R^m\frac{2}{2m+1}\end{aligned}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = A_2 R^2 \frac{2}{5}$$
$$A_2 = \frac{2\phi_0}{3R^2}$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2A_0$$
$$A_0 = \frac{\phi_0}{3}$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2 P_2(\cos\theta)$$

For the potential outside of the sphere, we set $A_l = 0$ and obtain

$$\int_{-1}^{1} dx\,\phi(R,x)P_m(x) = \int_{-1}^{1} dx\left[\sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta)\right]P_m(x)$$
$$= B_m R^{-(m+1)}\frac{2}{2m+1}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = B_2 R^{-3}\frac{2}{5}$$
$$B_2 = \frac{2\phi_0}{3}R^3$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2\frac{B_0}{R}$$
$$B_0 = \frac{\phi_0}{3}R$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}\frac{R}{r}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

b) The electric field inside the sphere is then given by

$$\vec{E} = -\nabla\phi(r,\theta) = -\hat{r}\frac{\partial\phi(r,\theta)}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}$$
$$= -\hat{r}\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta) - \hat{\theta}\frac{1}{r}\frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2[-3\sin\theta\cos\theta]$$
$$= -\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta)\hat{r} + 2\phi_0\frac{r}{R^2}[\sin\theta\cos\theta]\hat{\theta}$$

c) Using Gauss' law inside the shere

$$\oint \vec{E}\cdot d\vec{A} = -\frac{4\phi_0}{3}\frac{r}{R^2}r^2\int d\varphi\int_{-1}^{1} d(\cos\theta)P_2(\cos\theta) = 0$$

Thus, no charges are contained inside the sphere.

Príklad 4

It is reasonable to assume that P depends on the weight of the helicopter mg, its linear size L, and the density of air ρ, as a certain power-law:

$$P \propto (mg)^{\alpha} \times L^{\beta} \times \rho^{\gamma}.$$

The combination of exponents that gives the correct dimension of P is unique:

$$P \propto (mg)^{3/2} L^{-1} \rho^{-1/2}.$$

Since $mg \propto L^3$, this entails

$$P \propto L^{9/2} L^{-1} = L^{7/2}.$$

Therefore, the required power is $2^{7/2} P = 8\sqrt{2}\, P$.