METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima21 – Príklady 1

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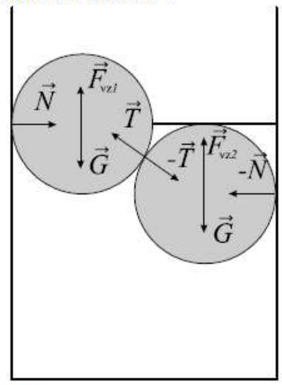
Príklad 1

$\boxed{15.}$ Na každé brvno pôsobí sila zo strany nádoby $\vec{N}$, tiažová sila $\vec{G}$, vztlaková sila $\vec{F}_{vz}$ a sila zo strany druhého brvna $\vec{T}$.



Smer sily $\vec{T}$ určíme z pravouhlého trojuholníka, ktorého prepona je spojnicou stredov kružníc a jeho jedna odvesna je kolmá na hladinu. Prepona má dĺžku $2r$, odvesna dĺžku r, teda sila $\vec{T}$ zviera s horizontálou uhol 30°. V rovnováhe je súčet síl nulový. Pre ľavé brvno dostávame rovnice

$$\begin{aligned} V\rho_0 - mg - T\sin 30^\circ &= 0 \\ N - T\cos 30^\circ &= 0 \end{aligned}$$

Pre pravé brvno

$$\frac{V}{2}\rho_0 g - mg + T\sin 30^\circ = 0$$

druhá rovnica je identická ako pre ľavé brvno. Riešením týchto rovníc dostávame $T = mg/\sqrt{3}$.

Príklad 2

Let r be the distance of R_1 and R_2 to B. Given $r \gg \lambda$.

Let amplitude of waves be E_0

At R_1: $E_{10} = E_0 e^{ik(r-\lambda/2)} + E_0 e^{ikr} + E_0 e^{ik(r+\lambda/2)} + E_0 e^{ik\sqrt{r^2+\lambda^2/4}}$

At R_2: $E_{20} = E_0 e^{ikr} + E_0 e^{ik(r+\lambda/2)} + 2E_0 e^{ik\sqrt{r^2+\lambda^2/4}}$

But $k\lambda = 2\pi \Rightarrow e^{ik\lambda/2} = \underline{-1}$

For $r \gg \lambda$, $\sqrt{r^2+\lambda^2/4} \approx r \Rightarrow e^{ik\sqrt{r^2+\lambda^2/4}} \to e^{ikr}$

Plugging in

$E_{10} \approx 0$ and $E_{20} \approx 2E_0 e^{ikr}$

Intensity $I \propto |E|^2$

$I_1 = 0$ and $I_2 = 4E_0^2$

R_2 picks up greater signal.

(b) If B is turned off

$E_1 \approx -E_0 e^{ikr}$ $E_2 \approx E_0 e^{ikr}$

$I_1 = I_2 \sim E_0^2$

Two receivers pick up same intensity

(c) If source D is turned off

$E_1 \approx -E_0 e^{ikr}$ $E_2 \approx 3E_0 e^{ikr}$

$I_1 \sim E_0^2$ $I_2 \sim 9E_0^2$

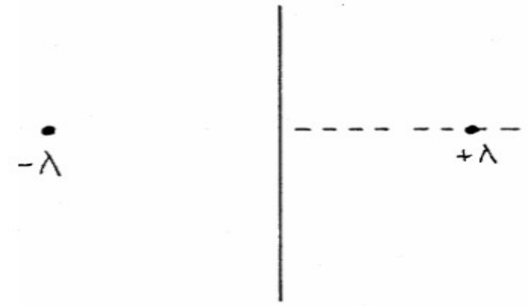
R_2 picks up nine times greater signal.

(d) R_2 is the only receiver sensitive to the B and D source conditions.

Príklad 3

a) Because $a \ll d$, we can treat the wire as if it is a carrier of charge of linear density λ.

Use the method of images to account for the induced charge on the surface of the conducting sheet, so imagine linear charge density $-\lambda$ a distance d past the conducting sheet.



Then, if we choose the electrostatic potential, φ, to be zero on the sheet, along the line passing through the charges,

$$\Delta\varphi = -\int_0^{d-a} dx\; E(x)$$

$$E(x) = \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{-x+d} + \frac{1}{x+d}\right)$$

$$\text{So}\quad \Delta\varphi = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(x+d) - \ln(d-x)\right)\Big|_0^{d-a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(\frac{2d-a}{d}\right) + \ln\left(d/a\right)\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right)$$

The capacitance per unit length is the charge per unit length $/ |\Delta\varphi|$

$$\frac{C}{L} = \lambda / |\Delta\varphi| = \frac{2\pi\varepsilon_0}{\ln\left(\frac{2d-a}{a}\right)}$$

b) The magnitude of the electric field from the wire at $+d$ at the surace of the plane is

$$|E_+(y)| = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\sqrt{d^2+y^2}}$$

The component $\perp$ to the plane is the above multiplied by $d/\sqrt{d^2+y^2}$.

The components $\parallel$ to the plane from the wire and its image cancel of course & the $\perp$ component is doubled:

$$|E| = \frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2} \quad\rightarrow\quad E(y) = -\frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2}\,\hat{x}$$

Then, the charge density is $\sigma = E \varepsilon_0$

So $\sigma(y) = \frac{-\lambda d}{\pi(d^2+y^2)}$

PRÍKLAD 3: overenie, že $\int_{-\infty}^{\infty} dy\, \sigma(x,y) = -\lambda$

(premyslieť prečo?)

$$-\int_{-\infty}^{\infty} dy \frac{\lambda d}{\pi(d^2+y^2)} = -\frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{dy}{d} \frac{1}{1+\frac{y^2}{d^2}} = -\frac{\lambda}{\pi} \arctan \frac{y}{d} \Big|_{-\infty}^{\infty} =$$

$$= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \boxed{-\lambda}$$

Príklad 4

a) Integrals of motion for a central potential $V(r)$:

Angular momentum $L = r V_t = r^2 \dot{\phi}$ (V_t = tangential velocity)

Energy per unit mass $E = \frac{1}{2}(\dot{r}^2 + V_t^2) + V(r) = \frac{1}{2}\dot{r}^2 + V_{eff}(r)$,

$V_{eff}(r) = V(r) + \frac{L^2}{2r^2}$.

Circular orbit: $\dot{r} = 0 \rightarrow \frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{dV}{dr} = \frac{L^2}{r^3} = \frac{V_t^2}{r} = r\dot{\phi}^2$

$\Rightarrow \dot{\phi} = \omega_\phi = \frac{L}{r^2} = \left(\frac{1}{r}\frac{dV}{dr}\right)^{1/2}$, $\boxed{P_\phi = \frac{2\pi}{\omega_\phi} = 2\pi\left(\frac{1}{r}\frac{dV}{dr}\right)^{-1/2}}$

b) $r(t) = r_0 + \epsilon(t)$ with $(dV_{eff}/dr)(r_0) = 0$, $\epsilon^2 \ll r_0^2$.

Energy per unit mass: $E = \frac{1}{2}\dot{\epsilon}^2 + V_{eff}(r_0+\epsilon)$.

Taylor expand: $V_{eff}(r_0+\epsilon) = V_{eff}(r_0) + \cancelto{0}{\left(\frac{dV_{eff}}{dr}\right)_{r_0}}\epsilon + \frac{1}{2}\left(\frac{d^2V_{eff}}{dr^2}\right)_{r_0}\epsilon^2 + O(\epsilon^3)$

$\therefore \; E - V_{eff}(r_0) = \frac{1}{2}\dot{\epsilon}^2 + \frac{1}{2}\omega_r^2\epsilon^2 + O(\epsilon^3) = \text{constant}$

$\boxed{\omega_r \equiv \left(\frac{d^2V_{eff}}{dr^2}\right)^{1/2}_{r_0}}$ this is the simple harmonic oscillator equation, and its general solution is

$\boxed{\epsilon(t) = \frac{\sqrt{2(E-E_0)}}{\omega_r}\cos[\omega_r(t-t_0)]}$ where $E_0 \equiv V_{eff}(r_0)$ and t_0 is an arbitrary constant.

Now write ω_r in terms of $V(r)$ instead of $V_{eff}(r)$:

$$\omega_r^2 = \frac{d^2V_{eff}}{dr^2} = \frac{d^2V}{dr^2} + \frac{3L^2}{r^4} = \frac{d^2V}{dr^2} + 3\omega_\phi^2 = \frac{d^2V}{dr^2} + \frac{3}{r}\frac{dV}{dr}$$

$$\boxed{\omega_r = \left[\frac{d^2V}{dr^2} + \frac{3}{r}\frac{dV}{dr}\right]^{1/2}_{r_0} = \left[\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right)\right]^{1/2}_{r_0}} \quad \boxed{\text{Radial period } P_r = \frac{2\pi}{\omega_r}}$$

c) Stability is determined by the sign of ω_r^2: $\omega_r^2 > 0$ for stability.

$\omega_r^2 = \frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right)$, $V(r) = -\frac{GM}{r}e^{-kr}$

$\rightarrow \omega_r^2 = \frac{GM}{r^3}e^{-kr}\left[1 + kr - (kr)^2\right]$

$1 + kr - (kr)^2 = \left(\frac{\sqrt{5}-1}{2} + kr\right)\left(\frac{\sqrt{5}+1}{2} - kr\right) > 0$ only if $kr < \frac{\sqrt{5}+1}{2}$

$\therefore$ $\boxed{\text{The circular orbits are unstable for } kr > \frac{\sqrt{5}+1}{2}}$