METÓDY RIEŠENIA FYZIKÁLNYCH ÚLOH zima25 – Príklady 6

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Cvičenie 9.12.2025

Príklad 1

SOLUTION: For a non-interacting ideal gas,

$$E = -\frac{\partial}{\partial\beta} N \ln\zeta \,,$$

where ζ is the single-molecule partition function

$$\zeta = \sum_{n=0}^{\infty} (n+1) \exp(-\beta n \varepsilon)\,.$$

This partition function can be evaluated as follows ($x \equiv \beta\varepsilon$):

$$\zeta = -e^x \frac{d}{dx} \sum_{n=0}^{\infty} \exp\big(-(n+1)x\big) = -e^x \frac{d}{dx} \frac{e^{-x}}{1-e^{-x}} = [1-\exp(-\beta\varepsilon)]^{-2}.$$

Hence, the sought contribution to the energy is

$$E = \frac{2N\varepsilon}{\exp(\varepsilon/kT) - 1}\,.$$

Alternatively, one can reproduce this result as follows. One can imagine that every molecule has two independent internal degrees of freedom of harmonic oscillator type, with energy spacing ε each. It is easy to see that this model gives the same spectrum and degeneracies if the energy is counted from the ground state. With this convention, the average energy of a single harmonic oscillator is $\varepsilon n_B(\varepsilon)$, where $n_B(\varepsilon)$ is the Bose-Einstein occupation number. Therefore, for the entire gas we get $E = 2N\varepsilon n_B(\varepsilon)$, in agreement with the first derivation.

Príklad 2

(a)

$$\Phi_0(\bar{r}) = \frac{q}{|\bar{r} - a\bar{z}|} + \frac{q}{|\bar{r} + a\bar{z}|} - \frac{2q}{|\bar{r}|}$$

$$= \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q}{\sqrt{r^2 + a^2 + 2ar\cos\theta}} - \frac{2q}{r}$$

$$= \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2 - 2(a/r)\cos\theta}} + \frac{q}{r}\cdot\frac{1}{\sqrt{1+(a/r)^2 + 2(a/r)\cos\theta}} - \frac{2q}{r}$$

$$\approx \frac{q}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right\}^2\right]$$

$$+ \frac{q}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 - \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right\}^2\right] - \frac{2q}{r}$$

$$\approx \frac{q}{r}\left[2 - \left(\frac{a}{r}\right)^2 + \frac{3}{4}\left\{2\frac{a}{r}\cos\theta\right\}^2 - 2\right], \text{ to second order in } a/r$$

$$\approx \frac{qa^2}{r^3}\left(3\cos^2\theta - 1\right) = \frac{Q}{r^3}\left(3\cos^2\theta - 1\right)$$

(b) On spherical shell $\Phi(r = b) = 0$. Inside shell we have $\Phi_{in}(r) = \Phi_0(r) + \Phi_1(r)$ where $\Phi_1(r)$ satisfies $\nabla^2\Phi_1 = 0$. Expand potential in Legendre polynomials:

$$\Phi_1(r,\theta) = \sum_\ell \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$= a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta + \left(a_2 r^2 + \frac{b_2}{r^3}\right)P_2(\cos\theta)$$

Now at r=0 potential should not diverge, and hence $b_i = 0$. At r=b we have

$$0 = \frac{Q}{b^3}\left(3\cos^2\theta - 1\right) + a_0 + a_1 b\cos\theta + a_2 b^2\,\frac{(3\cos^2\theta - 1)}{2}, \text{therefore } a_0 = a_1 = 0$$

and $a_2 = -\dfrac{2Q}{b^5}$. Therefore the interior potential is given by

$$\Phi_{in}(r) = \Phi_0(r) + \Phi_1(r) = \frac{Q}{r^3}\left(3\cos^2\theta - 1\right)\left[1 - \left(\frac{r}{b}\right)^5\right].$$

Príklad 3

SOLUTION: **Solution:** The oscillator changes from frequency ω to $\lambda\omega$. In terms of wavefunctions, the initial wavefunction is

$$\psi_i = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \tag{1}$$

The new ground state wavefunction is

$$\psi_0 = \left(\frac{m\lambda\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\lambda\omega x^2}{2\hbar}} \tag{2}$$

The probability is

$$p = |\langle\psi_0|\psi\rangle|^2 = \left|\int_{-\infty}^{\infty} \mathrm{d}x\ \psi_0^*(x)\psi_i(x)\right|^2 = \frac{2\sqrt{\lambda}}{1+\lambda} \tag{3}$$

It is convenient to reconstruct the ladder operators for the harmonic oscillator. We write

$$a = \frac{x}{\ell} + \frac{i\ell p}{2\hbar}\ ,$$

with arbitrary ℓ. This satisfies $\left[a, a^\dagger\right] = 1$, by construction. We furthermore have that

$$a^\dagger a + \tfrac{1}{2} = \frac{x^2}{\ell^2} + \frac{\ell^2 p^2}{4\hbar^2}\ .$$

Comparing with H, we equate the ratios of the coefficients of x^2 and p^2, resulting in $\ell = \sqrt{2\hbar/m\omega}$. The position operator is given by

$$x = \tfrac{1}{2}\ell\left(a + a^\dagger\right)\ .$$

We therefore must calculate

$$\langle\psi_0|x^4|\psi_0\rangle = \frac{\ell^4}{16}\left\langle 0\right|\left(a + a^\dagger\right)^4\left|0\right\rangle$$

The operator $\left(a + a^\dagger\right)^4$, when expanded, has $2^4 = 16$ terms. However, only two of these will yield any finite expectation value in the ground state $\left|0\right\rangle$. Clearly the only terms which survive must have an a on the left, so as not to annihilate $\left\langle 0\right|$, and an $a^\dagger$ on the right, so as not to annihilate $\left|0\right\rangle$. Furthermore, the number of creation $(a^\dagger)$ operators must be the same as the number of annihilation (a) operators. This leaves

$$\begin{aligned}\langle\psi_0|x^4|\psi_0\rangle &= \frac{\ell^4}{16}\left\langle 0\right|a\,a\,a^\dagger a^\dagger + a\,a^\dagger a\,a^\dagger\left|0\right\rangle \\ &= \frac{3\ell^4}{16} = \frac{3\hbar^2}{4m^2\omega^2}\ .\end{aligned}$$

Príklad 4

a) $\dfrac{(2m|\epsilon_0|)^{1/2}}{\hbar}$

b) $K = e^{\alpha d}/\alpha^{1/2}$ $\qquad \left(\int_d^\infty |\psi_0|^2 dx = \frac{1}{2} \right)$

c) $\left(\dfrac{V_0}{\hbar}\right) e^{-2\alpha d}$

or $|\epsilon_0|/\hbar$

or $\hbar/ma^2$

The main point is the $e^{-2\alpha d}$

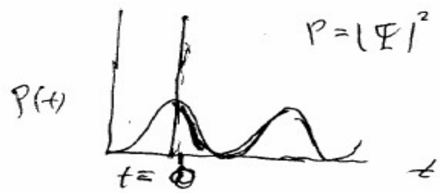
In our problem
$a \sim 1/\alpha$
$\langle KE \rangle \sim \langle P.E \rangle/2$
$\epsilon_0 \sim V_0/2$
so any dimension analysis "guess" gives same answer

d) $\dfrac{\Psi_0(x) + \Psi_0(d-x)}{\sqrt{2}} \equiv \psi_{0s}$

e) $\delta\epsilon = V_0 e^{-\alpha d}$ $\qquad \int_0^a V(x)\, \psi_0(x)\, \psi_0(d-x)$ $\qquad \int_0^a \frac{V_0}{a} e^{-d\alpha} \; dx$

$\Psi = \psi_{0s} e^{-i\delta\epsilon t/\hbar} + \psi_{\epsilon s} e^{\delta\epsilon t/\hbar}$ $\qquad \psi_{0s} = \dfrac{\psi_0(x) - \psi_0(d-x)}{\sqrt{2}}$

f) $P = |\Psi|^2 \propto \psi_0^2 \cos^2 \delta\epsilon/\hbar \; + \psi_0^2(d-x) \sin^2 \delta\epsilon t/\hbar$



$\underset{x \lesssim a}{P(t)} = \cos^2 t/\tau$ $\qquad \dfrac{1}{\tau} = \dfrac{\delta\epsilon}{\hbar}$

Ratio of times τ $\sim e^{-\alpha d}$ for c) and $\hbar/\delta\epsilon$