

METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto21 – Príklady 6

VZOROVÉ RIEŠENIA

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Príklad 1

Assume at time t the bug is distance $x(t)$ from the wall. At the same time the free end of the band is distance $L + Vt$ from the wall and is still moving with speed V . Since the band stretches uniformly, the speed of stretching at the position of the bug is

$$V \frac{x(t)}{L + Vt}$$

and relative to the wall the bug has speed

$$V \frac{x(t)}{L + Vt} - u = \frac{dx(t)}{dt}$$

Making substitution

$$x(t) = (L + Vt)f(t)$$

we have equation for $f(t)$:

$$Vf(t) + (L + Vt)\frac{df}{dt} = Vf(t) - u \quad \Rightarrow \quad \frac{df}{dt} = -\frac{u}{L + Vt}$$

which is easy to solve with initial condition $f(0) = 1$,

$$f(t) = 1 - \frac{u}{V} \ln \frac{L + Vt}{L}$$

The bug reaches wall when $f(t) = 0$ which will happen at time

$$t = \frac{L}{V} (e^{V/u} - 1)$$

Bug will always reach the wall, even if the free end of the band is at first pulled faster than it can crawl! It will take the bug exponentially long time, but with every step the bug will be entering region where the “stretching wind” blows slightly slower against it. Of course for very fast bug we have $V/u \ll 1$ and $t = L/u$, as expected.

“This is a nifty analogy to kinematics in an expanding universe and demonstrates how we eventually receive light that is traveling at $u = c$ from a point that is constantly moving away from us at a speed V greater than c .

... in a uniformly expanding universe, if light, moving at $u = c$, is emitted one billion years after the Big Bang from a point that has always been expanding away from us at $V = 2c$ (so it starts $L = 2$ billion light years away), it will take the light $10^9(e^2 - 1)$, or 6.38 billion years, to get to us, but it will eventually arrive.”

Príklad 2

(a) Applying the law of conservation of energy for a bead of mass m and charge Q in the field of a dipole with dipole moment P gives

$$\frac{1}{2}mv^2 + QP\frac{\cos\theta}{r^2} = \frac{1}{2}mv_0^2 + QP\frac{\cos(\pi/2)}{r^2} = 0$$

This gives the expression for the velocity of the bead v at angle θ

$$v = \sqrt{\frac{-2QP\cos\theta}{mr^2}}.$$

The bead moves along a circular path until it reaches the point opposite its starting position. The bead stops there and then goes back executing a periodic motion.

(b) The radial component of the force on the charge due to the dipole $F_{r-dipole}$ can be calculated as the derivative of the electric potential energy with respect to r

$$\frac{\partial}{\partial r} \left(QP\frac{\cos\theta}{r^2} \right) = -2QP\frac{\cos\theta}{r^3}.$$

For the circular motion $mv^2/r = F_{r-dipole} + F_{r-string}$. Substituting v and $F_{r-dipole}$ gives the normal force exerted by the string on the bead $F_{r-string} = 0$. If the string were not there, the bead would move along the circular path as with the string.

Príklad 3

(a) Let the length of string on the table be r and the length hanging below be y . The mass m_1 is described by the polar coordinates (r, ϕ) . The fixed length constraint is $y + r = \ell$. The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{y}^2 + m_2gy \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1r^2\dot{\phi}^2 + m_2g(\ell - r). \end{aligned}$$

(b) The angular momentum is conserved: $\dot{p}_\phi = 0$, with

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1r^2\dot{\phi}.$$

The equation of motion for r yields

$$\begin{aligned} (m_1 + m_2)\ddot{r} &= m_1r\dot{\phi}^2 - m_2g \\ &= \frac{p_\phi^2}{m_1r^3} - m_2g \equiv -\frac{\partial U_{\text{eff}}}{\partial r}, \end{aligned}$$

where the effective potential is

$$U_{\text{eff}} = \frac{p_\phi^2}{2m_1r^2} + m_2gr.$$

The condition for stationary m_2 is $\dot{r} = \ddot{r} = 0$, which requires $U'_{\text{eff}}(r) = 0$. This, in turn, has the solution $r = a$, with

$$a = \left(\frac{p_\phi^2}{m_1 m_2 g} \right)^{1/3} .$$

(c) The tension in the string along the table is radial, hence there are no torques, and p_ϕ is conserved.

(d) We write $r = a + \eta$ and expand the equation of motion:

$$(m_1 + m_2) \ddot{\eta} = -U''_{\text{eff}}(a) \eta + \mathcal{O}(\eta^2) .$$

The solution is

$$\eta(t) = \eta_0 \cos(\omega t + \delta) ,$$

where the oscillation frequency is

$$\omega = \sqrt{\frac{U''_{\text{eff}}(a)}{m_1 + m_2}} = \sqrt{\frac{3m_2 g}{(m_1 + m_2)a}} .$$