METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto24 – Príklady 6

VZOROVÉ RIEŠENIA

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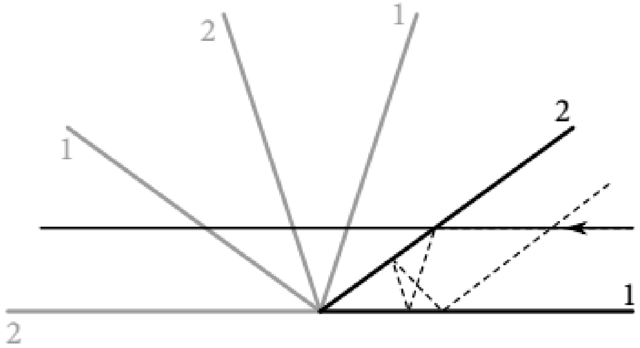
Príklad 1

17 Namiesto počítania komplikovanej geometrie si uvedomíme, že nám priestor medzi zrkadlami stačí niekoľkokrát odzrkadliť cez jedno zo zrkadiel. Potom totiž môžeme uvažovať, že lúč svetla putuje po rovnej trajektórii ako na obrázku.

Na konci musí byť lúč svetla rovnobežný s druhým zrkadlom. To znamená, že uhol 180° musí byť uhlami α rozdelený na celý počet častí. Zdalo by sa teda, že prípustné hodnoty α budú hodnoty $\frac{180°}{n}$, kde n sú prirodzené čísla. Avšak na to, aby bol lúč na konci rovnobežný s druhým zrkadlom, je potrebné, aby zrkadlo, ktoré sa zobrazí na uhol 180°, bolo práve to druhé zrkadlo.



Obrázok 17.1: *Riešenie s práve štyrmi odrazmi*

Rýchlo si uvedomíme, že táto podmienka nám hovorí, že číslo n musí byť nepárne. Prípustné hodnoty uhla α sú teda

$$\alpha = \frac{180°}{2n+1}, \tag{17.1}$$

kde $n \in \mathbb{N}$.

Príklad 2

Príklad 3

(a) Let the length of string on the table be r and the length hanging below be y. The mass m_1 is described by the polar coordinates (r, ϕ). The fixed length constraint is $y + r = \ell$. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \tfrac{1}{2}m_2\dot{y}^2 + m_2 g y \\ &= \tfrac{1}{2}(m_1 + m_2)\dot{r}^2 + \tfrac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g(\ell - r) \ . \end{aligned}$$

(b) The angular momentum is conserved: $\dot{p}_\phi = 0$, with

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \ .$$

The equation of motion for r yields

$$\begin{aligned}(m_1 + m_2)\ddot{r} &= m_1 r \dot{\phi}^2 - m_2 g \\ &= \frac{p_\phi^2}{m_1 r^3} - m_2\, g \equiv -\frac{\partial U_{\text{eff}}}{\partial r} \ ,\end{aligned}$$

where the effective potential is

$$U_{\text{eff}} = \frac{p_\phi^2}{2m_1 r^2} + m_2 g r \ .$$

The condition for stationary m_2 is $\dot{r} = \ddot{r} = 0$, which requires $U'_{\text{eff}}(r) = 0$. This, in turn, has the solution $r = a$, with

$$a = \left(\frac{p_\phi^2}{m_1 m_2\, g} \right)^{1/3} \ .$$

(c) The tension in the string along the table is radial, hence there are no torques, and p_ϕ is conserved.

(d) We write $r = a + \eta$ and expand the equation of motion:

$$(m_1 + m_2)\, \ddot{\eta} = -U''_{\text{eff}}(a)\, \eta + \mathcal{O}(\eta^2) \ .$$

The solution is

$$\eta(t) = \eta_0 \cos\left(\omega t + \delta\right) \ ,$$

where the oscillation frequency is

$$\omega = \sqrt{\frac{U''_{\text{eff}}(a)}{m_1 + m_2}} = \sqrt{\frac{3m_2\, g}{(m_1 + m_2)a}} \ .$$

Príklad 4

You start to skid when the required force to accelerate along your path is equal to the maximum possible friction force:

$$m|\ddot{\mathbf{r}}| = \mu m g \tag{1}$$

As we'll see the point of skidding is quite close to the pole, so we can neglect the Earth's curvature and assume we are on a flat surface of a skate rink. Let's work in cylindrical coordinates with origin at the pole, radius r and angle ϕ. The radius-vector is $\mathbf{r} = r\hat{r}$ and we have to remember that the unit vectors in cylindrical coordinates are function of angle: $\hat{r} = \hat{r}(\phi)$ and $\hat{\phi} = \hat{\phi}(\phi)$. This means we have to differentiate them as well, when finding the acceleration. Velocity is

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$$\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\dot{\phi} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

and since the velocity is always pointing in NW direction (45 degrees to both parallels and meridians) we can write

$$\dot{r} = -\frac{v}{\sqrt{2}} \qquad\qquad r\dot{\phi} = -\frac{v}{\sqrt{2}}$$

For acceleration we find

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}] = [\ddot{r} - r\dot{\phi}^2]\hat{r} + [\dot{r}\dot{\phi} + \frac{d}{dt}(r\dot{\phi})]\hat{\phi}$$

Using the values of velocity components and their time-invariance, we get

$$\ddot{\mathbf{r}} = -r\dot{\phi}^2\hat{r} + \dot{r}\dot{\phi}\hat{\phi} = -\frac{v^2}{2r}\hat{r} - \frac{v^2}{2r}\hat{\phi} \qquad \Rightarrow \qquad |\ddot{r}| = \frac{v^2}{2r}\sqrt{2}$$

which means that the distance where skidding starts is

$$\boxed{R = \frac{v^2}{\sqrt{2}\mu g} = 649\ m\,.}$$

Príklad 5

Solution:

1.

$$J = \rho v \pi r^2; \quad v = \sqrt{2gh};$$

2.

$$-\rho\frac{d}{dt}\left(\frac{\pi h^3 \tan^2\alpha}{3}\right) = J;$$

3.

$$\rho\frac{d}{dt}\left(\frac{\pi h^3 \tan^2\alpha}{3}\right) = -\rho v \pi r^2 = -\rho\pi r^2\sqrt{2gh};$$

$$\dot{h}h^{3/2} = -\frac{r^2}{\tan^2\alpha}\sqrt{2g};$$

4.

$$h(t) = \left(h(t=0)^{5/2} - t\frac{5r^2\sqrt{2g}}{2\tan^2\alpha}\right)^{2/5}$$

Príklad 6

(a) For the case where there is no dielectric, we can use Gauss' law to find the electric field at a radius r

$$\int E\,dA = \frac{Q_{enclosed}}{\epsilon}$$
$$E2\pi r\ell = \frac{\ell}{L}\left(\frac{Q}{\epsilon}\right)$$
$$E(r) = \frac{Q}{2\pi\epsilon L} \times \frac{1}{r}$$

(b) To find the capacitance, we use

$$C = \frac{Q}{V}$$

We can use the solution to (a) to find V from
$V = -\int E\,dr$ (where r goes from a to r)

$$V = -\frac{Q}{2\pi\epsilon L}\int\left(\frac{dr}{r}\right)$$
$$V_{ab} = -\frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$$

Therefore

$$C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$$

(c) When we pull the dielectric out, then we will have a length x which has no dielectric in it, and a length $L - x$ which has dielectric in it. So the net capacitance will be that of two capacitors in parallel

$$C = C_{x-no\ dielectric} + C_{L-x-with\ dielectric}$$

Where the capacitance with no dielectric has a similar form to that found in part (b) except that we use ϵ_0 instead of ϵ. So

$$C = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left[x + \frac{\epsilon}{\epsilon_0}(L - x)\right]$$

Where the change in capacitance is therefore given by

$$dC = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left(1 - \frac{\epsilon}{\epsilon_0}\right)dx$$

The change in the work will be equal to the change in the potential energy. Recall that V remains constant since it is attached to a battery. So

$$Fdx + VdQ = \frac{1}{2}V^2dC$$

Where

$$dQ = VdC$$

So if we substitute that in to the work equation, we get

$$Fdx + V(VdC) = \frac{1}{2}V^2 dC$$

$$Fdx = -\frac{1}{2}V^2 dC$$

So we can substitute in for dC and then substitute in the work equation to get

$$Fdx = -\frac{1}{2}V^2 \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}(1 - \frac{\epsilon}{\epsilon_0})dx$$

$$F = V^2 \frac{\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}(\frac{\epsilon}{\epsilon_0} - 1)$$