METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 1

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Príklad 1

Let r be the distance to bead

$x = r\sin\theta\cos\omega t$

$y = r\sin\theta\sin\omega t$

$z = A - r\cos\theta$

$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\sin\theta\cos\omega t - \omega r\sin\theta\sin\omega t)^2 + (\dot{r}\sin\theta\sin\omega t + \omega r\sin\theta\cos\omega t)^2 + (-\dot{r}\cos\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\sin^2\theta + \omega^2 r^2\sin^2\theta + \dot{r}^2\cos^2\theta\right]$$

$$= \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\theta)$$

V assuming $V=0$ at xy plane

$V = mg(A - r\cos\theta)$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\theta) - mg(A - r\cos\theta)$$

b. $\frac{\partial L}{\partial r} = m\omega^2 r\sin^2\theta + mg\cos\theta$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ so

$$m\ddot{r} - m\omega^2 r\sin^2\theta - mg\cos\theta = 0$$

$$\ddot{r} - \omega^2\sin^2\theta\, r = g\cos\theta$$

c. General solution if $g\cos\theta$ set to zero: $a_1 e^{\omega\sin\theta t} + a_2 e^{-\omega\sin\theta t}$

$g\cos\theta$ is a constant so the particular solution is $\frac{-g\cos\theta}{\omega^2\sin^2\theta}$

$$\therefore \quad r = a_1 e^{\omega\sin\theta t} + a_2 e^{-\omega\sin\theta t} - \frac{g\cos\theta}{\omega^2\sin^2\theta}$$

Príklad 2

(a) The electric potential V satisfies the Laplace equation, $\nabla^2 V = 0$. Given the boundary conditions

$$V(x, y = 0) = 0 = V(x, y = a), \qquad \text{and} \qquad V(x = 0, y) = V_0,$$

the solution is of the form

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{ikx}.$$

Inserting this solution into the Laplace equation, we have

$$-\left(\frac{\pi}{a}\right)^2 - k^2 = 0,$$

or $k = \pm i\pi/a$. Thus, the solution (for $x \geq 0$) is

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{-\pi x/a}.$$

(We can ignore $x \geq L/2$ since $e^{-piL/2a} \ll 1$ for $L/a \gg 1$.)

(b) To find the charge density σ at the surface of the conductors, we need the electric field $\vec{E}$ at the surface. The latter can be obtained from the potential $V(x, y)$ as

$$\vec{E} = -\vec{\nabla} V = \frac{\pi V_0}{a}\left[\sin\left(\frac{\pi y}{a}\right)\hat{x} - \cos\left(\frac{\pi y}{a}\right)\hat{y}\right] e^{-\pi x/a}.$$

At the surfaces of the conducting plates at $y = 0$ and $y = a$, the induced charge densities are the same, with

$$\sigma(x, y = 0) = \sigma(x, y = a) = \epsilon_0 \vec{E} \cdot \hat{n} = -\frac{\epsilon_0 \pi V_0}{a} e^{-\pi x/a}, \qquad x \geq 0$$

for both plates.

(c) Force exerted on a conductor is given by

$$\vec{F} = \int \sigma \, \vec{E}_{\text{ext}} \, dA,$$

integrated over the surface area of the conductor, with $E_{\text{ext}} = E_{\text{self}} = E/2$.

On the upper plate (and $x \geq 0$),

$$\begin{aligned}
\vec{F} &= L \int_0^{L/2 \to \infty} dx \, \sigma(x, y = a) \cdot \frac{1}{2}\vec{E}(x, y = a) \\
&= -\frac{\epsilon_0 \pi^2 V_0^2 L}{2a^2}\left[\int_0^{\infty} e^{-2\pi x/a} dx\right]\hat{y} \\
&= -\frac{\pi}{4}\epsilon_0 V_0^2 L \, \hat{y}
\end{aligned}$$

Including also the part from $x \leq 0$, the total force exerted on the top plate is

$$\vec{F}_{\text{upper}} = -\frac{\pi}{2}\epsilon_0 V_0^2 L\,\hat{y},$$

i.e., the top plate is attracted towards the lower plate.

By symmetry, the lower plate is attracted towards the upper plate with force of the same magnitude, i.e.,

$$\vec{F}_{\text{lower}} = +\frac{\pi}{2}\epsilon_0 V_0^2 L\,\hat{y}.$$

Príklad 3

The potential due to a uniformal spherical volume with net charge e and radius r_0 is

$$\begin{aligned} U &= -\frac{er^2}{2r_0^3} + \frac{3e}{2r_0}, \qquad (r < r_0) \\ &= \frac{e}{r}, \qquad (r > r_0) \end{aligned}$$

where the constant of integration in the first expression has been chosen to make U continuous at r_0.

The perturbation V in the Hydrogen atom potential is

$$\begin{aligned} \Delta V &= \frac{e^2r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}, \qquad (r < r_0) \\ &= 0, \qquad (r > r_0) \end{aligned}$$

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The energy shift is

$$\Delta E = \langle \Delta V \rangle$$

to first order in perturbation theory. For the $1s$ state,

$$\Delta E = \int |\psi|^2\,\Delta V = \int_0^{r_0} 4\pi r^2 \mathrm{d}r\ |\psi|^2\,\Delta V$$

Since $r_0 \ll a_0$, the typical scale of variation of the wavefunction, $\psi \approx \psi(0)$, and

$$\Delta E \approx |\psi(0)|^2 \int_0^{r_0} 4\pi r^2 \mathrm{d}r \left[\frac{e^2r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}\right] = \frac{2}{5}e^2\pi r_0^2\,|\psi(0)|^2$$

For the $1s$ state,

$$\psi = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0}$$

so

$$\Delta E \approx \frac{2}{5}\frac{e^2 r_0^2}{a_0^3} = \frac{2}{5}\frac{e^2}{a_0}\frac{r_0^2}{a_0^2}$$

The ground state energy of H is $-e^2/(2a_0) = -13.6$ eV, and $a_0 = 0.529 \times 10^{-10}$ m, so

$$\Delta E = \frac{2}{5}(2 \times 13.6\,\text{eV})\frac{r_0^2}{a_0^2} = 3.9 \times 10^{-9}\,\text{eV}$$

The wavefunction of the $2p$ state vanishes at the orgin. This suppresses ΔE by an additional factor of $r^2/a_0^2 \sim 10^{-10}$

Príklad 4

SOLUTION: For a non-interacting ideal gas,

$$E = -\frac{\partial}{\partial \beta} N \ln \zeta \,,$$

where ζ is the single-molecule partition function

$$\zeta = \sum_{n=0}^{\infty} (n+1) \exp(-\beta n \varepsilon) \,.$$

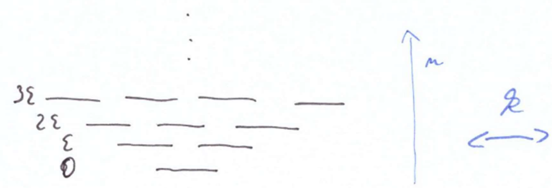
This partition function can be evaluated as follows ($x \equiv \beta\varepsilon$):

$$\zeta = -e^x \frac{d}{dx} \sum_{n=0}^{\infty} \exp\big(-(n+1)x\big) = -e^x \frac{d}{dx}\frac{e^{-x}}{1-e^{-x}} = [1 - \exp(-\beta\varepsilon)]^{-2}.$$

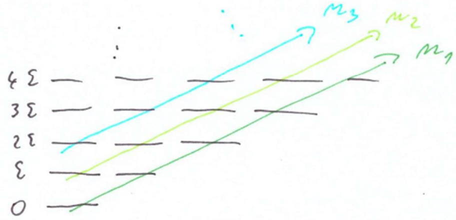
Hence, the sought contribution to the energy is

$$E = \frac{2N\varepsilon}{\exp(\varepsilon/kT) - 1} \,.$$

Alternatively, one can reproduce this result as follows. One can imagine that every molecule has two independent internal degrees of freedom of harmonic oscilator type, with energy spacing ε each. It is easy to see that this model gives the same spectrum and degeneracies if the energy is counted from the ground state. With this convention, the average energy of a single harmonic oscillator is $\varepsilon n_B(\varepsilon)$, where $n_B(\varepsilon)$ is the Bose-Einstein occupation number. Therefore, for the entire gas we get $E = 2N\varepsilon n_B(\varepsilon)$, in agreement with the first derivation.



$$Z = \sum_{n=0}^{\infty} \sum_{k=0}^{n} e^{-\beta n \varepsilon}$$



$$Z = \underbrace{\sum_{n_1=0}^{\infty} e^{-\beta n_1 \varepsilon}}_{Z_{LHO}} + e^{-\beta\varepsilon} \underbrace{\sum_{n_2=0}^{\infty} e^{-\beta n_2 \varepsilon}}_{Z_{LHO}} + e^{-2\beta\varepsilon} \underbrace{\sum_{n_3=0}^{\infty} e^{-\beta n_3 \varepsilon}}_{Z_{LHO}} + \ldots =$$

$$= Z_{LHO} \cdot \underbrace{\left(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + \ldots\right)}_{Z_{LHO}} = \boxed{Z_{LHO}^2}$$