METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima21 – Príklady 3

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Príklad 1

You start to skid when the required force to accelerate along your path is equal to the maximum possible friction force:

$$m|\ddot{\mathbf{r}}| = \mu m g \tag{1}$$

As we'll see the point of skidding is quite close to the pole, so we can neglect the Earth's curvature and assume we are on a flat surface of a skate rink. Let's work in cylindrical coordinates with origin at the pole, radius r and angle ϕ. The radius-vector is $\mathbf{r} = r\hat{r}$ and we have to remember that the unit vectors in cylindrical coordinates are function of angle: $\hat{r} = \hat{r}(\phi)$ and $\hat{\phi} = \hat{\phi}(\phi)$. This means we have to differentiate them as well, when finding the acceleration. Velocity is

$$\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\dot{\phi} = \dot{r}\hat{r} + r\hat{\phi}\dot{\phi}$$

and since the velocity is always pointing in NW direction (45 degrees to both parallels and meridians) we can write

$$\dot{r} = -\frac{v}{\sqrt{2}} \qquad r\dot{\phi} = -\frac{v}{\sqrt{2}}$$

For acceleration we find

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}] = [\ddot{r} - r\dot{\phi}^2]\hat{r} + [\dot{r}\dot{\phi} + \frac{d}{dt}(r\dot{\phi})]\hat{\phi}$$

Using the values of velocity components and their time-invariance, we get

$$\ddot{\mathbf{r}} = -r\dot{\phi}^2\hat{r} + \dot{r}\dot{\phi}\hat{\phi} = -\frac{v^2}{2r}\hat{r} - \frac{v^2}{2r}\hat{\phi} \qquad \Rightarrow \qquad |\ddot{r}| = \frac{v^2}{2r}\sqrt{2}$$

which means that the distance where skidding starts is

$$\boxed{R = \frac{v^2}{\sqrt{2}\mu g} = 649\ m\,.}$$

Príklad 2

Solution

a) $r > R$ $\quad \phi_o(r\theta) = -E_o r\cos\theta + \sum_0^\infty a_n r^{-(n+1)} P_n(\cos\theta)$

$S \le r \le R$ $\quad \phi_i(r\theta) = \sum_0^\infty \left(b_n r^n + c_n r^{-(n+1)}\right) P_n$

You can see only terms 0,1 are required to match boundary conditions.

$\phi_i(S\ \theta) = 0 \Rightarrow b_0 + \frac{c_0}{S} = 0 \qquad c_0 = -b_0 S \qquad P_0$

$b_1 S + \frac{c_1}{S^2} = 0 \qquad c_1 = -b_1 S^3 \qquad P_1$

$\underline{\phi_o(R) = \phi_i(R)}$ $\qquad b_o(1 - S/R) = a_o/R \qquad P_o$

① $\boxed{b_1 R\left(1 - \frac{S^3}{R^3}\right) = -E_o R + \frac{a_1}{R^2}} \qquad P_1$

$\underline{-\frac{\partial \phi_o}{\partial r}\Big|_R = -\varepsilon_r \frac{\partial \phi_i}{\partial r}\Big|_R}$ (D-field normal component continuous)

$-S/R^2\, b_o \varepsilon_r = a_o/R^2 \qquad P_o$

② $\boxed{-b_1 \varepsilon_r - 2\frac{S^3}{R^3} b_1 \varepsilon_r = E_o + \frac{2a_1}{R^3}} \qquad P_1$

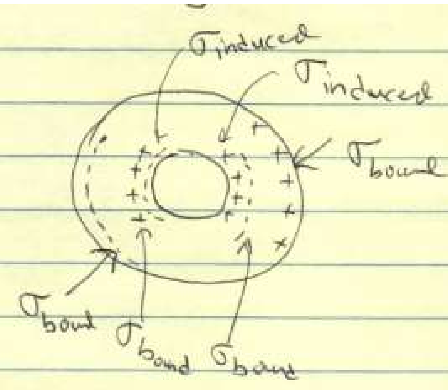
$\Rightarrow a_o = b_o = 0$

From ①+② $b_1 = \dfrac{-E_o}{1 - \frac{S^3}{R^3} - \frac{1}{3}\left[(1-\varepsilon_r) - \frac{S^3}{R^3}(1+2\varepsilon_r)\right]}$

Ans 2 $\quad \phi_o(r\theta) = -E_o r\cos\theta \left\{ 1 + \dfrac{\frac{R^3}{3r^3}\left[(1-\varepsilon_r) - \frac{S^3}{R^3}(1+2\varepsilon_r)\right]}{1 - \frac{S^3}{R^3} - \frac{1}{3}\left[(1-\varepsilon_r) - \frac{S^3}{R^3}(1+2\varepsilon_r)\right]} \right\}$

A quick check shows that for $S \to 0$ this approaches the dielectric sphere in uniform $\vec{E}$. And for $R \to S$, $\varepsilon_r \to 1$ it approaches the conducting shell in uniform $\vec{E}$.

b)



All charges decreasing towards $\theta = \pi/2$

Príklad 3

The x, y, z coordinates of masses m, m and M are $(-l\sin\theta, 0, -l\cos\theta)$, $(l\sin\theta, 0, -l\cos\theta)$ and $(0, 0, -2l\cos\theta)$.

The velocity is given by $\dot{\vec{r}}' = \dot{\vec{r}} + \vec{\omega}_0 \times \vec{r}$

Note that $\vec{\omega}_0 = (0, 0, \omega_0)$.

The corresponding velocities are

$(-l\dot{\theta}\cos\theta, l\omega_0\sin\theta, l\dot{\theta}\sin\theta)$,

$(l\dot{\theta}\cos\theta, -l\omega_0\sin\theta, l\dot{\theta}\sin\theta)$,

$(0, 0, -2l\dot{\theta}\sin\theta)$

Kinetic energy is

$$T = ml^2\omega_0^2\sin^2\theta + m^2l^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta$$

Potential energy is

$$V = -2mgl\cos\theta - 2Mgl\cos\theta$$

$$L = T - V = ml^2\omega_0^2\sin^2\theta + ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta + 2(m+M)gl\cos\theta \quad (1)$$

Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$2(m + 2M\sin^2\theta)l\ddot{\theta} - 2Ml\dot{\theta}^2\sin 2\theta - ml\omega_0^2\sin 2\theta + 2(m+M)g\sin\theta = 0$$

At equilibrium, $\ddot{\theta} = \dot{\theta} = 0$ and $\theta_0 = \theta$.

$$ml\omega_0^2\sin 2\theta_0 = 2(m+M)g\sin\theta_0$$

Two solutions:

(i) $\theta_0 = 0$

(ii) $\cos\theta_0 = \dfrac{(m+M)g}{ml\omega_0^2}$

$m\lambda \omega_0$

The distance of mass M from the top are

(i) $\theta_0 = 0 \Rightarrow 2l$

(ii) $2\lambda \cos\theta_0 = \dfrac{2(m+M)g}{m\omega_0^2}$

(b) For small oscillations,

$\theta' = \theta - \theta_0$ and $\dot{\theta}' = \dot{\theta}$ and $\theta' \ll \theta_0$

$$\sin\theta \approx \sin\theta_0 + \theta' \cos\theta_0$$

$$\sin 2\theta \approx \sin 2\theta_0 + 2\theta' \cos 2\theta_0$$

Keeping only first order terms, the equation of motion becomes

$$2(m+2M\sin^2\theta_0)\lambda\ddot{\theta}' - m\lambda\omega_0^2 \sin 2\theta_0 - 2m\lambda\omega_0^2\theta'\cos 2\theta_0$$
$$+ 2(m+M)g\sin\theta_0 + 2(m+M)g\,\theta'\cos\theta_0 = 0$$

2^{nd} and 4^{th} term cancel due to equilibrium condition.

$$(m+2M\sin^2\theta_0)\lambda\ddot{\theta} + [(m+M)g\cos\theta_0 - m\lambda\omega_0^2\cos 2\theta_0]\theta' = 0$$

Oscillation frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{(m+M)g\cos\theta_0 - m\lambda\omega_0^2\cos 2\theta_0}{(m+2M\sin^2\theta_0)\lambda}}$$