METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 2

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Príklad 1

(a) Start with equations of x_1 and x_2:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) \end{aligned}$$

The sum of these equations describe the free particle motion of the center of mass variable $x_{\mathrm{cm}} = (m_1x_1 + m_2x_2)/(m_1 + m_2)$:

$$\ddot{x}_{\mathrm{cm}} = 0$$

If the first equation is multiplied by m_2 and subtracted from the second multiplied by m_1, we find a simple harmonic equation for the variable $y = x_2 - x_1 - l$:

$$m_1m_2\ddot{y} = -(m_1 + m_2)ky \tag{1}$$

Thus, if we define $\omega_0 = \sqrt{k/\mu}$ with $\mu = m_1m_2/(m_1 + m_2)$ we have the solution:

$$\begin{aligned} y(t) &= -\Delta l \cos(\omega_0 t) \\ x_{\mathrm{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2} \end{aligned}$$

(b) The extra friction force does not change the structure of the equations:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma(\dot{x}_1 - \dot{x}_2) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) - \sigma(\dot{x}_2 - \dot{x}_1) \end{aligned}$$

so they are solved the same variables x_{cm} and y:

$$\begin{aligned} y(t) &= e^{-\gamma t/2}\Delta l\left\{-\cos(\omega t) + \frac{\gamma}{2\omega}\sin(\omega t)\right\} \\ x_{\mathrm{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2}, \end{aligned}$$

where $\gamma = \sigma/\mu$ and $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$.

(c) The equations become less familiar if friction is introduced between the table and m_1:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma\dot{x}_1 \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l). \end{aligned}$$

Now the center of mass motion will couple with the oscillating variables and the four frequencies present in this system of two coupled second order equations can be found by solving:

$$\begin{aligned} 0 &= \det\begin{pmatrix} -m_1\omega^2 + k + i\sigma\omega & -k \\ -k & -m_2\omega^2 + k \end{pmatrix} \\ &= m_1 m_2\omega^4 - (m_1 + m_2)k\omega^2 + i\sigma\omega(k - m_2\omega^2). \end{aligned}$$

If $\sigma = 0$, these have the double root $\omega = 0$ and the two roots $\omega = \pm\omega_0$ corresponding to the $x_{\text{cm}}(0) + \dot{x}_{\text{cm}}(0)t$ cm mass and oscillatory motion above. These zeroth-order results can then be substituted in the above equation to find the frequencies to first order in σ:

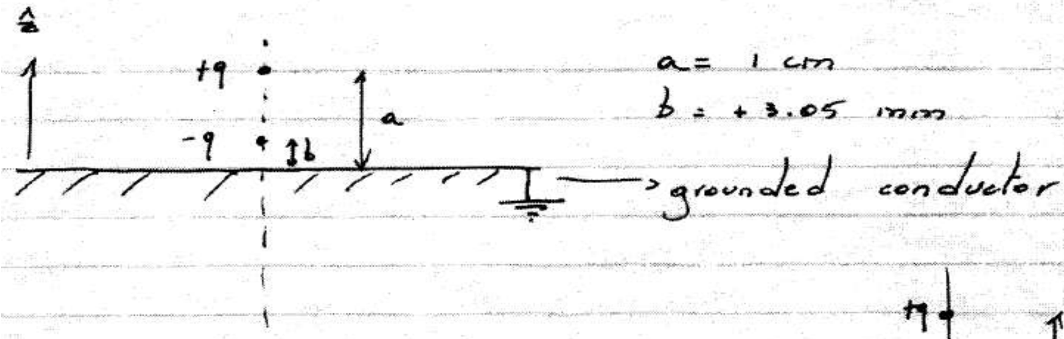
$$\begin{aligned} \omega &= \pm\omega_0 + i\sigma\frac{m_2}{m_1(m_1 + m_2)} \\ \omega &= 0, \quad \omega = +i\frac{\sigma}{m_1 + m_2}. \end{aligned}$$

The $\omega = 0$ root corresponds to equilibrium with an arbitrary cm location, while $i\sigma/(m_1+m_2)$ describes non-oscillatory behavior with non-zero cm velocity, decreasing exponentially to zero. Finally $\pm\omega_0 + i\sigma m_2/(m_1[m_1 + m_2])$ corresponds to oscillatory motion damped by the motion of m_1.

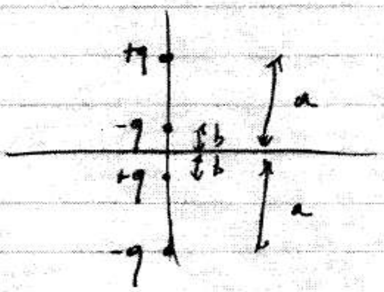
Príklad 2

$a = 1$ cm

$b = +3.05$ mm



This is an image problem: take



The force on the charge at b is then

$$F_z = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{(a-b)^2} - \frac{1}{(2b)^2} + \frac{1}{(a+b)^2}\right]$$

due to +q at b, due to +q at -b, due to -q at -a

$$F_z = 0 \quad \text{if} \quad \frac{1}{(a-b)^2} - \frac{1}{(2b)^2} + \frac{1}{(a+b)^2} = 0$$

Both a and b are > 0, and we will assume $a \neq b$

$$\Rightarrow (a+b)^2(2b)^2 - (a-b)^2(a+b)^2 + (a-b)^2(2b)^2 = 0$$

$$\Leftrightarrow 4a^2b^2 + 4ab^3 + 4b^4 - a^4 + 2a^3b - a^2b^2 - 2a^3b + 4a^2b^2 - 2ab^3 - a^2b^2 + 2ab^3 - b^4 = 0$$

$$\Leftrightarrow 10a^2b^2 + 7b^4 - a^4 = 0$$

$$\Leftrightarrow 7b^4 + 10a^2b^2 - a^4 = 0$$

Sol. $$x = \frac{-10a^2 \pm \sqrt{100a^4 + 28a^4}}{14}$$

$$a = 1 \Rightarrow x = \frac{-10 \pm \sqrt{128}}{14} = \frac{-10 \pm 11.3}{14}$$

The charge $-q$ is above the plane, (and b is a real number) so we take the positive solution:

$$x = \frac{1.3}{14} \Rightarrow b = \sqrt{\frac{1.3}{14}} = 0.305$$

$\Rightarrow b = 3.05$ mm implies the force on $-q = 0$

What is the surface charge density on the conductor?

· For a conductor, $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ at the surface

$$\leadsto \sigma = \epsilon_0 E_{\hat{n}} = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{surface}$$

$\rightarrow$ in this case, $\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$

$$\cdot V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\Delta r_i|}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{q}{\sqrt{x^2+y^2+(z+a)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+b)^2}}\right]$$

$$\leadsto \frac{\partial V}{\partial z} = -\frac{1}{2}\frac{2q}{4\pi\epsilon_0}\left[\frac{(z-b)}{(x^2+y^2+(z-b)^2)^{3/2}} - \frac{(z-a)}{(x^2+y^2+(z-a)^2)^{3/2}} + \frac{(z+a)}{(x^2+y^2+(z+a)^2)^{3/2}} - \frac{(z+b)}{(x^2+y^2+(z+b)^2)^{3/2}}\right]$$

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0} = \frac{q}{4\pi}\left[\frac{2a}{(x^2+y^2+a^2)^{3/2}} - \frac{2b}{(x^2+y^2+b^2)^{3/2}}\right]$$

Príklad 3

3. (a) At 10^7 K we estimate KE per neutron
$k_B T = 1.38 \times 10^{-23}$ J/K $\times 10^7$ K $\approx 10^{-16}$ J
compare to neutron rest energy
$mc^2 = 1.6 \times 10^{-27} \times 9 \times 10^{16}$ kg m/sec$^2 \approx 10^{-10}$ J $>> k_B T$
these neutrons are nonrelativistic (by a factor of 10^6 or so)

(b) compute Fermi energy for nonrelativistic gas

$$N = \frac{2 \cdot 4\pi V}{h^3}\frac{p_F^3}{3}$$

so

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{h^2}{2m}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}$$

plug in numbers, compute density $\rho = 10^{14}$ g/cm^3 = 10^{17} kg/m^3
$N/V = \rho/m = (10^{17}\ \text{kg/m}^3)/(1.6 \times 10^{-27}\ \text{kg}) \approx 10^{44}/\text{m}^3$ so
$\epsilon_F \approx 10^{-67}\ \text{J}^2\ \text{sec}^2 \times (10^{43}\ \text{m}^{-3})^{2/3})/(10^{-27}\ \text{kg}) \approx 10^{-11}\ \text{J} >> k_B T$
so neutron star can be considered a zero-temperature Fermi gas

(c) ground state energy of Fermi gas

$$E = \frac{3}{5} N\epsilon_F \approx N\epsilon_F$$

pressure is volume derivative

$$p = -\left(\frac{\partial E}{\partial V}\right) = \frac{2}{3}\frac{E}{V} = \frac{2}{5} N\epsilon_F/V$$

estimate
$p \approx \epsilon_F N/V \approx 10^{-11}\ \text{J} \times 10^{44} \text{m}^{-3} = 10^{33}$ Pa

plug in to find
$M \approx (10^{33}\text{N/m}^2/10^{-10}\ \text{N m}^2/\text{kg}^2)^{3/2}/(10^{34}\text{kg}^2/\text{m}^6) \approx 10^{29}$ kg
is the mass needed to generate this pressure.

Note that this mass corresponds to a neutron star size of $R \approx 10^4$ m, and is roughly the mass of our sun (which is 2×10^{30} kg).

More careful analysis shows that the critical mass needed to create a neutron star (to force electrons to combine with protons) is about 3×10^{30} kg.

Príklad 4

a) The ground state for particle is:

$$\psi_0(x) = \sqrt{\tfrac{2}{L}} \cos\left(\tfrac{\pi x}{L}\right)$$

and first excited state:

$$\psi_1(x) = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{2\pi x}{L}\right)$$

b) After the expansion, the final eigenfunctions are, for the ground state:

$$\psi_0'(x) = \tfrac{1}{\sqrt{L}} \cos\left(\tfrac{\pi x}{2L}\right)$$

and for the first excited state:

$$\psi_1'(x) = \tfrac{1}{\sqrt{L}} \sin\left(\tfrac{\pi x}{L}\right)$$

In the sudden approximation, define P_{0j} as the probability that the particle starts in the ground state 0 and ends in the final state *j*, so that:

$$P_{0j} = \left|I_{0j}\right|^2$$

with:

$$I_{0j} = \int_{-L/2}^{L/2} dx\, \psi_j'(x)\psi_0(x)$$

So the amplitude for the particle to remain in the ground state is:

$$I_{00} = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2} dx \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi x}{L}\right)$$

$$= \frac{1}{L\sqrt{2}}\int_{-L/2}^{L/2} dx\left(\cos\left(\frac{\pi x}{2L}\right) + \cos\left(\frac{3\pi x}{2L}\right)\right)$$

$$= \frac{1}{L\sqrt{2}}\frac{4L}{\pi}\left(\sin\left(\frac{\pi x}{2L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2L}\right)\right)\Big|_{x=L/2}$$

$$= \frac{8}{3\pi}$$

So the probability P_{00} is:

$$P_{00} = \left(\frac{8}{3\pi}\right)^2$$

c) For the transition between the initial ground state and the final excited state ψ_1', the amplitude for the transition is:

$$I_{01} = \int_{-L/2}^{L/2} dx\, \psi_1'(x)\psi_0(x)$$

$$= \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2} dx \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)$$

Since this is an odd function in *x* and we are integrating over $-\frac{L}{2} < x < +\frac{L}{2}$ then $I_{01} = 0$ and the probability P_{01} is:

$$P_{01} = 0$$