METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto21 – Príklady 2

VZOROVÉ RIEŠENIA

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Príklad 1

2. (a) Since elements are independent, we can compute the partition function for one element, $Z_1 = 1 + e^{-\beta\epsilon}$, and then compute $Z_N = (1 + e^{-\beta\epsilon})^N$. Since the energy is just proportional to the number of elements in state b, we can compute

$$\langle N_B \rangle = -\frac{\partial \ln Z}{\partial \beta\epsilon} = \frac{N}{e^{\beta\epsilon} + 1}$$

Note that $\langle N_A \rangle = N - \langle N_B \rangle$, so

$$\langle L \rangle = a \langle N_A \rangle + b \langle N_B \rangle = Na + \frac{N(b-a)}{e^{\beta\epsilon} + 1}$$

(b) $\langle L^2 \rangle = \left\langle (aN_A + bN_B)^2 \right\rangle = \left\langle (aN + (b-a)N_B)^2 \right\rangle$

$$= a^2N^2 + 2a(b-a)N \langle N_B \rangle + (b-a)^2 \langle N_B^2 \rangle$$

$$\langle L \rangle^2 = [aN + (b-a) \langle N_B \rangle]^2 = a^2N^2 + (b-a)^2 \langle N_B \rangle^2 + 2a(b-a)N \langle N_B \rangle$$

so

$$\langle L^2 \rangle - \langle L \rangle^2 = (b-a)^2 \left[\langle N_B^2 \rangle - \langle N_B \rangle^2 \right]$$

and

$$\langle N_B^2 \rangle - \langle N_B \rangle^2 = \frac{\partial^2}{\partial(\beta\epsilon)^2} \ln Z = \frac{Ne^{\beta\epsilon}}{(e^{\beta\epsilon} + 1)^2}$$

giving RMS fluctuation

$$\sqrt{\langle L^2 \rangle - \langle L \rangle^2} = \frac{\sqrt{N}(b-a)}{e^{\beta\epsilon/2} + e^{-\beta\epsilon/2}}$$

(c) If we force length to be L we force $N_B = (L - Na)/(b-a)$. The energy is $E = N_B\epsilon$, and the entropy is just the log of the number of ways to choose N_B of the monomers to be in state b, $S = k_B \ln \binom{N}{N_B}$. So,

$$S = k_B \ln \frac{N!}{N_B!(N - N_B)!}$$

which can be written in the large N and N_B limit as

$$S = k_B \left[N_B \ln \frac{N}{N_B} + (N - N_B) \ln \frac{N}{N - N_B} \right]$$

so

$$F = E - ST = N_B\epsilon + k_BT \left[N_B \ln \frac{N_B}{N} + (N - N_B) \ln \frac{N - N_B}{N} \right]$$

where $N_B/N = (L - Na)/[N(b-a)]$ and where $1 - N_B/N = (Nb - L)/[N(b-a)]$.

(d) Force is given by L derivative of free energy at fixed temperature

$$f = \left(\frac{\partial F}{\partial L}\right)_T = \left(\frac{\partial N_B}{\partial L}\right)\left(\frac{\partial F}{\partial N_B}\right)$$

$$f = \left(\epsilon + k_B T \left[\ln N_B/N - \ln(1 - N_B/N)\right]\right)/(b-a)$$

or

$$f = \frac{\epsilon + k_B T \ln \frac{L - Na}{Nb - L}}{b - a}$$

Note force diverges to infinite tension if $L \to Nb$; also force diverges to infinite compression if $L \to Na$.

Príklad 2

(a) Easily checked by substitution.

(b) We find

$$a = \frac{\pi_x + i\pi_y}{\sqrt{2}\,\hbar}\,\ell\,, \quad a^\dagger = \frac{\pi_x - i\pi_y}{\sqrt{2}\,\hbar}\,\ell\,.$$

(c) Let $\varepsilon_1 = \sqrt{2}\hbar v/\ell$, then the Schrödinger equation becomes

$$\frac{\varepsilon}{\varepsilon_1}\psi_A = a^\dagger \psi_B\,, \quad \frac{\varepsilon}{\varepsilon_1}\psi_B = a\psi_A\,, \quad \text{or} \quad \frac{\varepsilon^2}{\varepsilon_1^2}\psi_A = a^\dagger a \psi_A\,, \quad \frac{\varepsilon^2}{\varepsilon_1^2}\psi_B = a a^\dagger \psi_B\,.$$

(d) By the indicated analogy $\varepsilon_n^2/\varepsilon_1^2$ must be a nonnegative integer. We can write the set of energy levels (which includes $\varepsilon_n < 0$) as

$$\varepsilon_n = \text{sign}(n)\frac{\hbar v}{\ell}\,|2n|^{1/2}\,, \quad n \in \mathbb{Z}\,.$$

Príklad 3

(a) Show that the potential $\Phi = VR/r$ satisfies the required boundary conditions on the plane $\mathcal{C}$ separating dielectrics as well as on the sphere.

Solution:

$\boldsymbol{D} = \varepsilon_{1,2}\boldsymbol{E}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi = VR\hat{\boldsymbol{r}}/r^2$.

On the plane: $(D_1)_n = (D_2)_n = 0$ and $(E_1)_t = (E_2)_t = VR/r^2$.

On the sphere: $\Phi|_{r=R} = V$.

(b) Find the free charge density σ on the surface of the conducting sphere and the total amount of free charge Q on it.

Solution: $\sigma = D_r = \varepsilon_{1,2}V/R$.

Total charge $Q = 2\pi R^2(\sigma_1 + \sigma_2) = 2\pi(\varepsilon_1 + \varepsilon_2)VR$.

(c) Find the bound charge densities σ_b on the spherical boundaries $\mathcal{A}$ and $\mathcal{B}$ of the dielectrics.

Solution: $\sigma_b|_{\mathcal{A},\mathcal{B}} = -P_r = (\varepsilon_0 E - D)_r = (\varepsilon_0 - \varepsilon_{1,2})E_r = -(\varepsilon_{1,2} - \varepsilon_0)V/R$.

(d) Find the bound charge density σ_b on the flat boundary $\mathcal{C}$ between the dielectrics.

Solution: ($[\ldots]$ denotes discontinuity) $\sigma_b|_\mathcal{C} = -[P_n] = [\varepsilon_0 E_n] = 0.$

Príklad 4

In cylindrical coordinates the length element is $ds = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}$ and since the mountain is described by the equation $z = -r$ then the length element on the mountain is given by

$$ds = \sqrt{2dr^2 + r^2 d\phi^2}. \tag{1}$$

To find the optimal path we need to minimize the functional

$$\mathcal{P}[r(\phi)] = \int \sqrt{2dr^2 + r^2 d\phi^2} = \int d\phi \, \sqrt{r^2 + 2\dot{r}^2} = \int d\phi \, \mathcal{L}(r, \dot{r}, \phi) . \tag{2}$$

Consequently, the shortest path is the solution of

$$\frac{d}{d\phi}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{3}$$

$$\frac{2\ddot{r}}{\sqrt{r^2 + 2\dot{r}^2}} - \frac{2\dot{r}\left(\dot{r}r + 2\ddot{r}\dot{r}\right)}{\left(r^2 + 2\dot{r}^2\right)^{3/2}} - \frac{r}{\sqrt{r^2 + 2\dot{r}^2}} = 0 \tag{4}$$

$$2\ddot{r}\left(r^2 + 2\dot{r}^2\right) - 2\dot{r}\left(\dot{r}r + 2\ddot{r}\dot{r}\right) - r\left(r^2 + 2\dot{r}^2\right) = 0 \tag{5}$$

$$r^2 + 4\dot{r}^2 - 2\ddot{r}r = 0. \tag{6}$$

With $r = 1/u$ then $\dot{r} = -\dot{u}/u^2$ and $\ddot{r} = 2\dot{u}^2/u^3 - \ddot{u}/u^2$ which leads to

$$\frac{1}{u^2} + \frac{4\dot{u}^2}{u^4} - \frac{2}{u}\left(\frac{2\dot{u}^2}{u^3} - \frac{\ddot{u}}{u^2}\right) = 0, \quad \frac{1}{u^2} + \frac{2\ddot{u}}{u^3} = 0, \quad \ddot{u} + \tfrac{1}{2}u = 0. \tag{7}$$

The general solution for the optimal path is thus $u(\phi) = A\cos\frac{\phi}{\sqrt{2}} + B\sin\frac{\phi}{\sqrt{2}}$. From the initial position constraint $(r_0, \phi_0) = (1, 0)$, we obtain $u(0) = A$ which leads to $A = 1$. Similarly, from the final position constraint $(r_1, \phi_1) = (1, \pi)$ we obtain $u(\pi) = \cos\frac{\pi}{\sqrt{2}} + B\sin\frac{\pi}{\sqrt{2}}$ giving $B = \left(1 - \cos\frac{\pi}{\sqrt{2}}\right)/\sin\frac{\pi}{\sqrt{2}}$. Putting everything together we obtain that the shortest path to the refuge is the one described by the equation

$$r(\phi) = \cos\tfrac{\pi}{2\sqrt{2}} \sec\tfrac{\pi - 2\phi}{2\sqrt{2}}. \tag{8}$$