METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima21 – Príklady 5

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Cvičenie 23.11.2021

Príklad 1

Solution

Apply Gauss' Law to find the E field inside the sphere:

$$\begin{aligned}
\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \frac{Q_{enc}}{\epsilon_0}, \\
\oint E dA &= \frac{Q}{\epsilon_0} \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}, \\
E 4\pi r^2 &= \frac{Q}{\epsilon_0} \frac{r^3}{R^3}, \\
E &= \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}.
\end{aligned}$$

Apply circular motion physics,

$$\begin{aligned}
m\frac{4\pi^2 r}{T^2} &= eE, \\
m\frac{4\pi^2 r}{T^2} &= e\frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}, \\
T^2 &= \frac{16\pi^3 \epsilon_0 m R^3}{eQ}, \\
T &= 2\pi\sqrt{\frac{4\pi\epsilon_0 m R^3}{eQ}}.
\end{aligned}$$

Yes, it is independent of r.

Apply Gauss' Law to find the E field outside the sphere:

$$\begin{aligned}
\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \frac{Q_{enc}}{\epsilon_0}, \\
\oint E dA &= \frac{Q}{\epsilon_0}, \\
E 4\pi r^2 &= \frac{Q}{\epsilon_0},
\end{aligned}$$

$$E = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}.$$

Apply circular motion physics,

$$\begin{aligned}
m\frac{4\pi^2 r}{T^2} &= eE, \\
m\frac{4\pi^2 r}{T^2} &= e\frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}, \\
T^2 &= \frac{16\pi^3 \epsilon_0 m r^3}{eQ}, \\
T &= 2\pi\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}.
\end{aligned}$$

Yes, it is dependent of r. You will hopefully recognize Kepler's law. It is okay to start from a statement like "outside a spherically symmetric charge distribution it is possible to treat the distribution as a point charge."

Use the results of above and find the potential difference between the center and $r = 2R$.

$$\begin{aligned}\Delta V &= -\int_{2R}^{0} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}, \\ &= \int_{2R}^{R} \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} + \int_{R}^{0} \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}, \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{-1}{2R} - \frac{-1}{R} + \frac{R^2}{2R^3}\right), \\ &= \frac{Q}{4\pi\epsilon_0 R}\end{aligned}$$

Then use work-energy,

$$\begin{aligned}v &= \sqrt{\frac{2}{m}e\Delta V}, \\ &= \sqrt{\frac{2eQ}{4\pi\epsilon_0 mR}}.\end{aligned}$$

Dimensional analysis is the way to go. a has dimensions of $[\mathrm{L}]/[\mathrm{T}]^2$, P has dimensions of $[\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3$, c has dimensions of $[\mathrm{L}]/[\mathrm{T}]$, q has dimensions of $[\mathrm{C}]$, and ϵ_0 has dimensions of $[\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3$.

Set up an equation such as

$$P = a^\alpha c^\beta {\epsilon_0}^\gamma q^\delta$$

or

$$\left([\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3\right) = \left([\mathrm{L}]/[\mathrm{T}]^2\right)^\alpha \left([\mathrm{L}]/[\mathrm{T}]\right)^\beta \left([\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3\right)^\gamma \left([\mathrm{C}]\right)^\delta$$

Charge is only balanced if $\gamma = -2\delta$. Mass is only balanced if $\gamma = -1$. Similar expressions exist for length and time, yielding

$$P = \frac{1}{6\pi} a^2 c^{-3} {\epsilon_0}^{-1} q^2$$

The energy radiated away is given by

$$\Delta E = -PT,$$

where T is determined in the previous sections.

It is possible to compute the actual energy of each orbit, and it is fairly trivial to do for regions $r > R$, but perhaps there is an easier, more entertaining way. Consider

$$\Delta E = \Delta K + \Delta U$$

and for small changes in r,

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies (correctly) that the potential energy increases with increasing r.

$$\frac{\Delta K}{\Delta r} \approx \frac{d}{dr}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}\frac{d}{dr}\left|r\frac{mv^2}{r}\right|$$

but $mv^2/r = F$, so

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF|$$

and then

$$\frac{\Delta K}{\Delta r} \approx \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies (correctly) that the kinetic energy increases with increasing r. Not a sur prise, since this region acts similar to a multidimensional simple harmonic oscillator. Combine, and

$$\frac{\Delta E}{\Delta r} \approx 2\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = 2ma$$

Finally,

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right)\left(\frac{1}{2ma}\right)$$

This can be simplified, so

$$\begin{aligned}
\Delta r &= -\left(\frac{1}{12\pi}\frac{a}{mc^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right), \\
&= -\left(\frac{1}{6}\frac{e^2}{m^2c^3\epsilon_0}\right)\left(\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right)\left(\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}\right), \\
&= -\frac{1}{6}\sqrt{\frac{e^5 Q}{4\pi\epsilon_0{}^3 R(mc^2)^3}}\frac{r}{R}
\end{aligned}$$

You might want to group these in terms of dimensionless groupings:

$$\Delta r = -\frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}}r$$

Pick up where we left off, and

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}.$$

This implies (correctly) that the potential energy increases with increasing r.

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF|$$

and so

$$\frac{\Delta K}{\Delta r} \approx -\frac{eQ}{8\pi\epsilon_0}\frac{1}{r^2}.$$

This implies (correctly) that the kinetic energy decreases with increasing r. Combine, and

$$\frac{\Delta E}{\Delta r} \approx \frac{1}{2}\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = \frac{ma}{2}$$

Follow the same type of substitutions as before, and

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}\right)\left(\frac{2}{ma}\right)$$

This can be simplified, so

$$\begin{aligned}\Delta r &= -\left(\frac{1}{3\pi}\frac{a}{mc^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}\right),\\ &= -\left(\frac{2}{3}\frac{e^2}{m^2c^3\epsilon_0}\right)\left(\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}\right)\left(\frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}\right),\\ &= -\frac{1}{3}\sqrt{\frac{e^5Q}{4\pi\epsilon_0{}^3 r(mc^2)^3}}\end{aligned}$$

You might want to group these in terms of dimensionless groupings:

$$\Delta r = -\frac{4}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}\frac{R^2}{r}}$$

Príklad 2

(a)

$$\Phi_0(\bar{r}) = \frac{q}{|\bar{r}-a\bar{z}|} + \frac{q}{|\bar{r}+a\bar{z}|} - \frac{2q}{|\bar{r}|}$$

$$= \frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} - \frac{2q}{r}$$

$$= \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} + \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2+2(a/r)\cos\theta}} - \frac{2q}{r}$$

$$\approx \frac{q}{r}\left[1-\frac{1}{2}\left(\frac{a}{r}\right)^2+\left(\frac{a}{r}\right)\cos\theta+\frac{3}{8}\left\{\left(\frac{a}{r}\right)^2-2\frac{a}{r}\cos\theta\right\}^2\right]$$

$$+\frac{q}{r}\left[1-\frac{1}{2}\left(\frac{a}{r}\right)^2-\left(\frac{a}{r}\right)\cos\theta+\frac{3}{8}\left\{\left(\frac{a}{r}\right)^2+2\frac{a}{r}\cos\theta\right\}^2\right]-\frac{2q}{r}$$

$$\approx \frac{q}{r}\left[2-\left(\frac{a}{r}\right)^2+\frac{3}{4}\left\{2\frac{a}{r}\cos\theta\right\}^2-2\right], \text{ to second order in } a/r$$

$$\approx \frac{qa^2}{r^3}\left(3\cos^2\theta-1\right) = \frac{Q}{r^3}\left(3\cos^2\theta-1\right)$$

(b) On spherical shell $\Phi(r=b)=0$. Inside shell we have $\Phi_{in}(r)=\Phi_0(r)+\Phi_1(r)$ where $\Phi_1(r)$ satisfies $\nabla^2\Phi_1=0$. Expand potential in Legendre polynomials:

$$\Phi_1(r,\theta)=\sum_\ell\left(a_\ell r^\ell+\frac{b_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$=a_0+\frac{b_0}{r}+\left(a_1 r+\frac{b_1}{r^2}\right)\cos\theta+\left(a_2 r^2+\frac{b_2}{r^3}\right)P_2(\cos\theta)$$

Now at r=0 potential should not diverge, and hence $b_i=0$. At r=b we have

$$0=\frac{Q}{b^3}\left(3\cos^2\theta-1\right)+a_0+a_1 b\cos\theta+a_2 b^2\frac{(3\cos^2\theta-1)}{2},$$ therefore $a_0=a_1=0$

and $a_2=-\frac{2Q}{b^5}$. Therefore the interior potential is given by

$$\Phi_{in}(r)=\Phi_0(r)+\Phi_1(r)=\frac{Q}{r^3}\left(3\cos^2\theta-1\right)\left[1-\left(\frac{r}{b}\right)^5\right].$$

Príklad 3

SOLUTION: **Solution:** The oscillator changes from frequency ω to $\lambda\omega$. In terms of wavefunctions, the initial wavefunction is

$$\psi_i = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \tag{1}$$

The new ground state wavefunction is

$$\psi_0 = \left(\frac{m\lambda\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\lambda\omega x^2}{2\hbar}} \tag{2}$$

The probability is

$$p = |\langle\psi_0|\psi\rangle|^2=\left|\int_{-\infty}^{\infty}\mathrm{d}x\ \psi_0^*(x)\psi_i(x)\right|^2=\frac{2\sqrt{\lambda}}{1+\lambda} \tag{3}$$

It is convenient to reconstruct the ladder operators for the harmonic oscillator. We write

$$a = \frac{x}{\ell} + \frac{i\ell p}{2\hbar} ,$$

with arbitrary ℓ. This satisfies $[a, a^\dagger] = 1$, by construction. We furthermore have that

$$a^\dagger a + \tfrac{1}{2} = \frac{x^2}{\ell^2} + \frac{\ell^2 p^2}{4\hbar^2} .$$

Comparing with H, we equate the ratios of the coefficients of x^2 and p^2, resulting in $\ell = \sqrt{2\hbar/m\omega}$. The position operator is given by

$$x = \tfrac{1}{2}\ell(a + a^\dagger) .$$

We therefore must calculate

$$\langle\psi_0|x^4|\psi_0\rangle = \frac{\ell^4}{16}\,\langle 0|(a + a^\dagger)^4|0\rangle$$

The operator $(a + a^\dagger)^4$, when expanded, has $2^4 = 16$ terms. However, only two of these will yield any finite expectation value in the ground state $|0\rangle$. Clearly the only terms which survive must have an a on the left, so as not to annihilate $\langle 0|$, and an $a^\dagger$ on the right, so as not to annihilate $|0\rangle$. Furthermore, the number of creation $(a^\dagger)$ operators must be the same as the number of annihilation (a) operators. This leaves

$$\begin{aligned}\langle\psi_0|x^4|\psi_0\rangle &= \frac{\ell^4}{16}\,\langle 0|\,a\,a\,a^\dagger a^\dagger + a\,a^\dagger a\,a^\dagger\,|0\rangle \\ &= \frac{3\ell^4}{16} = \frac{3\hbar^2}{4m^2\omega^2} .\end{aligned}$$