METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 4

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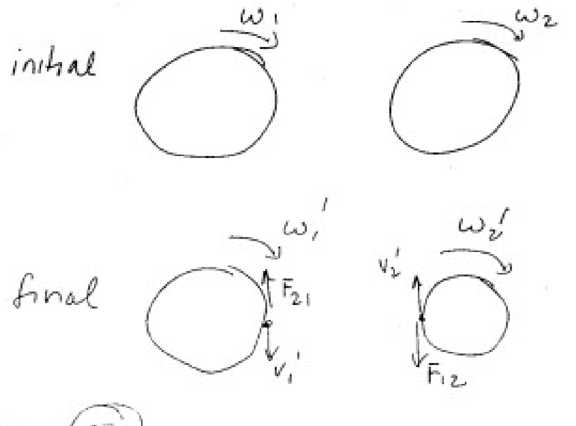
Príklad 1

Mechanics Solution - Tuts



Since the tangential final speeds must be the same (but opposite directions)

$\omega_1' r_1 = -\omega_2' r_2$ (5)

$\Rightarrow \omega_2' = -\frac{r_1}{r_2}\omega_1'$

Since the integral of the torque equals the change in angular momentum

$I_1(\omega_1' - \omega_1) = -\int |r_1||F_{21}|\,dt$ (5)

and $I_2(\omega_2' - \omega_2) = -\int |r_2||F_{12}|\,dt$ → but $|F_{21}| = |F_{12}|$ from Newton's 3rd Law

$\therefore \frac{I_1}{r_1}(\omega_1' - \omega_1) = \frac{I_2}{r_2}(\omega_2' - \omega_2)$

subst in for $\omega_2' = -\frac{r_1}{r_2}\omega_1'$

$\therefore \frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\cdot\frac{r_1}{r_2}\omega_1' = \frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2$

$$\omega_1' = \frac{\frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2}{\frac{I_1}{r_1} + I_2\frac{r_1}{r_2}}$$

with $I = \frac{1}{2}mr^2$ and some algebra (2)

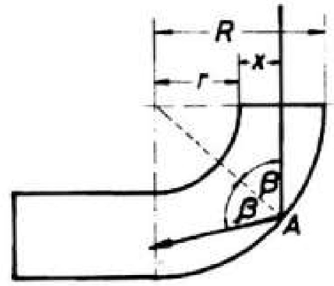
$$\boxed{\omega_1' = \frac{m_1 r_1 \omega_1 - m_2 r_2 \omega_2}{(m_1 + m_2) r_1}}$$ (3)

Príklad 2

Solution Example:

If β is larger than the angle of total internal reflection at point A, then the beam will stay inside the glass plate, consequently we have the condition:

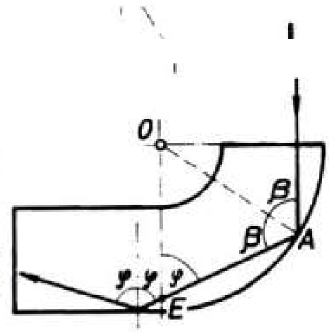
$$\sin\beta = \frac{x + r}{R} \geqslant \frac{1}{n}$$



$$\frac{r}{R} \geqslant \frac{1}{n} \quad \Rightarrow \quad \frac{R}{r} \leqslant n$$

The β angle is the is minimized when x = 0, providing the critical condition for the laser beam to stay inside the glass. Thus, the ratio of the larger radius to the smaller radius must remain below the index of refraction of the glass plate.

Consequently, the $\frac{R}{r} \leqslant n$ condition is always sufficient to ensure good quality for a light guide (e.g., for a radio where the front panel is far from the laser diode used as indicator on the PCB). LEDs have broader angular emission than lasers, therefore the question is slightly more complicated for LEDs. However the result is still a decent rule of thumb for nice laboratory design.



Príklad 3

Consider a three-dimensional box with sides of length L. It contains an ideal gas of non-interacting spin-less particles each with kinetic energy

$$\varepsilon = \frac{m}{2}\vec{v}^2$$

The temperature of the gas is T, and the particles are uniformly distributed throughout the box.

a) What is the normalized velocity distribution of the gas?

$$P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\vec{v}^2}{2k_B T}\right)$$

b) We now open one side of the box (the one facing the $+x$-direction) for a given time Δt. Using the result from (a), compute the number of particles that escape from the box in time Δt. To this end, consider these two steps: (i) Divide the box into slices of width dx and compute first the number of particles in a given slice at a distance x from the opening that have escape through the opening in time Δt.

The number of particles contained in this slice is given by

$$\frac{N}{L^3}L^2 dx = \frac{N}{L}dx$$

In order for a particle from this slice to escape in time Δt, its velocity in the $+x$-direction needs to satisfy $v_x \geq \frac{x}{\Delta t}$. The number of particles contained in the slice with velocity $v_x \geq \frac{x}{\Delta t}$ is given by

In order for a particle from this slice to escape in time Δt, its velocity in the $+x$-direction needs to satisfy $v_x \geq \frac{x}{\Delta t}$. The number of particles contained in the slice with velocity $v_x \geq \frac{x}{\Delta t}$ is given by

$$\frac{N}{L}dx \int_{\frac{x}{\Delta t}}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z P(\vec{v}) = \frac{N}{L}dx \int_{\frac{x}{\Delta t}}^{\infty} dv_x \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{m\vec{v}^2}{2k_B T}\right)$$

$$= \frac{N}{L}dx \left(\frac{m}{2\pi k_B T}\right)^{1/2} \frac{1}{2}\sqrt{\frac{\pi}{\frac{m}{2k_B T}}}\left[1 - Erf\left(\sqrt{\frac{m}{2k_B T}}\frac{x}{\Delta t}\right)\right]$$

$$= \frac{N}{2L}\left[1 - \mathrm{Erf}\left(\sqrt{\frac{m}{2k_BT}}\,\frac{x}{\Delta t}\right)\right]dx$$

(ii) In order to find the total number of escaped particles, integrate the result you obtained in (i).

$$N_{tot} = \int_0^L \left[1 - \mathrm{Erf}\left(\sqrt{\frac{m}{2k_BT}}\,\frac{1}{\Delta t}x\right)\right]dx$$

$$= \frac{N}{2L}\frac{1}{\sqrt{\frac{m}{2k_BT}}\,\frac{1}{\Delta t}}\left\{\frac{1}{\sqrt{\pi}}\left[1 - \exp\left(-\frac{m}{2k_BT}\left(\frac{L}{\Delta t}\right)^2\right)\right] + \sqrt{\frac{m}{2k_BT}}\,\frac{L}{\Delta t}\left[1 - \mathrm{Erf}\left(\sqrt{\frac{m}{2k_BT}}\,\frac{L}{\Delta t}\right)\right]\right\}$$

$$= \frac{N}{2}\frac{1}{z}\left\{\frac{1}{\sqrt{\pi}}[1 - \exp(-z^2)] + z[1 - \mathrm{Erf}(z)]\right\}$$

where I defined

$$z = \sqrt{\frac{m}{2k_BT}}\,\frac{L}{\Delta t}$$

c) How does the total number of escaped particles depend on Δt in the limit $\Delta t \to 0$?
This limit corresponds to $z \to \infty$ and I obtain

$$N_{tot} \approx \frac{N}{2}\frac{1}{z}\left\{\frac{1}{\sqrt{\pi}}[1 - \exp(-z^2)] + z\frac{\exp(-z^2)}{\sqrt{\pi}\,z}\right\} = \frac{N}{2\sqrt{\pi}}\frac{1}{z} = \frac{N}{2\sqrt{\pi}}\sqrt{\frac{2k_BT}{m}}\,\frac{\Delta t}{L}$$

Thus, the number of escaped particles is directly proportional to Δt.

Príklad 4

2. (a) Since A and B are Hermitian, we have that $(A+B)^2$ is also Hermitian. But $2\,AB$ is Hermitian if and only if A commutes with B. Therefore we have that $(A+B)^2 = A^2 + B^2 + 2\,AB = 2\,AB$ (where the first equality uses $[A, B] = \hat{0}$), and hence $A^2 + B^2 = \hat{0}$. Taking the expectation of both sides of this last equation in an arbitrary state $|\psi\rangle$ yields $\langle A\psi|A\psi\rangle + \langle B\psi|B\psi\rangle = 0$, where we have used the Hermiticity of A and B. Since both terms on the LHS of this equation are ≥ 0 for any A and $|\psi\rangle$, we see that $\langle A\psi|A\psi\rangle = \langle B\psi|B\psi\rangle = 0$. But $\langle\phi|\phi\rangle = 0$ if and only if the vector $|\phi\rangle$ is the zero vector $|0\rangle$. Hence $A\,|\psi\rangle = B\,|\psi\rangle = |0\rangle$ for all states $|\psi\rangle$, which shows that $A = B = \hat{0}$.

(b) Let $|\phi\rangle$ be a unit vector in $\mathcal{H}$ which is orthogonal to $|\psi\rangle$. (The vector $|\phi\rangle$ is unique up to an overall phase.) Then $\{|\psi\rangle\,, |\phi\rangle\}$ is an orthonormal basis for $\mathcal{H}$. We can now (for example) choose $A = \alpha\,(|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi|)$ and $B = \beta\,(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)$, where α and β are non-zero real numbers. The eigenvalues of A are $\pm\alpha$, and the state of the system after the first measurement is $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \pm |\phi\rangle)$ if the value $\pm\alpha$ is obtained (each occurs with probability 1/2). We want to show that if we now measure B (whose eigenvalues are $\pm\beta$), there will be a non-zero probability of obtaining the value $-\beta$, in which case the state of the system after the measurement will be $|\phi\rangle$ (which is orthogonal to $|\psi\rangle$). But it is immediate from the form of the states $|\chi_\pm\rangle$ that the probability of obtaining $-\beta$ in the measurement of B is 1/2, regardless of the outcome of the first measurement.