METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 2 zima23 – Príklady 1

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Príklad 1

$$\boxed{I = \frac{1}{9}M\ell^2}$$

We'll use scaling arguments and parallel axis theorem, but instead of taking out parts of the triangle, we'll reverse the procedure and we'll be building our shape starting from some tiny-tiny (presumably) trangle of mass m_0 with side length ℓ_0. The moment of inertia of this 'elementary' building block is $I_0 = Cm_0\ell_0^2$ - and it turns out that the exact coefficient here $C \sim 1$ does not matter in the end! (It kind of makes sense because after infinite number of triangle 'dilutions' it is not quite clear what kind of elementary block we get!)

Now we step by step will be building our shape, and in $n \to \infty$ limit we'll get the needed result. Assume at some building level n we know the mass, size and moment of inertia of our shape, so their values at the next level will be

$$m_{n+1} = 3m_n \qquad \ell_{n+1} = 2\ell_n \qquad I_{n+1} = 3(I_n + m_n a_n^2) = 3I_n + m_n\ell_n^2$$

where $a_n = \ell_n/\sqrt{3}$ is the distance between the center of the figure at level n and the center of the new figure.

For mass and size at level n we have

$$m_n = 3^n m_0 \underset{n\to\infty}{\longrightarrow} M \qquad \ell_n = 2^n\ell_0 \underset{n\to\infty}{\longrightarrow} \ell \qquad \Rightarrow \qquad m_n\ell_n^2 = 12^n m_0\ell_0^2 \underset{n\to\infty}{\longrightarrow} M\ell^2$$

The moment of inertia at this level is found by iterations,

$$\begin{aligned}
&I_1 = 3Cm_0\ell_0^2 + m_0^2\ell_0^2 = (3C+1)m_0\ell_0^2 \\
&I_2 = 3I_1 + m_1\ell_1^2 = (3^2C + 3 + 12)m_0\ell_0^2 \\
&I_3 = 3I_2 + m_2\ell_2^2 = (3^3C + 3^2 + 3 * 12 + 12^2)m_0\ell_0^2 \\
&\vdots \\
&I_n = (12^{n-1} + 12^{n-2}3 + \cdots + 3^{n-1} + 3^nC)m_0\ell_0^2 \qquad (1)
\end{aligned}$$

Take limit $n \to \infty$ of

$$I = \lim_{n\to\infty} I_n = \lim_{n\to\infty} 12^{n-1}m_0\ell_0^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}} + \frac{1}{4^n}C\right)$$

the last term, dependent on C, plays no role in the infinite series which results in

$$= \frac{1}{12}M\ell^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \ldots\right) = \frac{1}{12}M\ell^2\frac{1}{1-1/4} = \frac{1}{12}\frac{4}{3}M\ell^2 = \boxed{\frac{1}{9}M\ell^2}$$

As an exercise do a similar scaling analysis to determine numerical value of coefficient C in moment of inertias $I = CmL^2$ of solid triangle and solid square of mass m and length of the side L. Check your answer by direct integration.

Príklad 2

Laplace's equation in spherical polars

We now come to an equation that is very widely applicable in physical science, namely $\nabla^2 u = 0$ in spherical polar coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0. \qquad (21.38)$$

Our method of procedure will be as before; we try a solution of the form

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Substituting this in (21.38), dividing through by $u = R\Theta\Phi$ and multiplying by r^2, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \qquad (21.39)$$

The first term depends only on r and the second and third terms (taken together) depend only on θ and ϕ. Thus (21.39) is equivalent to the two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \qquad (21.40)$$

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -\lambda. \qquad (21.41)$$

Equation (21.40) is a homogeneous equation,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0,$$

which can be reduced, by the substitution $r = \exp t$ (and writing $R(r) = S(t)$), to

$$\frac{d^2S}{dt^2} + \frac{dS}{dt} - \lambda S = 0.$$

This has the straightforward solution

$$S(t) = A\exp\lambda_1 t + B\exp\lambda_2 t,$$

and so the solution to the radial equation is

$$R(r) = Ar^{\lambda_1} + Br^{\lambda_2},$$

where $\lambda_1 + \lambda_2 = -1$ and $\lambda_1\lambda_2 = -\lambda$. We can thus take λ_1 and λ_2 as given by ℓ and $-(\ell+1)$; λ then has the form $\ell(\ell+1)$. (It should be noted that at this stage nothing has been either assumed or proved about whether ℓ is an integer.)

Hence we have obtained some information about the first factor in the separated-variable solution, which will now have the form

$$u(r,\theta,\phi) = \left[Ar^{\ell} + Br^{-(\ell+1)}\right]\Theta(\theta)\Phi(\phi), \tag{21.42}$$

where Θ and Φ must satisfy (21.41) with $\lambda = \ell(\ell+1)$.

The next step is to take (21.41) further. Multiplying through by $\sin^2\theta$ and substituting for λ, it too takes a separated form:

$$\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \tag{21.43}$$

Taking the separation constant as m^2, the equation in the azimuthal angle ϕ has the same solution as in cylindrical polars, namely

$$\Phi(\phi) = C\cos m\phi + D\sin m\phi.$$

As before, single-valuedness of u requires that m is an integer; for $m = 0$ we again have $\Phi(\phi) = C\phi + D$.

Having settled the form of $\Phi(\phi)$, we are left only with the equation satisfied by $\Theta(\theta)$, which is

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = m^2. \tag{21.44}$$

A change of independent variable from θ to $\mu = \cos\theta$ will reduce this to a form for which solutions are known, and of which some study has been made in chapter 16. Putting

$$\mu = \cos\theta, \qquad \frac{d\mu}{d\theta} = -\sin\theta, \qquad \frac{d}{d\theta} = -(1-\mu^2)^{1/2}\frac{d}{d\mu},$$

the equation for $M(\mu) \equiv \Theta(\theta)$ reads

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]M = 0. \tag{21.45}$$

This equation is the *associated Legendre equation*, which was mentioned in subsection 18.2 in the context of Sturm–Liouville equations.

We recall that for the case $m = 0$, (21.45) reduces to Legendre's equation, which was studied at length in chapter 16, and has the solution

$$M(\mu) = EP_\ell(\mu) + FQ_\ell(\mu). \tag{21.46}$$

We have not solved (21.45) explicitly for general m, but the solutions were given in subsection 18.2 and are the associated Legendre functions $P_\ell^m(\mu)$ and $Q_\ell^m(\mu)$, where

$$P_\ell^m(\mu) = (1-\mu^2)^{|m|/2} \frac{d^{|m|}}{d\mu^{|m|}} P_\ell(\mu), \tag{21.47}$$

and similarly for $Q_\ell^m(\mu)$. We then have

$$M(\mu) = EP_\ell^m(\mu) + FQ_\ell^m(\mu); \tag{21.48}$$

here m must be an integer, $0 \le |m| \le \ell$. We note that if we require solutions to Laplace's equation that are finite when $\mu = \cos\theta = \pm 1$ (i.e. on the polar axis where $\theta = 0, \pi$), then we must have $F = 0$ in (21.46) and (21.48) since $Q_\ell^m(\mu)$ diverges at $\mu = \pm 1$.

It will be remembered that one of the important conditions for obtaining finite polynomial solutions of Legendre's equation is that ℓ is an integer ≥ 0. This condition therefore applies also to the solutions (21.46) and (21.48) and is reflected back into the radial part of the general solution given in (21.42).

Now that the solutions of each of the three ordinary differential equations governing R, Θ and Φ have been obtained, we may assemble a complete separated-

variable solution of Laplace's equation in spherical polars. It is

$$u(r,\theta,\phi) = (Ar^\ell + Br^{-(\ell+1)})(C\cos m\phi + D\sin m\phi)[EP_\ell^m(\cos\theta) + FQ_\ell^m(\cos\theta)], \tag{21.49}$$

where the three bracketted factors are connected only through the *integer* parameters ℓ and m, $0 \le |m| \le \ell$. As before, a general solution may be obtained by superposing solutions of this form for the allowed values of the separation constants ℓ and m. As mentioned above, if the solution is required to be finite on the polar axis then $F = 0$ for all ℓ and m.

a) Since the problem has an aziumuthal symmetry, we have

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)$$

Using that

$$P_0(\cos\theta) = 1$$
$$P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]$$

we obtain

$$\phi(\theta) = \phi_0\cos^2\theta = \frac{\phi_0}{3}[2P_2(\cos\theta)+1] = \frac{\phi_0}{3}[2P_2(\cos\theta)+P_0(\cos\theta)]$$

And thus

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\,\frac{\phi_0}{3}[2P_2(\cos\theta)+P_0(\cos\theta)]P_m(x) \\ &= \frac{\phi_0}{3}\left[2\frac{2}{2m+1}\delta_{m,2} + \frac{2}{2m+1}\delta_{m,0}\right] \\ &= \frac{\phi_0}{3}\left[\frac{4}{5}\delta_{m,2} + 2\delta_{m,0}\right]\end{aligned}$$

If we want to evaluate the potential inside of the sphere, we need to set $B_l = 0$ and obtain

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\left[\sum_l A_l R^l P_l(x)\right]P_m(x) \\ &= A_m R^m \frac{2}{2m+1}\end{aligned}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = A_2 R^2 \frac{2}{5}$$
$$A_2 = \frac{2\phi_0}{3R^2}$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2A_0$$
$$A_0 = \frac{\phi_0}{3}$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2 P_2(\cos\theta)$$

For the potential outside of the sphere, we set $A_l = 0$ and obtain

$$\int_{-1}^{1} dx\,\phi(R,x)P_m(x) = \int_{-1}^{1} dx\left[\sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta)\right]P_m(x)$$
$$= B_m R^{-(m+1)}\frac{2}{2m+1}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = B_2 R^{-3}\frac{2}{5}$$
$$B_2 = \frac{2\phi_0}{3}R^3$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2\frac{B_0}{R}$$
$$B_0 = \frac{\phi_0}{3}R$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}\frac{R}{r}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

b) The electric field inside the sphere is then given by

$$\vec{E} = -\nabla\phi(r,\theta) = -\hat{r}\frac{\partial\phi(r,\theta)}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}$$
$$= -\hat{r}\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta) - \hat{\theta}\frac{1}{r}\frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2[-3\sin\theta\cos\theta]$$
$$= -\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta)\hat{r} + 2\phi_0\frac{r}{R^2}[\sin\theta\cos\theta]\hat{\theta}$$

c) Using Gauss' law inside the shere

$$\oint\vec{E}\cdot d\vec{A} = -\frac{4\phi_0}{3}\frac{r}{R^2}r^2\int d\varphi\int_{-1}^{1} d(\cos\theta)P_2(\cos\theta) = 0$$

Thus, no charges are contained inside the sphere.

Príklad 3

SOLUTION: Find $\vec{B} = \hat{\phi}2I/rc$ and then $\vec{A} = -\hat{z}(2I/c)\ln(r/a)$, so the electrons have

$$L = -mc^2\sqrt{1 - v^2/c^2} + (2I|e|/c^2)v_z\ln(r/a).$$

The energy and p_z are conserved:

$$p_z = \frac{\partial L}{\partial v_z} = \gamma m v_z + (2I|e|/c^2)\ln(r/a) = \gamma_0 m v_0.$$

$$H = \gamma mc^2 = \gamma_0 mc^2$$

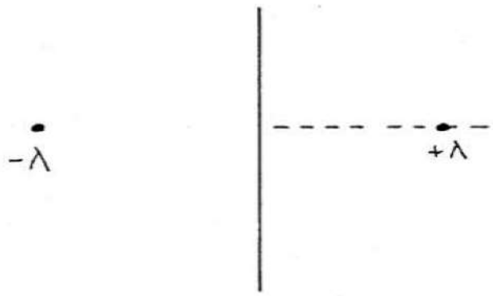
with $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\gamma_0 \equiv 1/\sqrt{1 - v_0^2/c^2}$. So $\gamma = \gamma_0$ and r_{max} is where $\dot{r} = 0$, which means that $v_z = -v_0$ (half-period of cyclotron rotation), which gives

$$r_{max} = a\exp(\gamma_0 m v_0 c^2/I|e|).$$

Príklad 4

a) Because $a \ll d$, we can treat the wire as if it is a carrier of charge of linear density λ.

Use the method of images to account for the induced charge on the surface of the conducting sheet, so imagine linear charge density $-\lambda$ a distance d past the conducting sheet.



Then, if we choose the electrostatic potential, φ, to be zero on the sheet, along the line passing through the charges,

$$\Delta \varphi = -\int_0^{d-a} dx \; E(x)$$

$$E(x) = \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{-x+d} + \frac{1}{x+d}\right)$$

So

$$\Delta\varphi = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(x+d) - \ln(d-x)\right)\Big|_0^{d-a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(\frac{2d-a}{d}\right) + \ln\left(\frac{d}{a}\right)\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right)$$

The capacitance per unit length is the charge per unit length $/ |\Delta\varphi|$

$$\frac{C}{L} = \lambda / |\Delta\varphi| = \frac{2\pi\varepsilon_0}{\ln\left(\frac{2d-a}{a}\right)}$$

b) The magnitude of the electric field from the wire at $+d$ at the surace of the plane is

$$|E_+(y)| = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\sqrt{d^2+y^2}}$$

The component $\perp$ to the plane is the above multiplied by $d/\sqrt{d^2+y^2}$.

The components $\parallel$ to the plane from the wire and its image cancel of course & the $\perp$ component is doubled:

$$|E| = \frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2} \quad \rightarrow \quad E(y) = -\frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2}\,\hat{x}$$

Then, the charge density is $\sigma = E\varepsilon_0$

So $\sigma(y) = \frac{-\lambda d}{\pi(d^2+y^2)}$

PRÍKLAD 3: overenie, že $\int_{-\infty}^{\infty} dy\, \sigma(x,y) = -\lambda$

(premyslieť prečo?)

$$-\int_{-\infty}^{\infty} dy\, \frac{\lambda d}{\pi(d^2+y^2)} = -\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{dy}{d}\,\frac{1}{1+\frac{y^2}{d^2}} = -\frac{\lambda}{\pi}\arctan\frac{y}{d}\Big|_{-\infty}^{\infty} =$$

$$= -\frac{\lambda}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \boxed{-\lambda}$$