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Príklad 1

4. Two blocks and two pulleys (Question from Dave, solution from Peter)

Most straightforward to use Lagrangian with constraint for fixed length of rope:

$$L = \frac{1}{2}(m_1\dot{x}^2 + m_2\dot{y}^2) + m_1gx \quad \text{where (referring to the diagram with the problem) } x$$

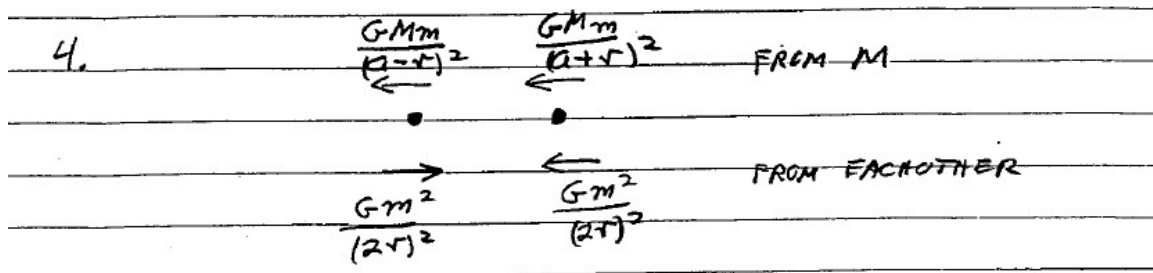
$$g(x,y) = 0 = x + 2y + d - l$$

increases downward and y increases to the right. The rope has length l and d accounts for all the rope not taken up by x and y . Also from the constraint, we have $\dot{x} = -2\dot{y}$. Using the Euler-Lagrange equations with undetermined multiplier gives

$$m_1\ddot{x} - m_1g + \lambda = 0 \quad -2m_1\ddot{x} + 2m_1g - 2\lambda = 0$$

$$m_2\ddot{y} + 2\lambda = 0 \quad \Rightarrow \quad -\frac{m_2\ddot{x}}{2} + 2\lambda = 0 \quad \Rightarrow \quad \ddot{x} = \frac{4m_1g}{4m_1 + m_2}$$

Check limits: $m_1 \gg m_2$, acceleration is g ; $m_1 \ll m_2$, acceleration goes to zero.



FOR STABILITY, FORCE TO THE LEFT ON THE LEFT OBJECT MUST BE LESS THAN THE FORCE TO THE LEFT ON THE RIGHT OBJECT

$$\frac{GMm}{(x-y)^2} - \frac{Gm^2}{4r^2} < \frac{GMm}{(x+y)^2} + \frac{GMm}{4r^2}$$

$$\frac{M}{(a^2-r^2)^2} (a+r)^2 - \frac{M}{(a^2-r^2)^2} (a-r)^2 < \frac{m}{2r^2}$$

$$4ar \frac{M}{a^4} < \frac{m}{2r^2}$$

$$\frac{8Mr}{a^3} < m = \frac{4}{3} \pi r^3 \rho$$

$$\frac{6}{\pi} \frac{M}{a^3} < \rho$$

Príklad 2

3. Current carrying wire (Question and solution from Peter)

a) Take loop of radius r around wire and use Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = 4\pi/c I \Rightarrow \vec{B} = B(r)\hat{\phi} = \frac{2I}{rc} \hat{\phi} \Rightarrow \vec{F} = q \frac{\vec{v}}{c} \times \vec{B} = \frac{2Ive}{rc^2} \hat{\rho} \text{ where } \hat{\rho} \text{ is radial direction in cylindrical coordinates.}$$

b) If F' moves along wire with v , then particle is at rest in F' and the Lorentz force is zero. Transform force from F :

$$F_{\perp} = \frac{dp_{\perp}}{dt} = \frac{dp'_{\perp}}{\gamma dt' - \beta \gamma dx'/dt'} = \frac{dp'_{\perp}/dt'}{\gamma + \beta \gamma dx'/dt'} = \frac{F'_{\perp}}{\gamma} \Rightarrow F'_{\perp} = \gamma \frac{2Ive}{rc^2}. \text{ Can also do}$$

problem by transforming currents and charge densities.

Príklad 3

SOLUTION: For a non-interacting ideal gas,

$$E = -\frac{\partial}{\partial \beta} N \ln \zeta,$$

where ζ is the single-molecule partition function

$$\zeta = \sum_{n=0}^{\infty} (n+1) \exp(-\beta n \epsilon).$$

This partition function can be evaluated as follows ($x \equiv \beta \epsilon$):

$$\zeta = -e^x \frac{d}{dx} \sum_{n=0}^{\infty} \exp(-(n+1)x) = -e^x \frac{d}{dx} \frac{e^{-x}}{1 - e^{-x}} = [1 - \exp(-\beta \epsilon)]^{-2}.$$

Hence, the sought contribution to the energy is

$$E = \frac{2N\epsilon}{\exp(\epsilon/kT) - 1}.$$

Alternatively, one can reproduce this result as follows. One can imagine that every molecule has two independent internal degrees of freedom of harmonic oscillator type, with energy spacing ε each. It is easy to see that this model gives the same spectrum and degeneracies if the energy is counted from the ground state. With this convention, the average energy of a single harmonic oscillator is $\varepsilon n_B(\varepsilon)$, where $n_B(\varepsilon)$ is the Bose-Einstein occupation number. Therefore, for the entire gas we get $E = 2N\varepsilon n_B(\varepsilon)$, in agreement with the first derivation.