METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH 1 leto24 – Príklady 6

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Príklad 1

22 Načrtnime si náš plán. Palička sa nehýbe, ak súčet síl a momentov síl na ňu pôsobiacich je nulový. Na paličku pôsobí gravitačná sila smerom nadol a elektrické sily od zvyšných nábojov v rôznych vodorovných smeroch, a nakoniec je tam strop, ktorý celú sústavu určite udrží. Momenty síl bude najprirodzenejšie počítať vzhľadom na bod na strope. V takom prípade na každú paličku pôsobí iba moment od tiažovej sily a od ostatných paličiek. Takže potrebujeme zistiť, o aký uhol sa palička musí vychýliť, aby bol ich súčet nulový.

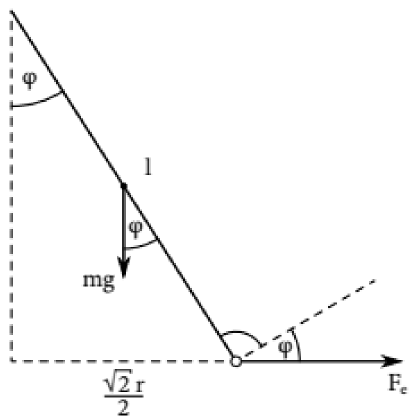
Budeme počítať so všeobecnými hodnotami, takže si označíme náboj q, hmotnosť paličky m a jej dĺžku d. Zo symetrie je jasné, že v rovnovážnej polohe tvoria bodové náboje vrcholy štvorca a stačí nám pozerať sa len na momenty síl pôsobiacich na jednu paličku. Nech má štvorec dĺžky strán r, čiže jeho uhlopriečka má dĺžku $\sqrt{2}r$. Vyberme si jednu paličku. Na náboj na nej pôsobia elektrické sily od susedných nábojov. Obe majú veľkosť

$$F_{e1} = \frac{q^2}{4\pi\varepsilon_0 r^2} \tag{22.1}$$

a sú navzájom kolmé. Preto keď ich vektorovo zložíme, dostaneme silu veľkosti $\sqrt{2}F_{e1}$ so smerom od stredu štvorca k náboju na našej paličke. Sila od náboja, ktorý leží uhlopriečne od nášho, má veľkosť

$$F_{e2} = \frac{q^2}{4\pi\varepsilon_0(\sqrt{2}r)^2} \tag{22.2}$$

a smer má tiež od stredu k nášmu náboju. Na náboj na našej paličke teda pôsobí celková elektrická sila veľkosti $F_e = \sqrt{2}F_{e1} + F_{e2}$ smerujúca od stredu štvorca. Gravitačná sila pôsobí samozrejme nadol a pôsobí v ťažisku paličky. Na obrázku 22.1 to celé vyzerá nasledovne:



Spodná strana trojuholníka je polovica uhlopriečky štvorca, preto je jej dĺžka $\frac{\sqrt{2}r}{2}$. A takisto z trojuholníka vidíme, že $r = \frac{2}{\sqrt{2}} d \sin\varphi$. Rovnica rovnosti momentov síl je teda

$$
\begin{aligned}
mg\frac{d}{2}\sin\varphi &= \frac{q^2}{4\pi\varepsilon_0 r^2}\left(\frac{1}{2}+\sqrt{2}\right) d\cos\varphi \\
mg\frac{d}{2}\sin\varphi &= \frac{q^2}{4\pi\varepsilon_0\left(\frac{2}{\sqrt{2}}d\sin\varphi\right)^2}\left(\frac{1}{2}+\sqrt{2}\right) d\cos\varphi \\
\frac{\sin^3\varphi}{\cos\varphi} &= \frac{q^2}{4\pi\varepsilon_0 d^2 mg}\left(\frac{1}{2}+\sqrt{2}\right).
\end{aligned} \tag{22.3}
$$

V tomto momente sa dostávame k tomu, že táto úloha nemá analytické riešenie, teda že nevieme vyjadriť, čomu sa rovná uhol φ. Musíme teda zobrať kalkulačku a s rozumom vyskúšať pár hodnôt φ. Dostatočne presný výsledok nájdeme napríklad binárnym vyhľadávaním. Výsledkom pre hodnoty zo zadania je $\varphi \doteq 68°$.

Príklad 2

Príklad 3

The pressure force from the water on the bottom is

$$F_w = \rho g H \frac{\pi D^2}{4}$$

it is pointing up and we may think it is applied to the center of the bottom. To separate the bottom from the cylinder we put the weight as close as possible to the side of the cylinder. Then the moments equation relative to the rotation axis at the diametrically opposite point will be

$$mgD - F_w\frac{D}{2} = 0 \qquad \Rightarrow \qquad \boxed{m = \frac{F_w}{2g} = \frac{1}{8}\rho H \pi D^2}$$

- only half the weight of the displaced liquid!

Príklad 4

(a). The electric field will be

$$\vec{E} = \frac{V_0}{L}\hat{z}$$

For the magnetic field,

$$2\pi s B(s) = \mu_0 I_{enc} = \mu_0 \pi s^2 J = \mu_0 \pi s^2 \sigma_c E,$$

where s is the radial coordinate, and J is the current density. Thus

$$\vec{B}(s) = \frac{\mu_0 s \sigma_c E}{2}\hat{\phi}.$$

Thus

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = -\frac{\sigma_c {V_0}^2 s}{2L^2}\hat{s},$$

and

$$P = 2\pi s L S = \frac{\pi\sigma_c {V_0}^2 s^2}{L}.$$

and so at $s = b/2$, the power flow will be

$$\vec{P} = -\frac{\pi b^2 \sigma_c {V_0}^2}{4L}\hat{s}$$

(i.e., flowing into the volume).

(b) For comparison, the Ohmic power will be

$$P = IV = (\tfrac{\pi b^2}{4})JV_0 = (\tfrac{\pi b^2}{4})\sigma_c E V_0 = \tfrac{\pi b^2 \sigma_c V_0{}^2}{4L},$$

and so they are the same.

Príklad 5

SOLUTION: A typical scale has atoms on lattice spacings around 2–3 Å, so let's say 2.5×10^{-10} m. Bond energy is a few electron volts; we'll say 2 eV, or about 3×10^{-19} J.

Approximating the trough of the potential as one that has a minimum at $(a,\ -\varepsilon)$, and intersects the x-axis at $a/2$, we get $U \approx -\varepsilon + \frac{4\varepsilon}{a^2}(x-a)^2$. The force is the negative gradient of potential, so $F = -\frac{8\varepsilon}{a^2}\Delta x$, where $\Delta x = (x-a)$ is the displacement from equilibrium. For this single-pair interaction, we associate the spring constant, $k = \frac{8\varepsilon}{a^2}$.

The cross section of the fiber has A/a^2 bonds, where A is the cross-sectional area. Therefore, the force required to separate adjacent lattice planes by Δx is multiplied by this ratio. We aim for a total spring constant, so that $F_{\text{tot}} = -k_{\text{tot}}\Delta X$, ΔX being the total fiber stretch that we seek. We note that each inter-atom displacement is a small fraction of the total fiber displacement by the ratio of inter-atomic spacing to total length: $\Delta x = \frac{a}{L}\Delta X$, where L is the fiber length. We therefore multiply the single-pair spring constant by $\frac{A}{a^2}\frac{a}{L}$ to get the total spring constant, resulting in $k_{\text{tot}} = \frac{8\varepsilon}{a^2}\frac{A}{a^2}\frac{a}{L} = \frac{8\varepsilon}{a^3}\frac{A}{L}$. The first factor is equivalent to an elastic modulus for the material, for which we would compute about 150 GPa (a reasonable value for metal). Multiplying by A/L results in approximately $k_{\text{tot}} \approx 1000$ N/m. Hanging a 1 kg (10 N) mass on this fiber would result in a 0.01 m displacement, which matches crude intuition, and also suggests that the material would probably snap for a strain this large.

Príklad 6

(a) Newton's equations are

$$m\partial_t^2\vec{x} = \vec{F} = -m\vec{\nabla}U\ .$$

Or we can use the Lagrangian

$$L = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mU$$

and the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}.$$

Using $\frac{\partial r}{\partial x} = \frac{x}{r}$, the result is

$$\ddot{x} = -\frac{\alpha x}{r^3} - \alpha\beta\left(\frac{15xz^2}{r^7} - \frac{3x}{r^5}\right)$$

$$\ddot{y} = -\frac{\alpha y}{r^3} - \alpha\beta\left(\frac{15yz^2}{r^7} - \frac{3y}{r^5}\right)$$

$$\ddot{z} = -\frac{\alpha z}{r^3} - \alpha\beta\left(\frac{15zz^2}{r^7} - \frac{3z}{r^5} - \frac{6z}{r^5}\right)$$

It will be useful for the next part to rewrite the equations of motion as

$$m\partial_t^2\vec{r} = -\alpha\vec{r}\left(\frac{1}{r^3} + 15\beta\frac{z^2}{r^7}\right) + 6\alpha\beta\frac{\hat{z}z}{r^5}.$$

(b) The angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \partial_t \vec{x}$$

and its time derivative is

$$\partial_t \vec{L} = \vec{r} \times \partial_t \vec{p}.$$

Only the non-central bit matters:

$$\partial_t \vec{L} = \vec{r} \times \hat{z} \frac{6\alpha\beta z}{r^5}.$$

(c) To evaluate the right hand side of the torque, we need to parametrize the orbit. If the plane of the orbit were not rotated away from the z-axis, it would be $\vec{r}_0 = a(\cos\varphi, 0, \sin\varphi)$. We need to rotate this vector by an angle θ about an axis (say $\hat{y}$) perpendicular to the the z-axis. The associated rotation matrix is

$$R(\hat{y}, \theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

This gives

$$\vec{r} = a(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta\cos\varphi).$$

$$\partial_t \vec{L} = \frac{-6\alpha\beta}{r^5}(-yz, xz, 0) = \frac{-6\alpha\beta}{a^3}(\cos\theta\sin\theta\cos\varphi\sin\varphi, -\cos\theta\sin\theta\cos^2\varphi, 0)$$

The average over a period of $\sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi$ is zero, while $\langle\cos^2\varphi\rangle = \frac{1}{2}$ so

$$\partial_t \vec{L} = \frac{3\alpha\beta}{a^3}\cos\theta\sin\theta\hat{y}.$$