METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 3

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Príklad 1

(a) Take $E \approx \frac{\Delta p^2}{2m} + c\Delta x \approx \frac{\hbar^2}{2m\Delta x^2} + c\Delta x$, using $\Delta p \Delta x \sim \hbar$. Minimizing, the characteristic length scale is $\Delta x = (\hbar^2/mc)^{1/3}$ and $E_{min} \approx \frac{3}{2}\left(\frac{\hbar^2 c^2}{m}\right)^{1/3}$.

(b) The trial wavefunction gives

$$\langle E \rangle = \int_0^\infty dx x e^{-ax} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + cx \right) x e^{-ax} / \int_0^\infty dx x^2 e^{-2ax},$$

$$= \frac{3c}{2a} + \frac{\hbar^2 a^2}{2m}.$$

Minimizing, the characteristic size is

$$a^{-1} = \left(\frac{2\hbar^2}{3cm}\right)^{1/3}$$

and the groundstate energy is bounded below as

$$E_{min} \geq \frac{9}{4}\left(\frac{2\hbar^2 c^2}{3m}\right)^{1/3}.$$

Príklad 2

Use the equations

$$\frac{dm}{dt} = kv\,, \quad \frac{dp}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt} = mg\,,$$

which imply

$$\left(\frac{dm}{dt}\right)^2 + m\frac{d^2m}{dt^2} = \frac{1}{2}\frac{d^2}{dt^2}m^2 = kgm\,.$$

Let $X \equiv m^2$, then

$$\frac{d^2X}{dt^2} = 2kgX^{1/2}.$$

This is analogous to the equation of motion of a particle subject to the external force that grows with coordinate as $X^{1/2}$. Such a particle would accelerate without bound so that the initial value of "position" X and "velocity" dX/dt would indeed be unimportant in the long-time limit.

Looking for a solution in the form $X(t) = At^\alpha$, we find $A = (kg/6)^2$, $\alpha = 4$, and so

$$m(t) = \frac{1}{6}kgt^2\,, \quad v(t) = \frac{1}{3}gt\,.$$

Effectively, the free fall acceleration gets reduced three times.

Príklad 3

$C_p = \left(\frac{\partial Q}{\partial T}\right)_p$. From the first Law $dU = dQ - pdV$, let $dQ = dU + pdV$.

Write dU in terms of V and T as $dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$.

The internal energy of an ideal gas depends only on temperature, so $\left(\frac{\partial U}{\partial V}\right)_T = 0$.

Combining these results leads to $dQ = \left(\frac{\partial U}{\partial T}\right)_V dT + pdV = C_V dT + pdV$.

Therefore, $C_p = \left(\frac{\partial Q}{\partial T}\right)_p = C_V + p\left(\frac{\partial V}{\partial T}\right)_p = C_V + Nk$, the latter is obtained from $pV = NkT$.

So, $C_p = \frac{7}{2}Nk$.

Paths acb, and adb consist of constant pressure and constant volume processes.

$$Q(acb) = \int_a^c C_V\, dT + \int_c^b C_p\, dT = \int_{p_1}^{p_2} \tfrac{5}{2}Nk\frac{V_1}{Nk}dp + \int_{V_1}^{V_2} \tfrac{7}{2}Nk\frac{p_2}{Nk}dV$$

$$= \tfrac{5}{2}Nk\frac{V_1}{Nk}(p_2 - p_1) + \tfrac{7}{2}Nk\frac{p_2}{Nk}(V_2 - V_1) = \tfrac{5}{2}Nk\frac{V_1}{Nk}(p_1) + \tfrac{7}{2}Nk\frac{2p_1}{Nk}(V_1) = \tfrac{19}{2}NkT_1$$

$$Q(adb) = \int_a^d C_p\, dT + \int_d^b C_V\, dT + = \int_{V_1}^{V_2} \tfrac{7}{2}Nk\frac{p_1}{Nk}dV + \int_{p_1}^{p_2} \tfrac{5}{2}Nk\frac{V_2}{Nk}dp$$

$$= \tfrac{7}{2}Nk\frac{p_1}{Nk}(V_2 - V_1) + \tfrac{5}{2}Nk\frac{V_2}{Nk}(p_2 - p_1) = \tfrac{7}{2}Nk\frac{p_1}{Nk}(V_1) + \tfrac{5}{2}Nk\frac{2V_1}{Nk}(p_1) = \tfrac{17}{2}NkT_1$$

The heat along path ab can be calculated by taking the difference between ΔU and W.

ΔU between states a and b can be calculated along any path because it is a state function.

W along path ab is also easy to calculate.

First, calculate $\Delta U = W + Q$ for any path. We already have $Q(adb)$, so

$$W(adb) = -\int_a^d p\,dV - \int_d^b p\,dV = -\int_{V_1}^{V_2} p_1\,dV = -p_1V_1 = -NkT_1$$

$$\Delta U = W(adb) + Q(adb) = \tfrac{17}{2}NkT_1 - NkT_1 = \tfrac{15}{2}NkT_1$$

Now, for $W(ab) = -\int_a^b p\,dV = -\int_{V_1}^{V_2} p_1\frac{V}{V_1}dV = -\frac{p_1}{2V_1}\left(V_2^2 - V_1^2\right) = -\frac{3p_1V_1}{2} = -\tfrac{3}{2}NkT_1$

$$Q(ab) = \Delta U - W(ab) = \tfrac{15}{2}NkT_1 - \left(-\tfrac{3}{2}NkT_1\right) = 9NkT_1$$

$C_{ab} = \left(\frac{dQ}{dT}\right)_{ab}$, so consider $dQ = dU + pdV = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV + pdV = C_V dT + pdV$

$\therefore C_{ab} = C_V + p\left(\frac{dV}{dT}\right)_{ab}$

Derive an expression for $\left(\frac{dV}{dT}\right)_{ab}$ using $p = \frac{p_1}{V_1}V$ and $p = \frac{NkT}{V} \Rightarrow V^2 = \frac{NkTV_1}{p_1}$

differentiate: $2VdV = \frac{NkV_1}{p_1}dT$ or $\left(\frac{dV}{dT}\right)_{ab} = \frac{NkV_1}{2Vp_1}$

$\therefore C_{ab} = C_V + p\frac{NkV_1}{2Vp_1} = C_V + \frac{Nk}{2} = 3Nk$

Príklad 4

The rotational kinetic energy is $KE(t) = \frac{1}{2}I_0\omega^2$ where $I_0 = \frac{1}{2}MR^2$ for the ring. When the ring is at an angle $\theta(t)$ with respect to the horizontal constant B field, there is a magnetic flux, $\Phi(t) = B\pi R^2 \cos\theta(t)$ through the loop with $d\theta/dt = \omega(t)$. Faraday's law says that the induced EMF $E(2\pi R) = I(t)\Omega = -1/c\ d\Phi/dt = \pi BR^2\omega(t)/c\ \sin\theta(t)$, where the ring resistance is $\Omega = (2\pi R)/(\sigma\pi r^2)$.

The induced alternating current $I(t)$ dissipates energy according to Joule heating at a rate $P = IV = \Omega I(t)^2 = (d\Phi/dt)^2/\Omega = (\pi BR^2\omega(t)/c\ \sin\theta(t))^2(\sigma\pi r^2)/(2\pi R)$. Over a period, the time averaged power dissipated using $< sin^2 >= \frac{1}{2}$ is $< P >= \frac{1}{2}(\pi BR^2\omega(t)/c)^2(\sigma\pi r^2)/(2\pi R)$.

The rotational kinetic energy decreases according to $dKE/dt = -P$. Using $M = \rho\pi r^2(2\pi R)$ in terms of the mass density ρ, $\frac{1}{2}\ \frac{1}{2}\rho\pi r^2(2\pi R)R^2(2\omega\dot{\omega}) = -(\pi BR^2\omega(t)/c)^2(\sigma\pi r^2)/(2\pi R)$.

Therefore , $\dot{\omega} = -\omega/\tau$ where the spin relaxation time is $\tau = 4\rho c^2/(B^2\sigma)$.

Check dimensions: $[rhoc^2] = En/Vol$, $[B^2] = En/Vol$ whereas $[\sigma] = 1/Time$.