METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima22 – Príklady 4

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Príklad 1

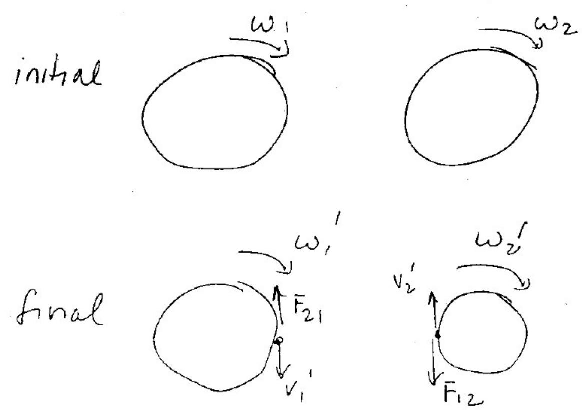
- Introduce the tension T in the rope and the force F which the mass M_h exerts to the right on the sphere. Use X, Y and x for the laboratory coordinates of M_h, M_v and m respectively.
- Write down equations for the acceleration of the cm of each mass:

$$\begin{aligned} M_h\ddot{X} &= T - F \\ m\ddot{a} &= F \\ M_v\ddot{Y} &= T - M_v g \end{aligned}$$

- Let θ represent the angular orientation of the sphere (increasing with clockwise motion) and write the equation for the angular acceleration of the sphere and the relation between $\ddot{\theta}$, $\ddot{a}$ and $\ddot{X}$:

$$\begin{aligned} \frac{2}{5}mR^2\ddot{\theta} &= -FR \\ \ddot{a} &= \ddot{X} + R\ddot{\theta} \end{aligned}$$

- These five equations can then be solved for T, F, $\ddot{\theta}$, $\ddot{X}$, $\ddot{Y}$,



Since the tangential final speeds must be the same (but opposite directions)

$$\omega_1' r_1 = -\omega_2' r_2 \qquad 5$$

$$\Rightarrow \omega_2' = -\frac{r_1}{r_2}\omega_1'$$

Since the integral of the torque equals the change in angular momentum

$$I_1(\omega_1' - \omega_1) = -\int |r_1||F_{21}|\,dt \qquad 5$$

and $I_2(\omega_2' - \omega_2) = -\int |r_2||F_{12}|\,dt$ → but $|F_{21}| = |F_{12}|$ from Newtons 3rd law

$$\therefore \quad \frac{I_1}{r_1}(\omega_1' - \omega_1) = \frac{I_2}{r_2}(\omega_2' - \omega_2)$$

Subst in for $\omega_2' = -\frac{r_1}{r_2}\omega_1'$

$$\therefore \frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\cdot\frac{r_1}{r_2}\omega_1' = \frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2$$

$$\omega_1' = \frac{\frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2}{\frac{I_1}{r_1} + I_2\frac{r_1}{r_2}} = \frac{\frac{L_1}{r_1} - \frac{L_2}{r_2}}{\frac{I_1}{r_1} + \frac{I_2 r_1}{r_2}}$$

with $I = \frac{1}{2}mr^2$ and some algebra (2)

$$\boxed{\omega_1' = \frac{m_1 r_1 \omega_1 - m_2 r_2 \omega_2}{(m_1+m_2) r_1}} \quad (3)$$

Príklad 2

(a) The Maxwell-Ampère law gives

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} .$$

With $\boldsymbol{j} = \sigma\boldsymbol{E}$, and with $\sigma \gg \omega$, we may drop the second term on the RHS. Taking the time derivative and invoking Faraday's law then gives

$$\begin{aligned} \frac{\partial}{\partial t}\nabla \times \boldsymbol{E} &\approx \frac{4\pi\sigma}{c}\frac{\partial \boldsymbol{E}}{\partial t} \\ &= -c\nabla \times \nabla \times \boldsymbol{E} \\ &= c\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} , \end{aligned}$$

whence Gauss's law results in

$$\nabla^2 \boldsymbol{E} = \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} .$$

Thus, we have

$$\frac{\partial^2 E_x}{\partial z^2} + \frac{4\pi i\sigma\omega}{c^2}E_x = 0 .$$

The solution is of the form

$$E_x(z,t) = A\,e^{i(kz-\omega t)} + B\,e^{-i(kz+\omega t)}$$

where

$$k^2 = \frac{4\pi i\sigma\omega}{c^2} \quad \Longrightarrow \quad k = (1+i)\,\frac{\sqrt{2\pi\sigma\omega}}{c}$$

Since $\mathrm{Im}(k) > 0$, we must set $B = 0$ to have a valid solution at $z \to \infty$, in which case $A = E_0$. The penetration depth of the electric field is then

$$\ell = \frac{1}{\mathrm{Re}(k)} = \frac{c}{\sqrt{2\pi\sigma\omega}} .$$

(b) The power dissipated per unit area is

$$\frac{P}{A} = \tfrac{1}{2}\sigma \int_0^\infty dz\,\left|E(z,t)\right|^2$$
$$= \frac{\sigma E_0^2}{2\,\mathrm{Re}(k)} = \sqrt{\frac{\sigma c^2}{8\pi\omega}}\,E_0^2\ .$$

Príklad 3

(a)

$$\Phi_0(\bar{r}) = \frac{q}{|\bar{r}-a\bar{z}|} + \frac{q}{|\bar{r}+a\bar{z}|} - \frac{2q}{|\bar{r}|}$$
$$= \frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} - \frac{2q}{r}$$
$$= \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} + \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2+2(a/r)\cos\theta}} - \frac{2q}{r}$$
$$\approx \frac{q}{r}\left[1-\frac{1}{2}\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right\}^2\right]$$
$$+\frac{q}{r}\left[1-\frac{1}{2}\left(\frac{a}{r}\right)^2 - \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right\}^2\right] - \frac{2q}{r}$$
$$\approx \frac{q}{r}\left[2-\left(\frac{a}{r}\right)^2 + \frac{3}{4}\left\{2\frac{a}{r}\cos\theta\right\}^2 - 2\right],\ \text{to second order in } a/r$$
$$\approx \frac{qa^2}{r^3}\left(3\cos^2\theta-1\right) = \frac{Q}{r^3}\left(3\cos^2\theta-1\right)$$

(b) On spherical shell $\Phi(r=b)=0$. Inside shell we have $\Phi_{in}(r) = \Phi_0(r)+\Phi_1(r)$ where $\Phi_1(r)$ satisfies $\nabla^2\Phi_1 = 0$. Expand potential in Legendre polynomials:

$$\Phi_1(r,\theta) = \sum_\ell \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$
$$= a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta + \left(a_2 r^2 + \frac{b_2}{r^3}\right)P_2(\cos\theta)$$

Now at r=0 potential should not diverge, and hence $b_i = 0$. At r=b we have

$$0 = \frac{Q}{b^3}\left(3\cos^2\theta-1\right) + a_0 + a_1 b\cos\theta + a_2 b^2\frac{(3\cos^2\theta-1)}{2},$$ therefore $a_0 = a_1 = 0$

and $a_2 = -\frac{2Q}{b^5}$. Therefore the interior potential is given by

$$\Phi_{in}(r) = \Phi_0(r)+\Phi_1(r) = \frac{Q}{r^3}\left(3\cos^2\theta-1\right)\left[1-\left(\frac{r}{b}\right)^5\right].$$