METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 3 leto22 – Príklady 6

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Cvičenie 5.5.2022

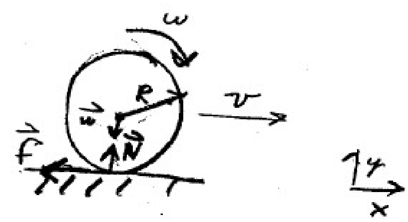
Príklad 1

#1

a) Torque caused by Friction leads to rolling

b) $V = \omega R$

c) Eqns of C.O.M.



① $m\ddot{y} = W - N = 0$

$\quad = Mg - N$

$\therefore N = Mg$

② $m\ddot{x} = -f$

$\quad = -\mu N$

$\quad = -\mu Mg$

$\therefore V = \dot{x} = \int_0^t dt' \ddot{x} = v_0 - \mu g t$

Eqn about C.O.M

$I\ddot{\theta} = Rf$

or $\frac{2}{5} MR^2 \ddot{\theta} = R\mu Mg$

$\ddot{\theta} = 5/2 \ \frac{\mu g}{R}$

$\omega = \dot{\theta} = \int_0^t dt' \ddot{\theta} = 5/2 \ \frac{\mu g}{R} t$

Rolling w/o slipping when $V = \omega R$, or

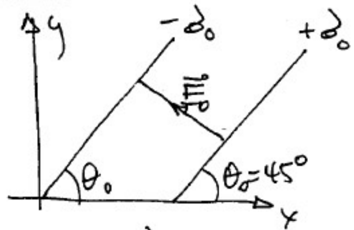
$v_0 - \mu g T = 5/2 \ \mu g T$

or $T = 2/7 \ v_0/\mu g$

Príklad 2

PART I : PROBLEM #3

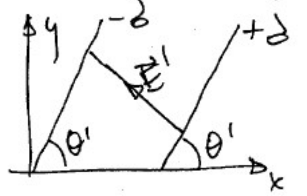
a)



In S_0, $|\vec{E}_0| = \frac{\sigma_0}{\epsilon_0}$, where

$$E_{0x} = -\frac{\sigma}{\sqrt{2}\,\epsilon_0}, \quad E_{0y} = +\frac{\sigma}{\sqrt{2}\,\epsilon_0}$$

b)



In S, $E_x' = E_{0x} = -\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}$ (∥ component preserved)

but $E_y' = \gamma E_{0y} = +\frac{\gamma\sigma_0}{\sqrt{2}\,\epsilon_0}$ (⊥ component to relative motion)

and $E' = \sqrt{\frac{1+\gamma^2}{2}}\,\frac{\sigma_0}{\epsilon_0}$.

c) $dy' = dy_0$ ⊥ to relative motion

$dx' = dx_0/\gamma$ length contracted along motion.

$\therefore \tan\theta' = \frac{dy'}{dx'} = \gamma \frac{dy_0}{dx_0}$, but $\frac{dy_0}{dx_0} = \tan 45^\circ = 1$

$\therefore \theta' = \tan^{-1}\gamma$ is angle of plates wrt x-axis

d) $$\frac{d\vec{r}'\cdot\vec{E}'}{r'E'} = \frac{(\frac{1}{\gamma}, 1)(-1, \gamma)}{\sqrt{1+\frac{1}{\gamma^2}}\sqrt{1+\gamma^2}} = \frac{\gamma^2 - 1}{\gamma^2 + 1} \neq 0$$

$\therefore \vec{E}'$ is <u>not</u> perpendicular to plates in S.

Príklad 2

SOLUTION:

(a) From Maxwell, we have

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}(\boldsymbol{j} + \boldsymbol{j}_{\rm d}) ,$$

and since $\nabla \cdot (\nabla \times \boldsymbol{B}) = 0$, it follows that $\nabla \cdot \boldsymbol{J} = 0$.

(b) From $\nabla \cdot \boldsymbol{E} = 4\pi\rho$, we have $\boldsymbol{E} = q\,\hat{\boldsymbol{r}}/r^2$ for a spherical distribution of charges. Thus,

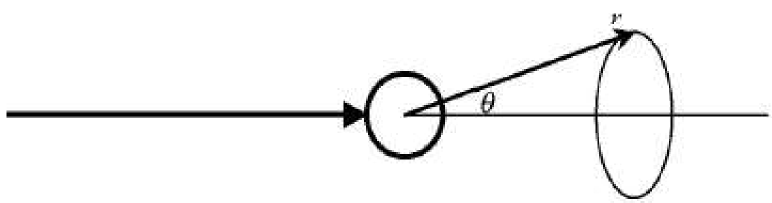
$$\boldsymbol{j}_{\rm d} = \frac{I}{4\pi r^2}\,\hat{\boldsymbol{r}} .$$

Note that $\nabla \cdot \boldsymbol{j}_{\rm d} = I\,\delta(\boldsymbol{r})$, which vanishes outside the sphere. Since $\nabla \cdot \boldsymbol{j} = 0$ outside the sphere as well, we have that $\nabla \cdot \boldsymbol{J} = 0$.

(c) From axial symmetry, we expect circular magnetic field lines. So use the integral form of Ampère's law,

$$\oint_{\partial\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{4\pi}{c}\int_{\Sigma} dA\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J} ,$$

where Σ is any two-dimensional surface, and $\hat{\boldsymbol{n}}$ is the local surface normal. Consider the $\boldsymbol{B}$ field along a circular loop a distance r from the center of the sphere, at an angle θ with respect to the wire's axis: Since there is no physical charge flowing through the loop, the total



current is just the displacement current from part (b). Let Σ be the

cap of a sphere of radius r, subtending a solid angle Ω. We therefore have

$$2\pi B r \sin\theta = \frac{4\pi}{c} \cdot \frac{\Omega}{4\pi} \cdot I = \frac{\Omega I}{c} ,$$

where $r\sin\theta$ is the radius of the loop, and $\Omega = 2\pi(1 - \cos\theta)$ is the solid angle subtended by the loop. We therefore have

$$B(r,\theta) = \frac{(1-\cos\theta)\,I}{cr\,\sin\theta} = \frac{I}{cr}\,\tan(\tfrac{1}{2}\theta) .$$

Note that there are two choices we could make for our cap. The complementary region Σ' would subtend solid angle $4\pi - \Omega$, and is pierced by the wire. In this case, both $\boldsymbol{j}$ and $\boldsymbol{j}_{\rm d}$ contribute to $\boldsymbol{J}$, and after considering the opposite orientation of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{r}}$ on Σ', we obtain

$$2\pi B r \sin\theta = \frac{4\pi}{c}\left\{ -\frac{4\pi - \Omega}{4\pi} \cdot I + I \right\} = \frac{\Omega I}{c} ,$$

as before.

(d) Near the wire, we have $\theta \to \pi$, and $\cos\theta \to 1$, and we recover the familiar expression

$$B(r,\theta) \approx \frac{2I}{cr\sin\theta} = \frac{2I}{cR} ,$$

where $R = r\sin\theta$ is the perpendicular distance from the wire.

Príklad 3

SOLUTION: The surface consists of N horizontal steps, $N_\uparrow$ upward steps, and $N_\downarrow$ downward steps. The degrees of freedom the system possesses are whether after each horizontal step the surface goes upward, downward, or remains at the same level. Let us represent these three possibilities by a scalar variable $\sigma = +1,\ 0$, or -1, respectively. We further label each step by a subscript $i \in \{1, \ldots, N\}$.

(a) With $H = \varepsilon \sum_i (1 + \sigma_i^2)$, the energy is written as a sum over the N columns. The contribution from each column is ε if $\sigma = 0$, *i.e.* if there is no step, and 2ε if $\sigma = \pm 1$, *i.e.* if there is a step in either direction. Since each step adds an extra lattice length to the length of the surface, this Hamiltonian properly accounts for the surface energy of ε per lattice length.

(b) The partition function is a sum over all configurations. This may be represented as a product over the steps, *viz.*

$$\begin{aligned} Z = \mathrm{Tr}\, e^{-H/k_\mathrm{B}T} &= \prod_{i=1}^{N} \sum_{\sigma_i=-1}^{1} e^{-(1+\sigma_i^2)/k_\mathrm{B}T} \\ &= e^{-N\varepsilon/k_\mathrm{B}T} \left(1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}\right)^N . \end{aligned}$$

(c) The free energy is

$$\begin{aligned} F &= k_\mathrm{B}T \ln Z \\ &= N\varepsilon - Nk_\mathrm{B}T \ln\left(1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}\right) . \end{aligned}$$

In the low temperature regime $k_\mathrm{B}T \ll \varepsilon$, we have $F \approx N\varepsilon$, which is the energy of a flat surface, whose length is the minimum value possible, N. In the high temperature regime $k_\mathrm{B}T \gg \varepsilon$, we have $-Nk_\mathrm{B}T \ln 3$, which reflects the fact that the surface is completely randomized, with 3^N equally probable configurations yielding an entropy $S = Nk_\mathrm{B} \ln 3$, as $T \to \infty$. The entropy term $-TS$ dominates the average energy E at these high temperatures.

(d) The total surface length is $L = N + N_\uparrow + N_\downarrow = N \cdot (1 + 2p)$, where p is the probability for an upward or downward step:

$$p = \frac{e^{-\varepsilon/k_\mathrm{B}T}}{1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}} .$$

Thus, $\langle L \rangle_{T \to 0} = N$, while $\langle L \rangle_{T \to \infty} = \frac{5}{3} N$.

Príklad 4

1. The state vector is

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$$

and the Schrödinger equation is

$$i\hbar\dot{\psi}_0 = ig\psi_1$$
$$i\hbar\dot{\psi}_1 = -ig\psi_0 + \Delta\,\psi_1$$

from which it follows that

$$-\hbar^2\ddot{\psi}_1 = g^2\psi_1 + i\hbar\Delta\,\psi_1$$

and therefore, since $\psi_0(0) = 1$ and $\psi_1(0) = 0$,

$$\psi_1(t) = A\left(e^{-i\omega_1 t} - e^{-i\omega_2 t}\right)$$

for some constant A and

$$\omega_{1,2} = \frac{\Delta}{2\hbar} \pm \sqrt{\frac{\Delta^2}{4\hbar^2} + \frac{g^2}{\hbar^2}}.$$

From the Schrödinger equation for ψ_0 we can now deduce that

$$\psi_0(t) = \frac{iAg}{\hbar}\left(\frac{1}{\omega_1}e^{-i\omega_1 t} - \frac{1}{\omega_2}e^{-i\omega_2 t}\right)$$

and therefore

$$\frac{iAg}{\hbar}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = 1$$

or

$$A = \frac{-i/2}{\sqrt{1 + \frac{\Delta^2}{4g^2}}}.$$

The desired probability is

$$|\psi_1(t)|^2 = |A|^2\left[2 - 2\cos(\omega_1 - \omega_2)t\right] = \frac{\sin^2 \frac{gt}{2\hbar}\sqrt{1 + \frac{\Delta^2}{4g^2}}}{1 + \frac{\Delta^2}{4g^2}}.$$

2. Continuous measurement of D forces the system to stay in one eigenstate of D. From the initial condition, this eigenstate is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If you have the solution to the first part in hand, then you can consider a sequence of measurements separated in time by ϵ; we eventually take the limit $\epsilon \to 0$. For sufficiently small ϵ, the probability that the first measurement yields one is

$$p_1 = \frac{g^2\epsilon^2}{4\hbar^2},$$

so the probability of remaining in the initial state is

$$p_0 = 1 - p_1 = 1 - \frac{g^2\epsilon^2}{4\hbar^2}.$$

All we really need is that $p_1 \propto \epsilon^2$, which is a consequence of the Schrödinger equation; the detailed solution is not necessary. After n measurements, the probability of still being in the initial state is at least p_0^n. For fixed time $t > 0$, choose $\epsilon = t/n$. Then the probability of being in initial state at time t is

$$P_0(t) \geq \left(1 - \frac{g^2 t^2}{4\hbar^2 n^2}\right)^n$$

and, for continuous measurement,

$$\lim_{n \to \infty} \log P_0(t) \geq n \log\left(1 - \frac{g^2 t^2}{4\hbar^2 n^2}\right) = 0.$$

We have $\lim_{n\to\infty} P_0(t) = 1$, and the system never leaves the initial state.