METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto21 – Príklady 6

VZOROVÉ RIEŠENIA

Cvičenie 6.5.2021

Príklad 1

Assume at time t the bug is distance $x(t)$ from the wall. At the same time the free end of the band is distance $L + Vt$ from the wall and is still moving with speed V. Since the band stretches uniformly, the speed of stretching at the position of the bug is

$$V \frac{x(t)}{L + Vt}$$

and relative to the wall the bug has speed

$$V \frac{x(t)}{L + Vt} - u = \frac{dx(t)}{dt}$$

Making substitution

$$x(t) = (L + Vt) f(t)$$

we have equation for $f(t)$:

$$V f(t) + (L + Vt) \frac{df}{dt} = V f(t) - u \qquad \Rightarrow \qquad \frac{df}{dt} = -\frac{u}{L + Vt}$$

which is easy to solve with initial condition $f(0) = 1$,

$$f(t) = 1 - \frac{u}{V} \ln \frac{L + Vt}{L}$$

The bug reaches wall when $f(t) = 0$ which will happen at time

$$\boxed{t = \frac{L}{V} \left(e^{V/u} - 1 \right)}$$

Bug will always reach the wall, even if the free end of the band is at first pulled faster than it can crawl! It will take the bug exponentially long time, but with every step the bug will be entering region where the "stretching wind" blows slightly slower against it. Of course for very fast bug we have $V/u \ll 1$ and $t = L/u$, as expected.

"This is a nifty analogy to kinematics in an expanding universe and demonstrates how we eventually recieve light that is traveling at $u = c$ from a point that is constantly moving away from us at a speed V greater than c.

... in a uniformly expanding universe, if light, moving at $u = c$, is emitted one billion years after the Big Bang from a point that has always been expanding away from us at $V = 2c$ (so it starts $L = 2$ billion light years away), it will take the light $10^9(e^2 - 1)$, or 6.38 billion years, to get to us, but it will eventually arrive."

Príklad 2

Príklad 3

SOLUTION:

Ohm's law for a wire of length l gives the voltage $V = I\rho l/A$. Therefore, the electric field strength in the wire is $E = V/l = I\rho/A$. Due to the difference in the resistivity of the materials the electric field strength has to be different in material 1 and material 2. According to Gauss's law, the difference in the electric field strengths implies an accumulation of charge at the boundary of the two materials. The net accumulated charge is

$$Q = \varepsilon_0 A(E_2 - E_1) = \varepsilon_0 I(\rho_2 - \rho_1), \tag{5}$$

where ε_0 is the electric constant.

Príklad 4

(a) Applying the law of conservation of energy for a bead of mass m and charge Q in the field of a dipole with dipole moment P gives

$$\tfrac{1}{2}mv^2 + QP\tfrac{\cos\theta}{r^2} = \tfrac{1}{2}mv_0^2 + QP\tfrac{\cos(\pi/2)}{r^2} = 0$$

This gives the expression for the velocity of the bead v at angle θ

$$v = \sqrt{\frac{-2QP\cos\theta}{mr^2}}.$$

The bead moves along a circular path until it reaches the point opposite its starting position. The bead stops there and then goes back executing a periodic motion.

(b) The radial component of the force on the charge due to the dipole $F_{r-dipole}$ can be calculated as the derivative of the electric potential energy with respect to r

$$\tfrac{\partial}{\partial r}\left(QP\tfrac{\cos\theta}{r^2}\right) = -2QP\tfrac{\cos\theta}{r^3}.$$

For the circular motion $mv^2/r = F_{r-dipole} + F_{r-string}$. Substituting v and $F_{r-dipole}$ gives the normal force exerted by the string on the bead $F_{r-string} = 0$. If the string were not there, the bead would move along the circular path as with the string.

Príklad 5

(a) Let the length of string on the table be r and the length hanging below be y. The mass m_1 is described by the polar coordinates (r, ϕ). The fixed length constraint is $y + r = \ell$. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \tfrac{1}{2}m_2\dot{y}^2 + m_2 gy \\ &= \tfrac{1}{2}(m_1 + m_2)\dot{r}^2 + \tfrac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g(\ell - r) \ . \end{aligned}$$

(b) The angular momentum is conserved: $\dot{p}_\phi = 0$, with

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \ .$$

The equation of motion for r yields

$$\begin{aligned}(m_1 + m_2)\ddot{r} &= m_1 r\dot{\phi}^2 - m_2 g \\ &= \frac{p_\phi^2}{m_1 r^3} - m_2\, g \equiv -\frac{\partial U_{\rm eff}}{\partial r} \ ,\end{aligned}$$

where the effective potential is

$$U_{\rm eff} = \frac{p_\phi^2}{2m_1 r^2} + m_2 g r \ .$$

The condition for stationary m_2 is $\dot{r} = \ddot{r} = 0$, which requires $U'_{\rm eff}(r) = 0$. This, in turn, has the solution $r = a$, with

$$a = \left(\frac{p_\phi^2}{m_1 m_2\, g}\right)^{1/3} \ .$$

(c) The tension in the string along the table is radial, hence there are no torques, and p_ϕ is conserved.

(d) We write $r = a + \eta$ and expand the equation of motion:

$$(m_1 + m_2)\, \ddot{\eta} = -U''_{\rm eff}(a)\, \eta + \mathcal{O}(\eta^2) \ .$$

The solution is

$$\eta(t) = \eta_0 \cos\left(\omega t + \delta\right) \ ,$$

where the oscillation frequency is

$$\omega = \sqrt{\frac{U''_{\rm eff}(a)}{m_1 + m_2}} = \sqrt{\frac{3m_2\, g}{(m_1 + m_2)a}} \ .$$

Príklad 6

a. There are two $+q$ - $-q$ pairs separated by a distance d, each having potential energy

$$-\frac{q^2}{4\pi\epsilon_0 d}$$

There are two $+q$ - $-q$ pairs separated by a distance r, each having potential energy

$$-\frac{q^2}{4\pi\epsilon_0 r}$$

There are a $+q$ - $+q$ pair and a $-q$ - $-q$ pair separated by a distance $\sqrt{r^2 + d^2}$, each having potential energy

$$\frac{q^2}{4\pi\epsilon_0\sqrt{r^2 + d^2}}$$

Note that the latter two terms go to zero as r becomes large, whereas the first term is not dependent on r. Thus the given zero convention will include only the latter two terms:

$$U = \frac{q^2}{4\pi\epsilon_0}\left(\frac{-2}{r} + \frac{2}{\sqrt{r^2 + d^2}}\right)$$

b. We have

$$U = \frac{2q^2}{4\pi\epsilon_0 r}\left(\frac{1}{\sqrt{1+\left(\frac{d}{r}\right)^2}} - 1\right)$$

Using the binomial approximation $(1+x)^n \approx 1 + nx$,

$$U \approx \frac{2q^2}{4\pi\epsilon_0 r}\left(1 - \frac{1}{2}\left(\frac{d}{r}\right)^2 - 1\right)$$

$$U \approx -\frac{q^2 d^2}{4\pi\epsilon_0 r^3}$$

or, in terms of p,

$$U \approx -\frac{p^2}{4\pi\epsilon_0 r^3}$$

c. We can infer by symmetry that the force must be in the direction along the line separating the dipoles. Since the potential energy decreases with decreasing distance, the force is attractive. Its magnitude can be determined by taking the derivative of the potential energy:

$$F = -\frac{dU}{dr} = -3\frac{p^2}{4\pi\epsilon_0 r^4}$$

with the negative sign confirming that the force is attractive.

One can, of course, also use an approach analogous to the previous one, *i.e.* write down the force exactly and use a binomial approximation as above. One must take care to account for the fact that the force between like-signed charges is not exactly in the same direction as that between opposite-signed charges.