METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 5

VZOROVÉ RIEŠENIA

Cvičenie 20.4.2022

Príklad 1

SOLUTION: For large x note that $\exp(-t - \frac{x^2}{4t})$ has a maximum at $t_0 = \frac{1}{2}x$. Set $t = \frac{1}{2}x(1+u)$ and expand $(1+u)^{-1}$ in the exponent to find

$$F_\nu(x) = \tfrac{1}{2}\int_{-1}^{\infty} \exp[-\tfrac{1}{2}x(1 + u + (1 - u + u^2 - u^3 + \dots))](1+u)^{-\nu-1}du.$$

For large x the integrand has an effective u range which is $O(1/\sqrt{x}) \ll 1$ around zero. Therefore,

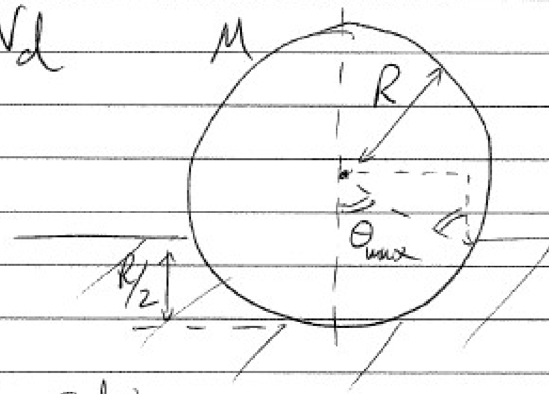
$$F_\nu(x) \approx \tfrac{1}{2}e^{-x}\int_{-\infty}^{\infty} e^{-\frac{1}{2}xu^2}du = \sqrt{\frac{\pi}{2x}}e^{-x},$$

independent of ν. (The function in question is the modified Bessel function $K_\nu(x)$.)

Príklad 2

First, determine the mass M in terms of ρ and R using displaced liquid volume V_d ;

Bouyancy $\Rightarrow M = \rho V_d$



$\theta_{max} = \cos^{-1}\left(\frac{R}{2R}\right) = \pi/3$

$d^3r = r\,dr\,d\phi\,dz$
$= 2\pi r\,dr\,dz$

$r = R\sin\theta \Rightarrow dr = R\cos\theta\,d\theta$

$z = R\cos\theta \Rightarrow dz = +R\sin\theta\,d\theta$

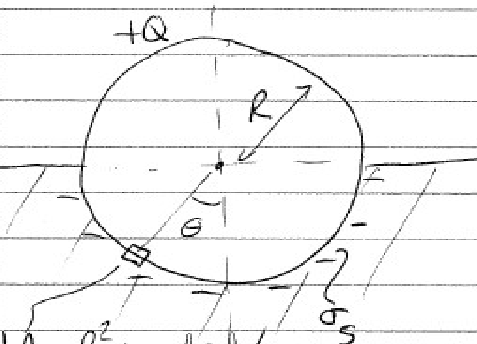
$\Rightarrow V_d = \int_0^{\pi/3} \pi(R\sin\theta)^2(+R\sin\theta\,d\theta) = +\pi R^3\int_0^{\pi/3} d\theta \sin^3\theta$

$$= +\pi R^3 \left\{ \left[-\cos\theta\right]_0^{\pi/3} + \left[\frac{1}{3}\cos^3\theta\right]_0^{\pi/3} \right\}$$

$$= +\pi R^3 \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) = \frac{5}{24}\pi R^3$$

$$\therefore \quad M = \frac{5}{24}\pi \rho R^3 \qquad \ldots \text{(A)}$$

When the sphere is charged, there is an additional electric force between the charge Q and the induced surface charge in the dielectric liquid;



$dA = R^2 \sin\theta \, d\theta \, d\phi$

$$dq = \sigma_s R^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow F_e = \frac{Q}{4\pi\epsilon_0} \int_0^{\pi/2} \frac{2\pi \sigma_s \cancel{R^2} \sin\theta \cos\theta \, d\theta}{\cancel{R^2}}$$

... downward component

$$\Rightarrow F_e = \frac{\sigma_s Q}{2\epsilon_0} \int_0^{\pi/2} d\theta . \frac{1}{2}\sin 2\theta = \frac{\sigma_s Q}{4\epsilon_0}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$$

$$= \frac{\sigma_s Q}{4\epsilon_0}\left(\frac{1}{2} - -\frac{1}{2}\right) = \frac{\sigma_s Q}{4\epsilon_0}$$

All that remains is to determine σ_s since we know that

$$F_e + Mg = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\rho g$$

... half sphere submerged

$$\Rightarrow \frac{\sigma_s Q}{4\epsilon_0} = \frac{11}{24}\pi R^3 \rho . g$$

... using (A)

Now, $D = \epsilon_0 \epsilon_r E = \epsilon_0 E + P$, where $P = \sigma_s$
and $\epsilon_0 E = \sigma = \frac{Q}{4\pi R^2}$, hence

$$\epsilon_r \sigma = \sigma + \sigma_s \Rightarrow \sigma_s = \frac{Q(\epsilon_r - 1)}{4\pi R^2 \epsilon_r}$$

Thus,

$$\frac{Q^2(\epsilon_r - 1)}{4\pi R^2 \epsilon_0 \epsilon_r} = \frac{11}{6}\pi R^3 \rho g$$

$$\therefore Q = \sqrt{\frac{22\pi^2 R^5 \epsilon_0 \epsilon_r \rho g}{3(\epsilon_r - 1)}}$$

Príklad 3

a)

$$P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\vec{v}^2}{2k_B T}\right)$$

b) Due to the equipartition theorem

$$\langle \varepsilon \rangle = \frac{m}{2}\left\langle \vec{v}^2 \right\rangle = \frac{3}{2}k_B T$$

and hence

$$E_0 = \langle E \rangle = \frac{3}{2} N k_B T$$

c)We instantaneously removed all particles with a kinetic energy

$$\varepsilon_{kin} = \frac{1}{2}mv^2 \geq n k_B T$$

The number of remaining particle, N_{new}, is given by

$$N_{new} = N_0\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int' d^3v \exp\left[-\frac{mv^2}{2k_B T}\right]$$
$$= N_0\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \int_0^{v_c} dv\, v^2 \exp\left[-\frac{mv^2}{2k_B T}\right]$$

where

$$v_c = \sqrt{\frac{2n k_B T}{m}}$$

I next perform the variable transformation

$$x = \frac{mv^2}{2k_BT} \Rightarrow dx = \frac{mv}{k_BT}dv = \frac{m}{k_BT}\sqrt{\frac{2k_BTx}{m}}\,dv = \sqrt{\frac{2mx}{k_BT}}\,dv$$

and thus

$$\begin{aligned} N_{new} &= N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2} 4\pi\sqrt{\frac{k_BT}{2m}}\int_0^n dx\,\frac{2k_BT}{m}x^{1/2}\exp[-x] \\ &= N_0\frac{2}{\sqrt{\pi}}\int_0^n dx\,x^{1/2}\exp[-x] \\ &= N_0\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\sqrt{n}\right) - \sqrt{n}\exp(-n)\right] \end{aligned}$$

Next, we compute the remaining energy that is contained in the system after the particles are removed.

$$\begin{aligned} E_{new} &= N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int^{\prime} d^3v\left(\frac{1}{2}mv^2\right)\exp\left[-\frac{mv^2}{2k_BT}\right] \\ &= N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{m}{2}4\pi\int_0^{v_c} dv\,v^4\exp\left[-\frac{mv^2}{2k_BT}\right] \end{aligned}$$

I next perform the variable transformation

$$x = \frac{mv^2}{2k_BT} \Rightarrow dx = \frac{mv}{k_BT}dv = \frac{m}{k_BT}\sqrt{\frac{2k_BTx}{m}}\,dv = \sqrt{\frac{2mx}{k_BT}}\,dv$$

and thus

$$\begin{aligned} E_{new} &= N_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{m}{2}4\pi\sqrt{\frac{k_BT}{2m}}\int_0^n dx\left(\frac{2k_BT}{m}\right)^2 x^{3/2}\exp[-x] \\ &= N_0\frac{2k_BT}{\sqrt{\pi}}\int_0^n dx\,x^{3/2}\exp[-x] \\ &= N_0\frac{k_BT}{2\sqrt{\pi}}\left[3\sqrt{\pi}\operatorname{erf}\left(\sqrt{n}\right) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)\right] \end{aligned}$$

After equilibration, the new temperature is given by

$$E_{new} = \frac{3}{2}N_{new}k_BT_{new}$$

or

$$\begin{aligned} T_{new}(n) &= \frac{2}{3}\frac{E_{new}}{k_BN_{new}} = \frac{2}{3}\frac{N_0\frac{k_BT}{2\sqrt{\pi}}[3\sqrt{\pi}\operatorname{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)]}{k_BN_0\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)\right]} \\ &= \frac{1}{6}\frac{3\sqrt{\pi}\operatorname{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)}{\frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)}T \end{aligned}$$

and thus

$$T_{new} = T/2$$

requires

$$\frac{1}{6}\frac{3\sqrt{\pi}\operatorname{erf}(\sqrt{n}) - 6\sqrt{n}\exp(-n) - 4n^{3/2}\exp(-n)}{\frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{n}) - \sqrt{n}\exp(-n)} = \frac{1}{2}$$

and thus

$$n = 1.527$$

Príklad 4

(a) The total energy is $E = (M + m\gamma)c^2$ where $\gamma \equiv 1/\sqrt{1-(v/c)^2}$, and the total momentum is $P = mv\gamma$. Since $E^2 - P^2c^2$ is Lorentz invariant, the energy in the center of mass frame E' is given by

$$E' = \sqrt{E^2 - P^2c^2} = c\sqrt{(M+m\gamma)^2c^2 - m^2v^2\gamma^2} = c^2\sqrt{M^2 + 2mM\gamma + m^2}\,.$$

After the fission, half of E' is carried by one of the daughter nuclei with momentum p' whose energy is $c\sqrt{(p')^2 + (M')^2c^2}$. Equating the two gives

$$p' = \frac{c}{2}\sqrt{M^2 + 2mM\gamma + m^2 - 4(M')^2}\,.$$

(b) Let the magnitude of the momentum of either e^- or $\bar{\nu}_e$ be p_2. From momentum conservation, we have $2p_2\cos\theta = p_1$ and energy conservation gives $2p_2c + c\sqrt{p_1^2 + m_p^2c^2} = m_nc^2$. Eliminating p_2 from the first equation, and using the second, we get

$$\cos\theta = \frac{p_1}{m_nc - \sqrt{(p_1)^2 + m_p^2c^2}}\,.$$

The expression on the right-hand side is a monotonically increasing function of p_1. Since $0 \le \cos\theta \le 1$, we have

$$0 \le p_1 \le \frac{(m_n^2 - m_p^2)c}{2m_N}\,.$$

When $p_1 = 0$, we have $\theta = \pi/2$ and e^- and $\bar{\nu}_e$ are back to back. In the other limit when $p_1 = (m_n^2 - m_p^2)c/(2m_N)$ we have $\theta = 0$ and $(e^-, \bar{\nu}_e)$ pair and the proton are back to back.