METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto21 – Príklady 5

VZOROVÉ RIEŠENIA

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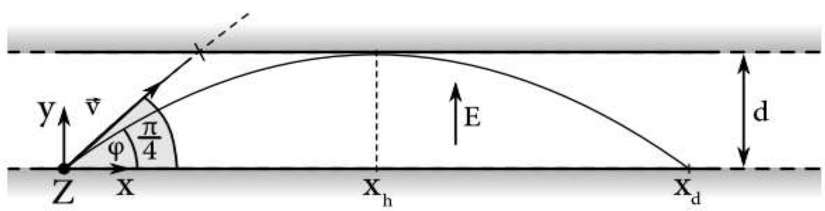
Príklad 1

22 Keďže uvažované elektrické pole je homogénne, na všetky vylietavajúce elektróny bude smerom nadol pôsobiť konštantná sila veľkosti Ee. Tá bude udeľovať elektrónom zrýchlenie $a = \frac{Ee}{m_e}$. Môžme si tiež uvedomiť, že gravitačná sila pôsobiaca na elektróny je oproti elektrickej sile zanedbateľná. Celá sústava je rotačne súmerná, takže vo výpočte sa stačí obmedziť na jeden rez obsahujúci zdroj. Analogicky k šikmému vrhu v homogénnom gravitačnom poli môžeme popísať súradnice elektrónu uniknuvšieho zo zdroja pod uhlom φ

$$x = v_x t = v \cos\varphi \cdot t,$$

$$y = v_y t - \frac{1}{2}at^2 = v \sin\varphi \cdot t - \frac{1}{2}at^2.$$



V závislosti na počiatočnej rýchlosti a elevačnom uhle elektrónu môžu vo všeobecnosti nastať tri prípady:

- elektrón narazí do hornej dosky kondenzátora;
- elektrón sa obtrie o hornú dosku kondenzátora;
- elektrón nedosiahne dostatočnú výšku na to, aby sa dotkol hornej dosky kondenzátora.

Pozrime sa na elektrón, ktorý sa práve obtrie o hornú dosku. Letiaci elektrón dosiahne svoju maximálnu výšku po čase

$$T = \frac{v_y}{a} = \frac{v \sin\varphi}{a}.$$

Ak tento čas dosadíme do vzorca pre y-súradnicu a položíme ju rovnú d, zistíme pod akým uhlom musí byť vypustený elektrón, aby svoju maximálnu výšku dosiahol práve vo výške dosky, čo teda znamená, že sa o ňu obtrie.

$$d = \frac{v^2 \sin^2\varphi}{a} - \frac{v^2 \sin^2\varphi}{2a} = \frac{v^2 \sin^2\varphi}{2a},$$

$$\varphi = \arcsin\frac{\sqrt{2ad}}{v}.$$

Ak do tohoto vzťahu dosadíme zadané hodnoty, vyjde uhol $\varphi \doteq 36{,}37°$. Tento uhol je menší ako 45°, čo znamená, že všetky elektróny, ktoré by mohli doletieť ďalej, narazia do hornej dosky. Ak dosadíme tento uhol spolu s časom T do rovnice pre x-súradnicu, zistíme v akej vzdialenosti od stredu sa budú elektróny obtierať o hornú dosku

$$x_h = \frac{v^2 \cos\varphi \sin\varphi}{a}.$$

Vieme teda, že vrchol trajektórie takéhoto elektrónu je vo vodorovnej vzdialenosti x_h od zdroja. Ak si uvedomíme, že elektrón dosiahne vrchol trajektórie presne v polovici prejdenej vodorovnej vzdialenosti, zistíme, že maximálny dolet na spodnej doske je $x_d = 2x_h$.

Po dosadení hodnôt zo zadania dostávame $x_h = 2{,}715$ m. Ak vezmeme do úvahy rotačnú súmernosť úlohy, zistíme, že elektróny budú na hornú dosku dopadať do kruhu s polomerom 2,715 m a na dolnú dosku do kruhu s polomerom 5,43 m. Pre celkovú plochu dopadu teda dostávame výsledok

$$S = \pi\left(x_h^2 + x_d^2\right) = 5\pi x_h^2 \doteq 116\ \text{m}^2.$$

Príklad 2

Príklad 3

Electron has maximum momentum when neutrinos move in opposite direction

$\longleftarrow p_{\bar{\nu}_e}$

$\longleftarrow p_{\nu_\mu} \qquad \longrightarrow p_e$ in μ rest frame, $c=1$

Let $\vec{p}_\nu = \vec{p}_{\nu_e} + \vec{p}_{\nu_\mu}$. max when $\vec{p}_\nu = -\vec{p}_e$

$|\vec{p}_\nu| = |\vec{p}_e| \equiv p$

$m_\mu = E_\nu + E_e \qquad E_\nu = p_\nu \quad (\nu \text{ massless})$

$m_\mu = p + (p^2 + m_e^2)^{1/2}$

$(m_\mu - p)^2 = p^2 + m_e^2 \qquad m_\mu^2 - 2m_\mu p + p^2 = p^2 + m_e^2$

$$p = \frac{m_\mu^2 - m_e^2}{2m_\mu}$$

$$E_e = (p^2 + m_e^2)^{1/2} = \left[\frac{m_\mu^4 - 2m_\mu^2 m_e^2 + m_e^4}{4m_\mu^2} + m_e^2\right]^{1/2}$$

$$= \left[\frac{(m_\mu^2 + m_e^2)^2}{4m_\mu^2}\right]^{1/2} = \frac{m_\mu^2 + m_e^2}{2m_\mu}$$

Príklad 4

Let y be the distance above the horizontal. The potential energy of the cylinder is just mgy and of the mass point is $mgy + mga\cos\theta/2$ and $y = a\theta\sin\alpha$, so the total potential energy is

$$V(\theta) = mg\left(2a\theta\sin\alpha + a\cos\theta/2\right).$$

The cylinder will roll freely when there are no stable points:

$$\begin{aligned}\frac{dV}{d\theta} &= mg\left(2\sin\alpha - \frac{1}{2}\sin\theta\right) = 0 \rightarrow \\ 4\sin\alpha &= \sin\theta\end{aligned}$$

so when $\sin\theta > 1/4$, the cylinder begins to roll.

Alternatively, one may take torques around the contact point.

Príklad 5

Solution 1.4. We will use the normal modes of the system to solve this problem. First we find the motion of the normal-mode coordinates subject to the applied force, and then transform from those coordinates to the ordinary spatial coordinates of the blocks. With some physics insight we could immediately write down the normal modes of the system (10.27). However, it is instructive to solve methodically for the normal modes, as we do below.

Let η_i be the displacement of block i from equilibrium. The potential energy of the system is

$$\mathcal{V} = \frac{1}{2}k(\eta_1 - \eta_2)^2 + \frac{1}{2}k(\eta_2 - \eta_3)^2, \tag{10.17}$$

and the kinetic energy is

$$\mathcal{T} = \frac{1}{2}m(\dot{\eta_1}^2 + \dot{\eta_2}^2 + \dot{\eta_3}^2). \tag{10.18}$$

The Lagrangian is $L = (\mathcal{T} - \mathcal{V})$, which we can write as

$$L = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j), \tag{10.19}$$

where

$$\mathbf{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}. \tag{10.20}$$

For our simple problem it may seem that writing $\mathbf{T}$ as a matrix is unnecessary and heavy-handed. However, it is useful to do this so that we could easily generalize to the case in which the masses of the particles are not equal.

Using Lagrange's equation, we find the equations of motion:

$$\mathbf{T}\ddot{\boldsymbol{\eta}} + \mathbf{V}\boldsymbol{\eta} = 0, \tag{10.21}$$

where we have defined the vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^T$. The normal modes are collective motions where all three blocks move with the same frequency. Since there are three degrees of freedom there will be three normal modes. For each one the solution is of the form

$$\boldsymbol{\eta}(t) = \mathbf{a}_j e^{i\omega_j t}, \tag{10.22}$$

where the $\mathbf{a}_j$ are time-independent. If we insert this form for $\boldsymbol{\eta}(t)$ into the equations of motion (10.21), we get a matrix equation for the vector $\mathbf{a}_j$:

$$(\mathbf{V} - \omega_j^2\mathbf{T})\mathbf{a}_j = 0. \tag{10.23}$$

In order for a nontrivial solution to exist, we must have

$$\det\left[\mathbf{V} - \omega_j^2\mathbf{T}\right] = 0. \tag{10.24}$$

This leads to a cubic equation in ω^2, with roots $\omega_1^2 = 0$, $\omega_2^2 = k/m$, and $\omega_3^2 = 3k/m$. Substituting these frequencies into equation (10.23) allows us to solve for the three normal modes, for which we choose the normalization prescription

$$\mathbf{a}_i^T\,\mathbf{T}\,\mathbf{a}_i = 1 \quad \text{(no summation on i)}. \tag{10.25}$$

In fact it can be shown (see Goldstein, chapter 6) that the vectors $\mathbf{a}_i$ may be chosen to satisfy the "orthogonality" condition

$$\mathbf{a}_i^T\,\mathbf{T}\,\mathbf{a}_j = \delta_{ij}, \tag{10.26}$$

and we will use this later on. Subject to this condition, our normal modes are

$$\mathbf{a}_1 = \frac{1}{\sqrt{3m}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \mathbf{a}_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \text{and} \quad \mathbf{a}_3 = \frac{1}{\sqrt{6m}}\begin{pmatrix}1\\-2\\1\end{pmatrix}. \tag{10.27}$$

We can use these vectors as a basis set to write an arbitrary displacement as

$$\boldsymbol{\eta}(t) = \xi_1 \mathbf{a}_1 + \xi_2 \mathbf{a}_2 + \xi_3 \mathbf{a}_3, \tag{10.28}$$

where the ξ_i are called normal coordinates.

Suppose we now apply a force $\mathbf{F}(t)$. Our equations of motion are

$$\mathbf{T}\left(\sum_{i=1}^{3} \ddot{\xi}_i \mathbf{a}_i\right) + \mathbf{V}\left(\sum_{i=1}^{3} \xi_i \mathbf{a}_i\right) = \mathbf{F}(t). \tag{10.29}$$

We use the matrix equation for a normal mode vector, (10.23), to rewrite $\mathbf{V}\mathbf{a}_i$ as $\omega_i^2 \mathbf{T}\,\mathbf{a}_i$. If we now multiply on the left by $\mathbf{a}_j^T$ and use the orthogonality condition (10.26), the normal modes decouple (which is why they are called normal modes) and we obtain the equations of motion for the normal coordinates:

$$\ddot{\xi}_j + \omega_j^2 \xi_j = f_j(t), \tag{10.30}$$

where we have defined

$$f_j(t) = \mathbf{a}_j^T \mathbf{F}(t). \tag{10.31}$$

In our particular problem, the force is given by

$$\mathbf{F} = \begin{pmatrix} f \cos \omega t \\ 0 \\ 0 \end{pmatrix}. \tag{10.32}$$

This gives us

$$\begin{aligned} f_1 &= \frac{1}{\sqrt{3m}} f \cos \omega t, & (10.33)\\ f_2 &= \frac{1}{\sqrt{2m}} f \cos \omega t, & (10.34)\\ f_3 &= \frac{1}{\sqrt{6m}} f \cos \omega t. & (10.35) \end{aligned}$$

It is now straightforward to solve the equations of motion (10.30), subject to the initial conditions

$$\dot{\xi}_i = 0, \quad \xi_i = 0. \tag{10.36}$$

The solution is

$$\xi_1 = \frac{f}{\sqrt{3m}\,\omega^2}(1-\cos\omega t), \quad (10.37)$$

$$\xi_2 = \frac{f}{\sqrt{2m}(\omega_2^2-\omega^2)}(\cos\omega t - \cos\omega_2 t), \quad (10.38)$$

$$\xi_3 = \frac{f}{\sqrt{6m}(\omega_3^2-\omega^2)}(\cos\omega t - \cos\omega_3 t). \quad (10.39)$$

(This can be verified by substitution.) Next we substitute the normal coordinates back into equation (10.28), to find the motion of mass C:

$$\eta_3 = \frac{f}{m}\left[\frac{1}{3\omega^2}(1-\cos\omega t) - \frac{1}{2(\omega_2^2-\omega^2)}(\cos\omega t - \cos\omega_2 t) + \frac{1}{6(\omega_3^2-\omega^2)}(\cos\omega t - \cos\omega_3 t)\right]. \quad (10.40)$$

Príklad 6

In the lab frame:

$E_z = 4\pi\sigma \qquad E_x = E_y = 0$
$B_x = B_y = B_z = 0$

In moving observer's frame:

$$\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma\,(E_y - \beta B_z) \\ E_z' &= \gamma\,(E_z + \beta B_y) \\ B_x' &= B_x \\ B_y' &= \gamma\,(B_y + \beta E_z) \\ B_z' &= \gamma\,(B_z - \beta E_y) \end{aligned} \quad \Rightarrow \quad \begin{aligned} E_x' &= E_y' = 0 \\ E_z' &= \gamma E_z = 4\pi\gamma\sigma \\ B_x' &= B_z' = 0 \\ B_y' &= \gamma\beta E_z = 4\pi\gamma\beta\sigma \end{aligned}$$

Notice that $E_z' = \gamma E_z = 4\pi\gamma\sigma$ is basically because the moving observer sees measured charge density due to Lorentz contraction.