METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto24 – Príklady 4

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Príklad 1

Let v = velocity of mass m and V = velocity of mass M

There are 2 external forces on the system of M and m, namely gravity, which is conservative, and the normal force of the table, which does no work.

Therefore the sum of the kinetic and gravitational potential energies is conserved:

$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

The external forces have no horizontal components so the horizontal component of the momentum is conserved.

$$mv - MV = 0$$

Combining these gives:

$$v = \sqrt{\frac{2gR}{1+\frac{m}{M}}} \qquad \text{and} \qquad V = \frac{m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}}$$

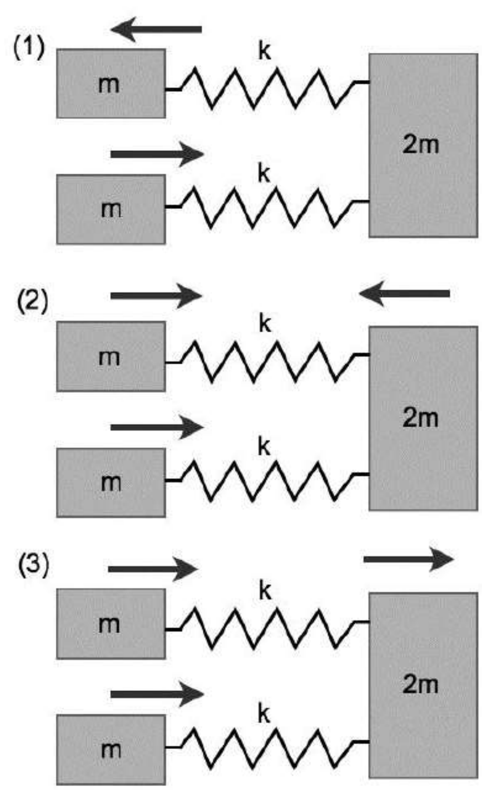
Príklad 2

Príklad 3

In a normal mode, all the masses oscillate with the same frequency and the same phase. There are three normal modes for a system with three masses. Because of high symmetry of the problem, in this case, all of these can be found without doing any calculations. First normal mode corresponds to the case where small masses on the left moves by equal and opposite amount and the mass 2m on the right is stationary. The frequency of this normal mode is the same as a single mass m connected to a spring with spring constant k i.e. $\omega_1 = \sqrt{\frac{k}{m}}$.

The second normal mode corresponds to the motion in which two small masses on the left move together by the same amount, and the mass 2m moves by the equal but opposite amplitude. Since the two small mass moves together, we can consider them as one mass with value 2m. So the problem boils down to two masses with value 2m joined together with a spring of total spring constant 2k. The frequency of this motion is $\omega_2 = \sqrt{\frac{2k}{m}}$.

Finally, the last normal mode corresponds to the case where all the masses are displaced by the same amount, they will not return back to their equilibrium positions, so $\omega_3 = 0$.



Príklad 4

SOLUTION: This problem is conveniently solved using the method of images.

(a) An equipotential $\phi = 0$ is achieved over the entire sphere by placing an image charge of strength $\tilde{Q} = -(a/b)\,Q$ a distance a^2/b from the center, also at $\theta = 0$. Even if we did not remember these values, they could easily be determined by supposing the image charge lies a distance d from the center, and then demanding that the potential vanish anywhere on the surface of the sphere:

$$\phi(R,\theta,\phi) = \frac{Q}{\sqrt{a^2+b^2-2ab\,\cos\theta}} + \frac{\tilde{Q}^2}{\sqrt{a^2+d^2-2ad\,\cos\theta}} = 0 \quad \forall\,\theta\,.$$

After pushing one radical over to the other side of the equation, inverting both sides, and squaring, one then separately equates the constant terms on both sides as well as the coefficients of $\cos\theta$. This yields two equations:

$$b\,\tilde{Q}^2 = d\,Q^2$$
$$(a^2+b^2)\,\tilde{Q}^2 = (a^2+d^2)\,Q^2\ ,$$

which yield the familiar results $\tilde{Q} = -aQ/b$ and $d = a^2/b$. The potential everywhere is then

$$\phi(r,\theta,\varphi) = \frac{Q}{\sqrt{r^2+b^2-2br\,\cos\theta}} - \frac{Q}{\sqrt{\left(\frac{br}{a}\right)^2 + a^2 - 2br\cos\theta}}\ .$$

(b) It is tempting to compute the potential due to the image charge at Q,

$$\phi_{\text{image}}(r)\big|_{\theta=0} = -\frac{Qa}{br-a^2} \qquad \Longrightarrow \qquad \phi_{\text{image}}(b) = -\frac{Qa}{b^2-a^2}\ ,$$

multiply by Q, and conclude that $W = aQ^2/(b^2-a^2)$ is the work required. This is wrong! The reason is that *the image charge moves with Q*. To get the right answer, integrate $F\,dr$, where F is the radial component of the force, $\boldsymbol{F} = Q\boldsymbol{E}$. The electric field due to the image at Q is

$$E(r) = -\frac{\partial\phi_{\text{image}}(r)}{\partial r}\bigg|_{b=r} = -\frac{Qar}{(r^2-a^2)^2}\ .$$

Next we multiply by Q and then integrate to get the work done *on* the charge:

$$W = -Q \int_b^\infty dr\, E(r) = a\,Q^2 \int_b^\infty \frac{r\,dr}{(r^2 - a^2)^2} = \frac{a\,Q^2}{2(b^2 - a^2)} \; .$$

The wrong answer we obtained by the simplistic analysis is a factor of two too large.

Príklad 5

Assume at time t the bug is distance $x(t)$ from the wall. At the same time the free end of the band is distance $L + Vt$ from the wall and is still moving with speed V. Since the band stretches uniformly, the speed of stretching at the position of the bug is

$$V \frac{x(t)}{L + Vt}$$

and relative to the wall the bug has speed

$$V \frac{x(t)}{L + Vt} - u = \frac{dx(t)}{dt}$$

Making substitution

$$x(t) = (L + Vt) f(t)$$

we have equation for $f(t)$:

$$V f(t) + (L + Vt)\frac{df}{dt} = V f(t) - u \qquad \Rightarrow \qquad \frac{df}{dt} = -\frac{u}{L + Vt}$$

which is easy to solve with initial condition $f(0) = 1$,

$$f(t) = 1 - \frac{u}{V} \ln \frac{L + Vt}{L}$$

The bug reaches wall when $f(t) = 0$ which will happen at time

$$\boxed{t = \frac{L}{V}\left(e^{V/u} - 1\right)}$$

Bug will always reach the wall, even if the free end of the band is at first pulled faster than it can crawl! It will take the bug exponentially long time, but with every step the bug will be entering region where the "stretching wind" blows slightly slower against it. Of course for very fast bug we have $V/u \ll 1$ and $t = L/u$, as expected.

"This is a nifty analogy to kinematics in an expanding universe and demonstrates how we eventually recieve light that is traveling at $u = c$ from a point that is constantly moving away from us at a speed V greater than c.

... in a uniformly expanding universe, if light, moving at $u = c$, is emitted one billion years after the Big Bang from a point that has always been expanding away from us at $V = 2c$ (so it starts $L = 2$ billion light years away), it will take the light $10^9(e^2 - 1)$, or 6.38 billion years, to get to us, but it will eventually arrive."