

Lecture Notes on

**ALGEBRAICALLY
SOLVABLE PROBLEMS
IN PHYSICS**

Following the lectures of Prof. V. Parameswaran Nair

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Introduction

The idea of this course arose from the fact that, within the regular courses, there is usually not enough time to cover certain interesting techniques for solving physical problems. Very often, it is more than a technique, these methods illustrate a new facet of the physics of the problem. There is a lot of such extra material, but in this course, we will focus on algebraic techniques. The quantum theory of any physical system can be defined as a single unitary representation of the algebra of observables and thus, in principle, every problem is algebraic. However, in practice, not every physical problem is easily solvable by algebraic methods. We will discuss a number of algebraically solvable problems as listed below. This choice of problems is made such that each one can be considered as the quintessential example of a physical principle as well as illustrating a novel mathematical technique.

In the classical theories, almost all problems are differential. This is because the underlying equations are differential equations, e.g. the Newton equation, Maxwell equations, etc. However in quantum mechanics, the physical observables form an algebra and theories are defined as unitary irreducible representations of this algebra. As an example we have many particle Heisenberg algebra, i.e. the set of operators \hat{x}_i, \hat{p}_i obeying

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (0.1)$$

where $[,]$ denotes commutator $[A, B] = AB - BA$. If we now have a vector space to act on with these operators, we have defined the physical theory. Quantities of interest, namely expectation values of the observables and their time dependence, can be computed purely from the structure of the algebra and the vector space, i.e. by algebraical methods.

The requirement of unitarity is very important in quantum theories. It implies probability conservation. We require that the operators, that transform the states of the vector space to be unitary, i.e. so that $U^\dagger U = \hat{1}$. Where $\hat{1}$ is just the identity operator. If we write U in terms of a generator $U = e^{iH}$ this condition translates into requiring H to be hermitian, i.e. $H^\dagger = H$.¹ Now, for two states $|\alpha\rangle, |\beta\rangle$ in the vector space, that transform as

$$|\alpha\rangle \rightarrow |\alpha'\rangle = U |\alpha\rangle \quad , \quad |\beta\rangle \rightarrow |\beta'\rangle = U |\beta\rangle . \quad (0.2)$$

¹Then $U^\dagger = e^{-iH^\dagger} = e^{-iH}$ so $U^\dagger U = e^{-iH+iH} = 1$.

The overlap between the states remains the same

$$\langle \alpha' | \beta' \rangle = \langle \alpha | U^\dagger U | \beta \rangle = \langle \alpha | \beta \rangle. \quad (0.3)$$

Transformations U of the vectors are then given by exponentiating corresponding observable by a parameter describing the transformation as

$$U = e^{i\mathcal{O}\theta}. \quad (0.4)$$

These notes are still work in progress. Your comments, as well as reports of mistakes and typos, are mostly welcomed at [juraj.tekel\(at\)gmail\(dot\)com](mailto:juraj.tekel@gmail.com). Thank you!

In the text, we adopt units, where $c = 1$ and $\hbar = 1$.² Also, I_n denotes the $n \times n$ identity matrix. We use the summation convention and a repeated index is understood to be summed over. Examples

$$\begin{aligned} r^2 &= x_1^2 + x_2^2 + x_3^2 = \sum_{i=1}^3 x_i^2 = x_i x_i, \\ \vec{A} \cdot \vec{B} &= \sum_{i=1}^n A_i B_i = A_i B_i, \\ (M\vec{A})_i &= \sum_{j=1}^n M_{ij} A_j = M_{ij} A_j, \\ (\vec{A} \times \vec{B})_i &= \varepsilon_{ijk} A_j B_k. \end{aligned}$$

²One could argue that taking the classical limit means taking $\hbar \rightarrow 0$ and $c \rightarrow \infty$, which is obviously impossible if we set them both to unity. The classical limit is obtained in the case when the number of quanta in the system is very large, i.e. $n/\hbar \rightarrow \infty$. Note that this is equivalent to taking the limit $\hbar \rightarrow 0$, but the spirit of the limit is different.

Chapter 1

The multidimensional oscillator and $su(N)$ -algebra

In this whole section we will set $m, \omega = 1$ for simplicity. Rescaling \hat{p}, \hat{x} or alternatively using a slightly different definition in (1.2) would do the job for general values.

1.1 One dimensional harmonic oscillator

This section should be familiar from the quantum mechanics course. The one dimensional LHO Hamiltonian is given by

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2. \quad (1.1)$$

If we introduce a new set of operators

$$\begin{aligned} \hat{a} &= \frac{\hat{x} + i\hat{p}}{\sqrt{2}}, \\ \hat{a}^\dagger &= \frac{\hat{x} - i\hat{p}}{\sqrt{2}}. \end{aligned} \quad (1.2)$$

we can easily check that from $[\hat{x}, \hat{p}] = i$ we get $[\hat{a}, \hat{a}^\dagger] = 1$ and that Hamiltonian becomes

$$H = \hat{a}^\dagger \hat{a} + \frac{1}{2}. \quad (1.3)$$

Now we observe, that if $|\alpha\rangle$ is an eigenvector of $\hat{N} = \hat{a}^\dagger \hat{a}$ with an eigenvalue $\alpha \neq 0$, then also $\hat{a}|\alpha\rangle$ is an eigenvector with an eigenvalue¹ $(\alpha - 1)$. Thus, in this notation we get

$$|\alpha - 1\rangle = \frac{1}{\sqrt{\alpha}} \hat{a} |\alpha\rangle. \quad (1.4)$$

¹This is a consequence of the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$

Similarly, we get

$$|\alpha + 1\rangle = \frac{1}{\sqrt{\alpha + 1}} \hat{a}^\dagger |\alpha\rangle. \quad (1.5)$$

Using these formulas, we can get new and new eigenstates of \hat{N} , with lower and lower eigenvalues. However, all the eigenvalues of this operator are positive, since

$$\alpha = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha - 1 | \alpha - 1 \rangle \geq 0. \quad (1.6)$$

We can always get a lower eigenvalue using (1.4), therefore we need to have a state with $\alpha = 0$, since then $\hat{a} | 0 \rangle = 0$. So we postulate existence of such state and using (1.5), we generate the complete set of eigenstates

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle \quad (1.7)$$

of the operator \hat{N} with the eigenvalue n . Since $H = \hat{N} + \frac{1}{2}$, these are also eigenstates of H with the energy

$$E_n = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots \quad (1.8)$$

In the general case, there would be a factor of $\hbar\omega$ in this expression.

1.2 Two dimensional harmonic oscillator

Or equivalently two uncoupled one dimensional oscillators. The Hamiltonian of such system is

$$H = \frac{1}{2} \hat{p}_1^2 + \frac{1}{2} \hat{p}_2^2 + \frac{1}{2} \hat{x}_1^2 + \frac{1}{2} \hat{x}_2^2. \quad (1.9)$$

We have taken both frequencies to be equal (to 1), however using a different definition for operators $\hat{a}_1, \hat{a}_2, \hat{a}_1^\dagger, \hat{a}_2^\dagger$ we could get the same expression also in the case of different frequencies in different directions². Now following an analogous approach as in the previous section we take

$$\begin{aligned} \hat{a}_1 &= \frac{\hat{x}_1 + i\hat{p}_1}{\sqrt{2}}, & \hat{a}_1^\dagger &= \frac{\hat{x}_1 - i\hat{p}_1}{\sqrt{2}}, \\ \hat{a}_2 &= \frac{\hat{x}_2 + i\hat{p}_2}{\sqrt{2}}, & \hat{a}_2^\dagger &= \frac{\hat{x}_2 - i\hat{p}_2}{\sqrt{2}}, \end{aligned} \quad (1.10)$$

and turn the Hamiltonian (1.9) into

$$H = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1.$$

²This way, the system is equivalent even to two coupled harmonic oscillators.

Using the commutation relations among the $\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2$ operators, we get

$$[\hat{a}_i, \hat{a}_j^\dagger] = 1 \quad , \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \quad , \quad i, j = 1, 2. \quad (1.11)$$

Again, with the same logic as before we get to the set of eigenstates of this Hamiltonian

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} \left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2} |0\rangle \quad , \quad n_1, n_2 = 0, 1, 2, \dots \quad , \quad \hat{a}_1 |0\rangle = \hat{a}_2 |0\rangle = 0 \quad (1.12)$$

with the eigenvalues

$$E_{n_1, n_2} = n_1 + n_2 + 1 \quad , \quad (1.13)$$

and

$$\begin{aligned} \hat{a}_1 |n_1, n_2\rangle &= \sqrt{n_1} |n_1 - 1, n_2\rangle \quad , \\ \hat{a}_2 |n_1, n_2\rangle &= \sqrt{n_2} |n_1, n_2 - 1\rangle \quad , \\ \hat{a}_1^\dagger |n_1, n_2\rangle &= \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle \quad , \\ \hat{a}_2^\dagger |n_1, n_2\rangle &= \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle . \end{aligned} \quad (1.14)$$

This way, we got a representation of the algebra of the operators $\hat{a}_1, \hat{a}_2, \hat{a}_1^\dagger, \hat{a}_2^\dagger$ in the vector space spanned by the vectors (1.12)³. Of course, this is not the only possible representation of this algebra. For example if we take

$$\hat{a}_1 = \frac{\partial}{\partial z_1} \quad , \quad \hat{a}_2 = \frac{\partial}{\partial z_2} \quad , \quad \hat{a}_1^\dagger = z_1 \quad , \quad \hat{a}_2^\dagger = z_2 \quad (1.15)$$

acting on the space of the complex functions of two variables, we can easily check, that we get the same commutation relations. So, at least in principle, we could get two, or more, different results describing the same system (1.9). However, the following theorem secures, that this will not happen.

Theorem 1.2.1 (Stone-Von Neuman). *For any finite set of $\hat{a}_i, \hat{a}_i^\dagger$, there exists only one representation of their algebra with commutation rules (1.11), on a simply connected space and up to a unitary transformation.*

Meaning. Since equivalent representations produce the same physics, we get a well defined physics. However note the two caveats in this theorem. First, if we have a theory on a space that is not simply connected we can get two non-equivalent representations of the algebra and this is going to be the case of Quantum Hall Effect. Second, if the number of $\hat{a}_i, \hat{a}_i^\dagger$ is infinite, we can again get a non-equivalent representation describing different physics. This is going to be the case of superconductivity and BCS theory.

³Representation of algebra is a mapping of the algebra to the space of linear operators on a vector space, which preserves the commutation relations.

1.3 Symmetries, conservation laws and degeneration

Let us transform the Hamiltonian by an unitary transformation

$$H \rightarrow U^\dagger H U \quad , \quad (1.16)$$

where we assume, that the transformation is continuous, thus U can be expressed as

$$U = e^{i\beta G} \quad (1.17)$$

with G being hermitian and β being the transformation parameter (eg. distance, angle). For the variation of the Hamiltonian, we get

$$\delta H = -i\beta[H, G]. \quad (1.18)$$

If U is to represent a symmetry, then $\delta H = 0$ and the generator of this symmetry must commute with the Hamiltonian. Inserting this into the Heisenberg equation of motion for the operator G we get

$$\frac{dG}{dt} = -i[H, G] = 0. \quad (1.19)$$

and the G is conserved. Therefor if there is a symmetry, there is a conserved quantity. The argumentation works also the other way around and we find out, that where is a conserved quantity, there is a symmetry. This is essentially the Noether's theorem.

Writing down the Jaccobi equation we also see, that if A, B are conserved, then also $[A, B]$ is conserved, since

$$[H, [A, B]] + [A, [B, H]] + [H, [B, A]] = 0 = [H, [A, B]]. \quad (1.20)$$

Now, lets have $|\alpha\rangle$ an eigenstate of H with eigenvalue α . Since $[H, G] = 0$, we get

$$HG|\alpha\rangle = GH|\alpha\rangle = \alpha G|\alpha\rangle. \quad (1.21)$$

Also $G|\alpha\rangle$ is an eigenstate of H , with the same eigenvalue. This way we find out that all the states $G^n|\alpha\rangle$ are degenerate and thus also states $|\alpha\rangle$ and $U|\alpha\rangle$ are degenerate. This means that with every symmetry comes degeneration and the eigenstates of H are grouped into multiplets of states with the same eigenvalue. These states are connected among each other by symmetry transformation, however not necessarily every couple of states is.

As an example, consider the hydrogen atom. The Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{r} \quad (1.22)$$

is obviously rotationally invariant and therefor

$$[L_i, H] = 0 \quad (1.23)$$

with L_i being generators of the rotations, i.e. the angular momentum operators.⁴ We expect the eigenstates of this Hamiltonian to be grouped into groups of the same energy. And this is indeed the case. The eigenstates of are labeled by three number $|n, l, m\rangle$, where $n = 1, 2, \dots$ and $l = 0, 1, \dots, n - 1$ and $m = -l, -l + 1, \dots, l - 1, l$. However only the number n determines the energy of the state, so all the states with the same n are degenerate. There is only one state with $n = 1$. There are four states with $n = 2$, namely

$$|2, 0, 0\rangle, |2, 1, -1\rangle, |2, 1, 0\rangle, |2, 1, 1\rangle. \quad (1.24)$$

The last three states are mixed together by some rotation $U = e^{i\theta_i L_i}$, but since states with different l have different total angular momentum, there is no way we can mix these with the first one. The natural question to ask is, whether there is a larger underlying symmetry beyond rotations, that would mix these states together. We will answer this question later in section 2.

This is an illustration of the Wigner theorem, which states, that if $[H, G_i] = 0$, then the states fall into multiplets forming an irreducible representation of the algebra of generators of the symmetry G_i .

There is however a catch to the previous argument. We have no guarantee that the state $G|\alpha\rangle$ is going to be an element of our Hilbert space. If this state is not normalizable, i.e. $|G|\alpha\rangle| > \infty$, we encounter phenomenon called spontaneous symmetry breaking. This is a different way to break the symmetry, where the generators still commute with the Hamiltonian, opposing to simply changing the Hamiltonian and lifting the degeneracy. We will discuss this case in the section 4.

1.4 Two dimensional harmonic oscillator and $su(2)$

We suggest that the reader reads also the appendix A.5 for some more information.

We see, that the n -th excited level states are $n + 1$ times degenerated, since all the states

$$|n, 0\rangle, |n - 1, 1\rangle, \dots, |0, n\rangle \quad (1.25)$$

have the same energy $n + 1$. Thus, we can expect some kind of symmetry.

⁴One could compute explicitly this commutator with $L_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$ and see that it vanishes, but that is not necessary. We see, that Hamiltonian (1.23) consist of only rotationally invariant quantities p^2 and r and thus is itself rotationally invariant.

1.4.1 The first excited state

Let us first deal with the first excited states, i.e. states $|1, 0\rangle = \hat{a}_1^\dagger |0\rangle$, $|0, 1\rangle = \hat{a}_2^\dagger |0\rangle$. It is easy to find the operator G , that does the work, i.e. $G|1, 0\rangle = |0, 1\rangle$ and $G^{-1}|0, 1\rangle = |1, 0\rangle$. For the reasons that will become clear later, let us denote $G \equiv G_-$ and $G^{-1} \equiv G_+$ and we have

$$G_- = \hat{a}_2^\dagger \hat{a}_1 \quad , \quad G_+ = \hat{a}_1^\dagger \hat{a}_2. \quad (1.26)$$

We can easily check that these two commute with the Hamiltonian⁵

$$\begin{aligned} \left[\hat{a}_2^\dagger \hat{a}_1, \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right] &= \left[\hat{a}_2^\dagger \hat{a}_1, \hat{a}_1^\dagger \hat{a}_1 \right] + \left[\hat{a}_2^\dagger \hat{a}_1, \hat{a}_2^\dagger \hat{a}_2 \right] = \hat{a}_2^\dagger \left[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_1 \right] + \left[\hat{a}_2^\dagger, \hat{a}_2^\dagger \hat{a}_2 \right] \hat{a}_1 = \\ &= \hat{a}_2^\dagger \left[\hat{a}_1, \hat{a}_1^\dagger \right] \hat{a}_1 + \hat{a}_2^\dagger \left[\hat{a}_2^\dagger, \hat{a}_2 \right] \hat{a}_1 = 0. \end{aligned} \quad (1.27)$$

Using (1.20) we see, that also $[G_+, G_-] = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 \equiv 2G_3$ is a symmetry generator. If we compute the following commutators, we get

$$\begin{aligned} [G_3, G_-] &= -G_- \quad , \\ [G_3, G_+] &= G_+ \end{aligned} \quad (1.28)$$

and the generators G_+, G_-, G_3 follow the angular-momentum-algebra commutation relations familiar from the quantum mechanics course. Therefor this is a particular representation of this algebra. Since it is 2-dimensional, it should be equivalent to the spin- $\frac{1}{2}$ representation.

This can be explicitly seen, when we use the following notation

$$A = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \quad , \quad A^\dagger = (\hat{a}_1^\dagger, \hat{a}_2^\dagger)$$

. In this notation

$$\begin{aligned} G_+ &= A^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A \quad , \\ G_- &= A^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} A \quad , \\ G_3 &= A^\dagger \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A. \end{aligned} \quad (1.29)$$

⁵We have used $[A, BC] = B[A, C] + [A, B]C$ in the calculation.

Using a different linear combination of the matrices we get

$$\begin{aligned} G_1 &= A^\dagger \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \quad , \\ G_2 &= A^\dagger \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} A \quad , \\ G_3 &= A^\dagger \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \end{aligned} \quad (1.30)$$

or

$$G_i = A^\dagger \left(\frac{\sigma_i}{2} \right) A \quad (1.31)$$

with σ_i being the Pauli matrices.⁶ For these

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i\varepsilon_{ijk} \frac{\sigma_k}{2}. \quad (1.32)$$

These matrices form $su(2)$ algebra of all 2×2 hermitian matrices with a vanishing trace and this algebra generates the group $SU(2)$, i.e. all 2×2 matrices, which are unitary and have determinant equal to 1. As should be familiar from the course of quantum mechanics, Pauli matrices form $\text{spin}-\frac{1}{2}$ representation of $su(2)$. Therefor the representation of matrices G_i is equivalent to the system of $\text{spin } \frac{1}{2}$ and so is the first excited state of the two dimensional harmonic oscillator.

1.4.2 The higher excited states

For third and higher excited states, we will get the representations of the same algebra with the same commutation rules in a higher dimensional space, i.e. representation equivalent to a corresponding spin representation.

For a fixed n , the states

$$|k, n-k\rangle = \frac{1}{\sqrt{k!(n-k)!}} \left(\hat{a}_1^\dagger \right)^k \left(\hat{a}_2^\dagger \right)^{n-k} |0\rangle \quad , \quad k = 0, 1, \dots, n \quad (1.33)$$

⁶The matrix multiplication indexes have been and will be suppressed in expressions like this. With the indexes explicit, it would be

$$G_i = (A^\dagger)_\alpha \left(\frac{\sigma_i}{2} \right)_{\alpha\beta} (A)_\beta$$

, which is a little cumbersome.

are going to be $(n + 1)$ -times degenerate. Again, the symmetry will be generated by the same operators G_-, G_+, G_3 given by (1.29). We compute

$$G_3 |k, n - k\rangle = \left(k - \frac{n}{2}\right) |k, n - k\rangle.$$

The possible eigenvalues of G_3 are therefor $-j, -j + 1, \dots, j - 1, j$ with $j = n/2$. We then take $G^2 = G_i G_i$, which commutes with G_3 and look for it's possible eigenvalues. We get

$$\begin{aligned} G^2 &= G_1^2 + G_2^2 + G_3^2 = G_+ G_- + G_3^2 - G_3 = \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 + \frac{1}{4}(\hat{N}_1 - \hat{N}_2)^2 - \frac{1}{2}(\hat{N}_1 - \hat{N}_2) = \\ &= \left(\frac{\hat{N}_1 + \hat{N}_2}{2}\right)^2 + \frac{\hat{N}_1 + \hat{N}_2}{2} \end{aligned} \quad (1.34)$$

and the eigenvalues are then $(n/2)^2 + n/2 = j(j + 1)$. As expected, we got spin $n/2$ representation.

1.4.3 The general treatment

One can easily check, that the following operator

$$\hat{M} = a_i^\dagger a_j M_{ij} \quad (1.35)$$

commutes with the Hamiltonian $H = a_i^\dagger a_i + 1$ for any matrix M_{ij} . Note that we have used the summation convention and that the indexes $i, j = 1, 2$. Also note that this would be true also for higher number of dimensionality with the corresponding $H = a_i^\dagger a_i + N/2$ and $i, j = 1, 2, \dots, N$.

This operators \hat{M} therefor represents a general symmetry and we now look for the multiplets of degenerate states.

In the present case, M_{ij} is a 2×2 hermitian matrix and thus \hat{M} can be expressed as

$$M = M_0 I_n + \frac{1}{2} \sigma_\alpha M_\alpha \quad (1.36)$$

with M_0, M_α some real numbers. Quite trivially

$$a_i^\dagger a_j (M_0 I_n)_{ij} = H_0 \left(\hat{N}_1 + \hat{N}_2\right) \quad (1.37)$$

This part of the operator \hat{M} is not interesting, since it is just the Hamiltonian itself and is not a real symmetry of the system. We therefor denote $\Sigma_\alpha = a_i^\dagger a_j \left(\frac{1}{2} \sigma_\alpha M_\alpha\right)_{ij}$ and compute

$$[\Sigma_\alpha, \Sigma_\beta] = i \varepsilon_{\alpha\beta\gamma} \Sigma_\gamma \quad (1.38)$$

Here, we derive a usefull formula. A short computation gives

$$\begin{aligned} Tr M &= 2M_0 \\ Tr \sigma_\beta M &= \dots \end{aligned} \quad (1.39)$$

Using this in (1.36) yields

$$\begin{aligned} M_{lk}\delta_{ij}\delta_{jk} &= \frac{1}{2}M_{lk} \left[\delta_{kl}\delta_{ij} + (\sigma_\alpha)_{ij}(\sigma_\alpha)_{kl} \right] \\ \delta_{il}\delta_{jk} &= \frac{1}{2}\delta_{kl}\delta_{ij} + \frac{1}{2}(\sigma_\alpha)_{ij}(\sigma_\alpha)_{kl} \end{aligned} \quad (1.40)$$

Using this formula we compute

$$\Sigma_\alpha\Sigma_\alpha = \left(\frac{\hat{N}_1 + \hat{N}_2}{2} \right)^2 + \frac{\hat{N}_1 + \hat{N}_2}{2} \quad (1.41)$$

1.5 Higher dimensional harmonic oscillator and representations of $su(N)$

Very similar approach could be explicitly used for the case $N = 3$. We have the set of operators $\hat{a}_i, \hat{a}_i^\dagger$ with $i = 1, 2, 3$ and corresponding commutations relations (1.11). The generators for this case are

$$\hat{a}_1^\dagger\hat{a}_2, \hat{a}_1^\dagger\hat{a}_3, \hat{a}_3^\dagger\hat{a}_2 \quad (1.42)$$

and their conjugates. Again, writing these in terms of A matrices in fashion similar to (1.29) and using matrices of the form $(B \pm iB^\dagger)/2$ we get these six generators

$$\begin{aligned} \lambda_1 &= A^\dagger \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A, & \lambda_2 &= A^\dagger \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A \\ \lambda_4 &= A^\dagger \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} A, & \lambda_5 &= A^\dagger \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} A \\ \lambda_6 &= A^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A, & \lambda_7 &= A^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} A \end{aligned} \quad (1.43)$$

Computing the commutators and adding the two matrices that are not linear combination of the previous ones

$$\lambda_3 = A^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A, \quad \lambda_8 = A^\dagger \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} A \quad (1.44)$$

we close our algebra. Now, we could compute commutators of these matrices. However there is a clever trick to figure it out without too much work. Note, that

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right]^\dagger = -\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] \quad (1.45)$$

so that the commutator itself can be written as $i \times$ (*hermitian matrix*). Due to the cyclicity of the trace this matrix needs to be traceless. And a traceless hermitian matrix is just a combination of the original λ matrices. We get the commutation relation

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = if_{ijk} \frac{\lambda_k}{2} \quad (1.46)$$

for some f_{ijk} . These need to be antisymmetric in the first two indexes, since the commutator is antisymmetric. However computing

$$Tr\left(\frac{\lambda_l}{2} \left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right]\right) = if_{ijk} Tr\left(\frac{\lambda_l \lambda_k}{2}\right) = if_{ijk} \frac{\delta_{kl}}{2} = i \frac{1}{2} f_{ijl} \quad (1.47)$$

and using the cyclic property of the trace again

$$Tr\left(\frac{\lambda_l}{2} \left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right]\right) = Tr\left(\frac{\lambda_j}{2} \left[\frac{\lambda_l}{2}, \frac{\lambda_i}{2}\right]\right) = Tr\left(\frac{\lambda_i}{2} \left[\frac{\lambda_j}{2}, \frac{\lambda_l}{2}\right]\right) \quad (1.48)$$

we see, that f_{ijk} needs to be antisymmetric in every pair of indexes.

We know, that the algebra closes, since we found 8 matrices, which are independent and thus form a basis in 8-dimensional space of $su(3)$.⁷

Then, characteristics of n dimensional representation of $su(3)$ could be found in a similar fashion, as we found characteristics of n dimension representations of $su(2)$ in the case of higher excited states.

We could in principle find description of the representations of $su(N)$ by eneralizing this procedure to an N dimensional oscillator. Finding $N(N - 1)$ generators in the form (1.42), rewriting them as $(B \pm iB^\dagger)/2$, computing the additional commutators and then finding the eigenvalues of commuting operators.

⁷It is space of 3×3 anti-hermitian matrices with $Tr A = 0$, thus it is 8 dimensional.

Chapter 2

The Hydrogen atom: $O(4)$ and $O(3, 1)$ symmetries and bound and scattering states.

In this chapter, we will show how one can solve the hydrogen atom using purely algebraical methods. Opposing to the approach where we explicitly solve the Schrodinger equation using the usual representation of the momentum operator. The presented method is due to Pauli and was actually the first used to determine the spectrum of the hydrogen.

2.1 Physics of rotations

Let us have a point in the 3D space, described by coordinates $x = (x_1, x_2, x_3)$. We now transform the coordinate system by rotating it around the x_3 axis by an angle θ_3 . This transformation can be given in a matrix form

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2.1)$$

Similarly, for rotations around the other two axes we get

$$\begin{aligned} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} &= \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \quad (2.2)$$

If we now make θ very small, we get the infinitesimal rotations

$$\begin{aligned} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} &= \begin{pmatrix} 1 & \theta_3 & 0 \\ -\theta_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = L_3 x \\ \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \theta_2 \\ 0 & 1 & 0 \\ -\theta_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = L_2 x \\ \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta_1 \\ 0 & -\theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = L_1 x \end{aligned} \quad (2.3)$$

or

$$\delta x_i \equiv x_i - x'_i = \varepsilon_{ijk} x_j \theta_k \quad (2.4)$$

with ε_{ijk} the antisymmetric tensor in the three dimensions^{1,2}. A general rotation is obtained from the infinitesimal ones by

$$U = e^{i\theta_k L_k} \quad (2.5)$$

with L_k given by (2.3). If we now take the quantum mechanical rotations, and evaluate $U^\dagger \hat{x}_i U$ for infinitesimal θ_k 's we get

$$\begin{aligned} U^\dagger \hat{x}_i U &= \hat{x}_i + i\theta_k [x_i, L_k] \\ U^\dagger \hat{p}_i U &= \hat{p}_i + i\theta_k [p_i, L_k] \end{aligned} \quad (2.6)$$

where the second formula follows from the fact, that momentum transforms as a vector. Comparing with (2.4) we obtain

$$\begin{aligned} [\hat{x}_i, L_k] &= -i\varepsilon_{ijk} \hat{x}_j \\ [\hat{p}_i, L_k] &= -i\varepsilon_{ijk} \hat{p}_j \end{aligned} \quad (2.7)$$

¹Given by $\varepsilon_{ijk} = 0$ if any of i, j, k are the same, and $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = -\varepsilon_{213} = -\varepsilon_{132} = -\varepsilon_{321} = 1$ or $\varepsilon_{\sigma(123)} = (-1)^{\text{sgn}(\sigma)}$, where σ is permutation of three elements and $\text{sgn}(\sigma)$ is its sign.

²Recall, that we use the convention of summing over a repeated index.

If we assume, that generators L_k are some combination of \hat{x}_i, \hat{p}_i only, we get that $L_i = \varepsilon_{ijk} x_j p_k$, the same formula as is usually obtained in quantum mechanics by correspondence principle. Using this to evaluate the commutator of generators we obtain³

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \quad (2.8)$$

Therefore, generators of rotations obey the $su(2)$ -algebra commutation relations. This algebra corresponds however to two groups, $SU(2)$ and $SO(3)$. Both do represent rotations, however only the second one represents rotations in the real space, due to the fact that the first one has half-integer and the second one integer spin representations.

2.2 Central potential problems

Such problems are described by a Hamiltonian in the form

$$H = \frac{\hat{p}^2}{2m} + V(r) \quad (2.9)$$

where r is the radial distance from the origin. As mentioned in the introduction, this Hamiltonian is rotationally invariant, since \hat{p}^2 transforms as a scalar and r is rotationally invariant, and this can be checked by computing

$$[L_i, H] = 0. \quad (2.10)$$

Thus any central symmetric problem has a $SO(3)$ symmetry. As mentioned in previous sections, this will lead to degeneracy, since if $H|\alpha\rangle = \alpha|\alpha\rangle$, then also $HL_i|\alpha\rangle = \alpha L_i|\alpha\rangle$ and the states $L_i^n|\alpha\rangle$ are all degenerate. Therefore, every energy level will form an irreducible representation of the $SO(3)$ group and states will fall into multiplets of degenerate energy.⁴

Now, we will deal with a special case of potential $V(r) = -\kappa/r$. For $\kappa = Ze^2$ we get an electron in the field of a nucleus, for $\kappa = GMm$ we get a gravitational two body problem⁵.

Recall the eigenstates of the hydrogen atom, which are characterized by three quantum numbers n, l, m . The states of a given n with the same l but different m have the same energy. This is the degeneracy due to the $SO(3)$ rotational symmetry and holds for any rotationally invariant potential. However the states of the same l have also the same energy, which is given

³The calculation uses the identity $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{km} - \delta_{im}\delta_{jl}$.

⁴This means, that these states will span a vector space, where this group represents irreducibly.

⁵Other reason to study this problem is Bertrand's theorem. This proves, that there are only two potentials in the three dimensions, for which every orbit of a bound motion closes. Namely isotropic harmonic oscillator potential r^2 and potential $1/r$.

just by the principal quantum n . This is an unexpected degeneracy, which suggests there is some additional symmetry we have not described yet.

This can be seen also at the classical level, where there is another conserved quantity for the $1/r$ potential and as we mentioned before, conserved quantities are in intimate relation to symmetries due to the Noether's theorem.

It can be checked, that classically, vector

$$R^c = \frac{1}{m} \vec{p} \times \vec{L} - \frac{\kappa}{r} \vec{x} \quad (2.11)$$

called the Runge-Lenz vector, is conserved for a bound motion. It follows from the fact, that the ellipse, that is the trajectory of such motion, does not change. The RL vector points to ellipse's perihelion⁶.

At classical level, this means that the Poisson brackets $\{R_i^c, H\}$ vanish. At quantum level, we want to write $[R_i, H] = 0$, however since p and L do not longer commute, it is not clear how to order these operators.

2.3 Quantized Runge-Lenz vector

One of the possible choices is to take

$$R_i = \frac{1}{2m} \varepsilon_{ijk} (\hat{p}_j L_k + L_k \hat{p}_j) - \frac{\kappa}{r} \hat{x}_i \quad , \quad (2.12)$$

which makes it hermitian. In the following, we will omit the hats and understand x, p as operators. For R in this form, we get

$$\begin{aligned} [R_i, H] &= 0 \quad , \\ [L_i, R_i] &= i \varepsilon_{ijk} R_k \quad , \\ [R_i, R_j] &= i \left(-\frac{2H}{m} \right) \varepsilon_{ijk} L_k \quad , \\ R^2 &= \frac{2H}{m} (L^2 + 1) + \kappa^2 \quad , \end{aligned} \quad (2.13)$$

where the second formula follows from the fact that R is a vector. The other three are results of rather painful calculations, that we shall now show. In what follows, we will often use what is often referred to as 'Davis Cup identity'⁷

$$\varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}. \quad (2.14)$$

⁶Note, that perihelion of the planet Mercury is precessing and thus the RL vector is not a conserved quantity. This is due to the perturbations and relativistic effects, which change the potential and it is no longer $1/r$.

⁷This name is due to the easy mnemonics for this formula : first day matches are the first seed vs. the first seed and the second seed vs. the second seed, the second day the matches are first-second and second-first. Proof of this formula can be done using the expression of matrix determinant in terms of the ε -symbol.

We will also often use the identity

$$[A, BC] = [A, B]C + B[A, C] \quad (2.15)$$

and the fact that x 's commute among themselves and with any function of x . We encourage readers to try to do the computation on their own as a very useful exercise.

And finally let us mention that we will be very loose in the positioning of the indexes. We just note that strictly speaking, some of the indexes should be upper and some lower and readers who wish to make this difference are probably able to do it on their own⁸.

- $[R, H]$

Since L_i 's commute with the Hamiltonian, we can write

$$[R_i, H] = \frac{1}{2m} \varepsilon_{ijk} \left([p_j, H] L_k + L_j [p_k, H] \right) - \frac{\kappa}{2m} \left[\frac{x_i}{r}, p^2 \right]. \quad (2.16)$$

We compute

$$\left[\frac{\partial}{\partial x_j}, \frac{1}{r} \right] = \frac{\partial}{\partial x_j} \frac{1}{r} - \frac{1}{r} \frac{\partial}{\partial x_j} = \left(\frac{\partial}{\partial x_j} \frac{1}{\sqrt{x_i x_i}} \right) = \frac{x_j}{r^3}. \quad (2.17)$$

If this is not clear, it is best to imagine the commutator acting on some function f . This yields

$$[p_j, H] = \left[p_j, -\frac{\kappa}{r} \right] = i\kappa \left[\frac{\partial}{\partial x_j}, \frac{1}{r} \right] = i\kappa \frac{x_j}{r^3}. \quad (2.18)$$

Similarly

$$\left[\frac{\partial}{\partial x_j}, \frac{x_i}{r} \right] = x_i \left[\frac{\partial}{\partial x_j}, \frac{1}{r} \right] + \left[\frac{\partial}{\partial x_j}, x_i \right] \frac{1}{r} = \frac{x_i x_j}{r^3} + \frac{1}{r} \delta_{ij} \quad (2.19)$$

and this gives

$$\left[\frac{x_i}{r}, p^2 \right] = p_j \left[\frac{x_i}{r}, p_j \right] + \left[\frac{x_i}{r}, p_j \right] p_j = i \left[p_i \frac{1}{r} + \frac{1}{r} p_i - \frac{1}{r^3} x_i x_j p_j - p_j x_i x_j \frac{1}{r^3} \right]. \quad (2.20)$$

These two together give

$$[R_i, H] = -\frac{i\kappa}{2m} \varepsilon_{ijk} \left(\frac{x_j}{r^3} L_k + L_k \frac{x_j}{r^3} \right) - \frac{i\kappa}{2m} \left[p_i \frac{1}{r} + \frac{1}{r} p_i - \frac{1}{r^3} x_i x_j p_j - p_j x_i x_j \frac{1}{r^3} \right]. \quad (2.21)$$

⁸The readers who did not understand these two sentences are advised to simply ignore them. The text will make perfect sense even without them. It turns out this can be done with almost any sentence that is not understood.

In the first term, we can take r^{-3} out, since it commutes with L_i . Then, we obtain

$$\begin{aligned}\varepsilon_{ijk} \frac{x_j}{r^3} L_k &= \varepsilon_{ijk} \varepsilon_{klm} \frac{x_j}{r^3} x_l p_m = \frac{x_j}{r^3} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) x_l p_m = \frac{1}{r^3} x_j x_i p_j - \frac{1}{r^3} x_j x_j p_i = \\ &= \frac{1}{r^3} x_i x_j p_j - \frac{1}{r} p_i\end{aligned}\quad (2.22)$$

and similarly we get

$$\begin{aligned}\varepsilon_{ijk} L_k \frac{x_j}{r^3} &= \varepsilon_{ijk} \varepsilon_{klm} x_l p_m \frac{x_j}{r^3} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) x_l p_m \frac{x_j}{r^3} = x_i p_j x_j \frac{1}{r^3} - x_j p_i x_j \frac{1}{r^3} = \\ &= p_j x_i x_j \frac{1}{r^3} + [x_i, p_j] x_j \frac{1}{r^3} - p_i x_j x_j \frac{1}{r^3} - [x_j, p_i] x_j \frac{1}{r^3} = p_j x_i x_j \frac{1}{r^3} - p_i \frac{1}{r}\end{aligned}\quad (2.23)$$

We put these two back in (2.21) and watch all the terms cancel like crazy, making the expression literally vanish, as desired. This shows that R 's are truly the symmetry of the theory.

- $[R, R]$

For the commutator of two R 's we get

$$\begin{aligned}[R_i, R_j] &= \frac{1}{m^2} [\varepsilon_{iab} p_a L_b, \varepsilon_{jmn} p_m L_n] - \frac{i}{m^2} \left([\varepsilon_{iab} p_a L_b, p_j] - \{i \leftrightarrow j\} \right) - \\ &\quad - \frac{\kappa}{m} \left([\varepsilon_{iab} p_a L_b, \frac{x_j}{r}] - \{i \leftrightarrow j\} \right) - \frac{\kappa}{m} \left([p_i, \frac{x_j}{r}] - \{i \leftrightarrow j\} \right).\end{aligned}\quad (2.24)$$

In the above expression $\{i \leftrightarrow j\}$ denotes the same term with indexes i, j interchanged. Note, that any symmetric part of the expression cancels. This way the whole last term vanishes, since the commutator contains terms proportional to δ_{ij} and $x_i x_j$. We now compute

$$[\varepsilon_{iab} p_a L_b, p_j] = \varepsilon_{iab} p_a i \varepsilon_{bjk} p_k = i (\delta_{ij} \delta_{ak} - \delta_{ik} \delta_{ja}) p_a p_k = i (\delta_{ij} p^2 - p_i p_j).\quad (2.25)$$

We have used that the p 's transform as a vector under rotations. This expression is also symmetric in i, j and therefor the corresponding term in (2.24) vanishes. We get the other commutators in a very similar fashion. We compute

$$\begin{aligned}&\frac{1}{m^2} [\varepsilon_{iab} p_a L_b, \varepsilon_{jmn} p_m L_n] = \\ &= \varepsilon_{iab} \varepsilon_{jmn} \left(p_a [L_b, p_m] L_n + p_a p_m [L_b, L_n] + [p_a, p_m] L_m L_b + p_m [p_a, L_n] L_b \right) \\ &= i \varepsilon_{iab} \varepsilon_{jmn} \left(\varepsilon_{bmk} p_a p_k L_n + \varepsilon_{bnk} p_a p_m L_k - \varepsilon_{nak} p_m p_k L_b \right) \\ &= i \varepsilon_{iab} \left((\delta_{nk} \delta_{jb} - \delta_{nb} \delta_{jk}) p_a p_k L_n + (\delta_{jk} \delta_{mb} - \delta_{jb} \delta_{mk}) p_a p_m L_k + (\delta_{jk} \delta_{ma} - \delta_{ja} \delta_{mk}) p_m p_k L_b \right) \\ &= i \varepsilon_{iaj} p_a p_k L_k - i \varepsilon_{iab} p_a p_j L_b + i \varepsilon_{iab} p_a p_b L_j - i \varepsilon_{iaj} p_a p_k L_k + i \varepsilon_{iab} p_a p_j L_b - i \varepsilon_{ijb} p_k p_k L_b \\ &= -i \varepsilon_{ijb} p^2 L_b.\end{aligned}\quad (2.26)$$

And we are left to compute (on our way, we drop a term symmetric in i, j couple of times)

$$\begin{aligned}
-\frac{\kappa}{m} \left[\varepsilon_{iab} p_a L_b, \frac{x_j}{r} \right] &= -\frac{\kappa}{m} i \varepsilon_{iab} \left\{ p_a \frac{1}{r} i \varepsilon_{bjk} x_k - \left(\frac{\delta_{aj}}{r} - \frac{x_j x_a}{r^3} \right) L_b \right\} = \\
&= -i \frac{\kappa}{m} \left(-p_j \frac{x_i}{r} - \frac{1}{r} \varepsilon_{ijk} L_k + \frac{1}{r^3} \varepsilon_{iab} \varepsilon_{bkl} x_j x_a x_k p_l \right) = \\
&= i \frac{\kappa}{m} \left(p_j \frac{x_i}{r} + \frac{1}{r} \varepsilon_{ijk} L_k + \frac{1}{r^3} x_j x_a x_a p_i \right) = i \frac{\kappa}{m} \left(p_j \frac{x_i}{r} + \frac{x_j}{r} p_i + \frac{1}{r} \varepsilon_{ijk} L_k \right) = \\
&= \frac{\kappa}{m} \left(p_j \frac{x_i}{r} + p_i \frac{x_j}{r} + \left[\frac{x_j}{r}, p_i \right] + \frac{1}{r} \varepsilon_{ijk} L_k \right) = \frac{\kappa}{m} \frac{1}{r} \varepsilon_{ijk} L_k \quad , \quad (2.27)
\end{aligned}$$

where we have again used that $[p_i, x_j/r]$ is symmetric in i, j . This way, we get the advertised expression for the commutator of two R 's

$$[R_i, R_j] = -i \frac{1}{m^2} \varepsilon_{ijb} p^2 L_b + 2 \frac{\kappa}{m} \frac{1}{r} \varepsilon_{ijk} L_k = i \left(-\frac{2H}{m} \right) \varepsilon_{ijk} L_k. \quad (2.28)$$

- R^2

And finally, we will sketch the calculation of R^2 . Filling in the details should be no problem by now. We get

$$\begin{aligned}
R^2 &= \frac{1}{4m^2} (p_j L_k + L_k p_j) (p_j L_k + L_k p_j - p_k L_j - L_j p_k) \\
&\quad - \frac{\kappa}{2m} \varepsilon_{ijk} (p_j L_k + L_k p_j) \frac{x_i}{r} - \frac{\kappa}{2m} \frac{x_i}{r} \varepsilon_{ijk} (p_j L_k + L_k p_j) + \kappa^2. \quad (2.29)
\end{aligned}$$

and after some more or less straightforward algebra, we find

$$\begin{aligned}
(p_j L_k + L_k p_j) (p_j L_k + L_k p_j) &= 4p^2 L^2 - i \varepsilon_{kjl} p_l (p_j L_k + L_k p_j) = 4p^2 L^2 + 2p^2 \quad , \\
-(p_j L_k + L_k p_j) (p_k L_j + L_j p_k) &= 2p^2 \quad , \\
-\varepsilon_{ijk} (p_j L_k + L_k p_j) \frac{x_i}{r} &= - \left(\frac{2L^2}{r} + \frac{2i}{r} p \cdot x \right) \quad , \\
-\frac{x_i}{r} \varepsilon_{ijk} (p_j L_k + L_k p_j) &= - \left(\frac{2L^2}{r} - \frac{2i}{r} p \cdot x \right). \quad (2.30)
\end{aligned}$$

These expressions, when inserted into (2.29) give the desired result.

To conclude, let us repeat the commutation relations again

$$\begin{aligned}
[L_i, R_i] &= i \varepsilon_{ijk} R_k \quad , \\
[R_i, R_j] &= i \left(-\frac{2H}{m} \right) \varepsilon_{ijk} L_k \quad , \\
R^2 &= \frac{2H}{m} (L^2 + 1) + \kappa^2. \quad (2.31)
\end{aligned}$$

2.4 Bound states and the $O(4)$ symmetry

We will constraint ourselves only to the states with a constant, negative value of $H < 0$. Then, it is legitimate to change to a different quantity⁹

$$K_i = \frac{1}{\sqrt{-\frac{2H}{m}}} R_i. \quad (2.32)$$

Now all the commutation rules are

$$\begin{aligned} [K_i, K_j] &= i\varepsilon_{ijk} L_k, \\ [L_i, L_j] &= i\varepsilon_{ijk} L_k, \\ [L_i, K_j] &= i\varepsilon_{ijk} K_k. \end{aligned} \quad (2.33)$$

which are the $O(4)$ algebra commutation relations. To see this, we need to look at infinitesimal rotations in four dimensions. We would write the 4×4 matrices corresponding to these rotations in a similar way as in the section 2.1 and compute the commutation relations.

Now, we introduce a different set of operators M_i, N_i

$$\begin{aligned} M_i &= \frac{K_i + L_i}{2}, \\ N_i &= \frac{K_i - L_i}{2}, \end{aligned} \quad (2.34)$$

so that (2.33) becomes

$$\begin{aligned} [M_i, M_j] &= i\varepsilon_{ijk} M_k, \\ [N_i, N_j] &= i\varepsilon_{ijk} N_k, \\ [M_i, N_j] &= 0 \end{aligned} \quad (2.35)$$

i.e. two independent $SU(2)$ algebras, thus $O(4) = SU(2) \times SU(2)$. Using (2.13) and (2.32), we get

$$R^2 = \frac{2H}{m}(L^2 + 1) + \kappa^2 = -\frac{2H}{m}K^2 \quad \Rightarrow \quad H = -\frac{\kappa^2 m}{2} \frac{1}{L^2 + K^2 + 1}. \quad (2.36)$$

⁹One could wonder how is an expression like $1/\sqrt{\text{operator}}$ defined. One could then imagine this as a Taylor series but then questions as convergence of a series of states arise. However if we constrain ourselves only to the eigenstates of this operator, is is quite clear that

$$\frac{1}{\sqrt{\text{operator}}} |eigenstate\rangle = \frac{1}{\sqrt{\text{eigenvalue}}} |eigenstate\rangle$$

We find out that

$$\begin{aligned} M^2 &= \frac{L^2 + K^2 + K \cdot L + L \cdot K}{4}, \\ N^2 &= \frac{L^2 + K^2 - K \cdot L - L \cdot K}{4}. \end{aligned} \quad (2.37)$$

We compute

$$R \cdot L = \frac{1}{2m} \varepsilon_{ijk} \varepsilon_{imn} (p_j L_k + L_k p_j) x_m p_n - \kappa \frac{1}{r} \varepsilon_{imn} x_i x_m p_n. \quad (2.38)$$

The second term vanishes, since it is both symmetric and antisymmetric in i and m indexes, or equivalently $x \cdot (x \times p) = 0$. The other two terms are also going to vanish due to the following

$$\begin{aligned} \varepsilon_{ijk} \varepsilon_{imn} p_j L_k x_m p_n &= p_j L_k x_j p_k - p_j L_k x_k p_j = (p \cdot x) \underbrace{(L \cdot p)}_0 - p_j \varepsilon_{kjr} x_r p_k - p_j \underbrace{(L \cdot x)}_0 p_j = \\ &= -i p_j \underbrace{\varepsilon_{jrk} x_r p_k}_{(x \times p)_j} = 0 \end{aligned} \quad (2.39)$$

So we see that $R \cdot L = 0$, thus $K \cdot L = 0$ and we get $M^2 = N^2$, since $K \cdot L = L \cdot K$. Finally, for the Hamiltonian we get

$$H = -\frac{\kappa^2 m}{2} \frac{1}{2(M^2 + N^2) + 1} \quad (2.40)$$

Both M^2 and N^2 are equivalent to J^2 operator of $SU(2)$, thus for the irreducible representations they are both equal to $k(k+1)$, with $k = 0, \frac{1}{2}, 1, \dots$, and we obtain

$$H = -\frac{\kappa^2 m}{2} \frac{1}{(2k+1)^2} = -\frac{\kappa^2 m}{2} \frac{1}{n^2}, \quad n = 1, 2, \dots \quad (2.41)$$

We have derived the well known spectrum of the hydrogen atom. The states are given by a product of two spin- k representations of the $SU(2)$, i.e. $|j_M = k, m_M\rangle \otimes |j_N = k, m_N\rangle$. The last thing we need to determine are degeneracies of the energy levels. This is the number of possible values of $L_3 = M_3 + N_3$, where M_3 and N_3 have possible values of k to $-k$. This is equivalent to the composition of two spins k and we know that the possible values of L_3 are from $j_M + j_N = 2k$ to $|j_M - j_N| = k - k = 0$. Thus, we have $2k + 1 = n$ possible states with the corresponding energy, given by n . This solves the bound states for the hydrogen atom.

Note, that the additional 1 in the denominator of (2.40) is the purely quantum mechanical term. If we used the \hbar units, this term would have been \hbar^2 and thus vanishing in the classical limit $\hbar \rightarrow 0^{10}$. This is expected, since the ground state of the classical Hamiltonian is just $r = 0, p = 0, H \rightarrow -\infty$. This however means an electron with sharp momentum and position, which as we know is not possible in the quantum theory due to the uncertainty principle. And we see that the lowest energy state has a finite energy.

¹⁰Recall the discussion about the meaning of this limit from the introduction.

2.5 Scattering states and the $O(3, 1)$ symmetry

We will try to follow a similar approach for this case. Here, we have $H > 0$, thus we define

$$\tilde{K}_i = \frac{1}{\sqrt{\frac{2H}{m}}} R_i \quad (2.42)$$

which yields similar commutation rules

$$\begin{aligned} [\tilde{K}_i, \tilde{K}_j] &= -i\varepsilon_{ijk} L_k \\ [L_i, L_j] &= i\varepsilon_{ijk} L_k \\ [L_i, \tilde{K}_j] &= i\varepsilon_{ijk} \tilde{K}_k \end{aligned} \quad (2.43)$$

If we now redefine once again $\tilde{K}_i = iQ_i$, then Q 's and L 's follow commutation relations analogous to (2.33). However, Q 's are no longer hermitian operators. Alternatively, in the definition of the symmetry transformation

$$U = e^{i(\theta_i L_i + \omega_i \tilde{K}_i)} \quad (2.44)$$

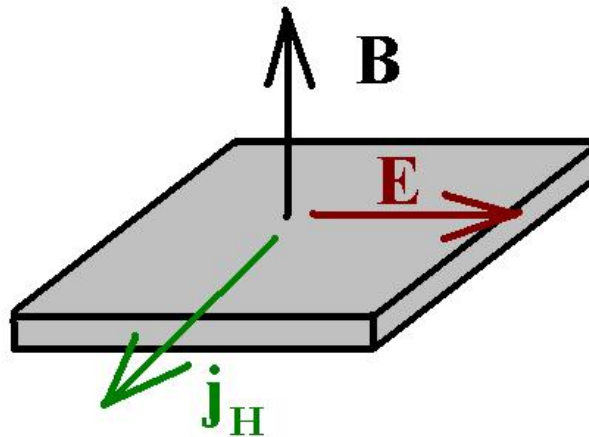
we could make parameters ω_i complex $\omega_i \rightarrow i\omega_i$. This is very similar to the special theory of relativity, where the time is rotated by a complex angle, resulting in a minus sign in the metric and $O(3, 1)$ as the symmetry group.

This is going to be the case also here and we get the Lorentz group $O(3, 1)$ as the symmetry group of the problem for the scattering states. This group, how we could see in the case of Q operators, does not have a finite dimensional irreducible unitary representation, since it is not a compact group. The finite dimensional operators were not hermitian, yielding a non-unitary group transformation after exponentiation. The only unitary representations, which are of our interest, are infinite dimensional. We would also find, that even though the theory is quantum, it does not have quantized energy spectrum, which is continuous.

Chapter 3

The Quantum Hall effect

The Hall effect is a classical effect discovered by Edwin Hall in 1879. It arises in a two-dimensional plate of conducting material that is placed into a homogeneous electric field parallel and magnetic field perpendicular to the plate. The free electrons start to move in the direction of the electric field and the current density is $j = \sigma E$.



However due to the magnetic field, there will be also magnetic force on the electrons perpendicular to the electric field and therefore a current in this direction. For this current, the density is

$$j_H = \sigma_H(B)E. \quad (3.1)$$

This magnetic conductivity is function of the magnetic field and the material used.

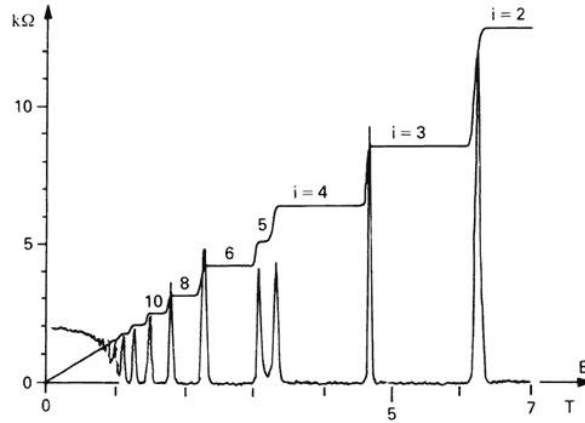
In the classical case, this dependence on the magnetic field B is linear. However in the case of a very large magnetic fields and low temperatures, the quantum effects kick in and this is going to change. This was predicted in 1975 by Ando, Matsumoto, and Uemura. Their calculations

suggested quantization of the the magnetic conductivity and the step-like dependence of $\sigma_H(B)$ on B , namely

$$\sigma_H(B) = \nu \frac{e^2}{2\pi\hbar} B \quad , \quad (3.2)$$

where ν is some integer. This was first measured by von Klitzing in 1980 and for this finding, he was awarded the Nobel Prize in Physics in 1985.

The magnetic conductivity is quantize in the units of $\frac{e^2}{2\pi\hbar}$. The effect can be measured with such precision, that it is currently source of the best value of the Plank constant and ration $h/e^2 = 25812.807557(18) \Omega$ is used as a standard for the electric resistance.



In the quantum case, it has been observed to have a step-function character at very low temperatures. This is called the integer QHE. Later, fractional steps has been found and this is called the fractional QHE.

3.1 The Physics of integer QHE

3.1.1 Electron in a magnetic field

Complete Hamiltonian of the N electron system in the presence of electromagnetic field is

$$H = \sum_{\alpha=1}^N \left(\frac{(\hat{p}_{\alpha} - e\hat{A}(\vec{x}_{\alpha}))^2}{2\mu} + \frac{1}{2} \sum_{\beta \neq \alpha} \frac{e^2}{|\vec{x}_{\alpha} - \vec{x}_{\beta}|} \right) \quad , \quad (3.3)$$

where μ is the effective mass of the free electron in the conducting solid. This expression does not contain the potential that confines the electrons to the sample, which we assume to be steep enough that the electrons can be considered to move inside a very steep potential well.

Now, as a first approximation, we neglect the second term and consider the electrons to be noninteracting. This way our Hamiltonian separates and we get just N independent problems with Hamiltonian

$$H_\alpha = \frac{(\hat{p}_\alpha - e\hat{A}_\alpha)^2}{2\mu}. \quad (3.4)$$

In our case¹

$$A = \left(-\frac{1}{2}Bx_2, \frac{1}{2}Bx_1, 0 \right) \quad (3.5)$$

and thus

$$H_\alpha = \frac{1}{2\mu} \left[\left(p_{\alpha 1} + \frac{1}{2}eBx_{\alpha 2} \right)^2 + \left(p_{\alpha 2} - \frac{1}{2}eBx_{\alpha 1} \right)^2 + p_{\alpha 3}^2 \right]. \quad (3.6)$$

Notice, that so far we have not introduced any electric field and at the moment we are dealing only with the case of electrons in the presence of magnetic field. Also note, that the motion in the 3-direction is trivial. Indeed, we can completely ignore it, as we assume the sample to be very thin and therefore the remaining problem to be essentially two dimensional.

To solve this Hamiltonian, we introduce an analogue of canonical transformation²

$$\begin{aligned} Q &= \frac{p_1 + \frac{1}{2}eBx_2}{\sqrt{eB}}, \\ P &= \frac{p_2 - \frac{1}{2}eBx_1}{\sqrt{eB}} \end{aligned} \quad (3.7)$$

which yields

$$H_\alpha = \frac{eB}{2\mu} [P^2 + Q^2]. \quad (3.8)$$

This is the Hamiltonian of a harmonic oscillator we can immediately write the resulting energies

$$E_n = \frac{eB}{\mu} \left(n + \frac{1}{2} \right). \quad (3.9)$$

This formula was expected. We are dealing with circular orbits of electrons around magnetic field lines. The quantization condition here is similar as the one in the case of hydrogen atom and the wave of electron needs to 'close' itself around the loop, yielding discrete momenta and thus discrete energies. These levels are called Landau levels.

¹We easily check that $\vec{\nabla} \times \vec{A} = (0, 0, B)$

²We easily check that $[Q, P] = i$

We have still yet to deal with all the canonical variables. There are four of them so we can introduce two more

$$\Pi_1 = \frac{p_1 - \frac{1}{2}eBx_2}{\sqrt{eB}} \quad , \quad \Pi_2 = \frac{p_2 + \frac{1}{2}eBx_1}{\sqrt{eB}}. \quad (3.10)$$

It is easy to check that given the commutators of p 's and x 's that these commute with P, Q . Also, we note that these commute with the Hamiltonian. Therefore they are symmetry of our problem, that results in the degeneracy. This corresponds to the fact, that we can move the center of the electron orbit around freely.

We are left to determine the degeneracies of these energy states. At the moment, we are going to work in semi-classical limit, where the number of the states is going to be very large and we can use the following formula for the density of states³

$$\text{number of states in the Hilbert space} = \frac{\text{phase volume}}{(2\pi)^{(\text{dimension of phase space})/2}}. \quad (3.11)$$

The phase volume is $dPdQd\Pi_1d\Pi_2$. However, we can constrain ourselves to the states of a given energy, since we are interested in a Landau level of a fixed n . This means $P^2 + Q^2 = \text{const}$. In the phase space volume, we can introduce spherical coordinates and get

$$dPdQd\Pi_1d\Pi_2 = r dr d\Pi_1 d\Pi_2 \int d\theta = r dr 2\pi d\Pi_1 d\Pi_2. \quad (3.12)$$

Since we are interested in the fixed r , we can disregard $r dr$ and we get for the number of states

$$\#states = \frac{2\pi d\Pi_1 d\Pi_2}{(2\pi)^2} = \frac{d\Pi_1 d\Pi_2}{2\pi}. \quad (3.13)$$

Now, still in the semi-classical limit we assume that the energies are very small and therefore

$$H \approx 0 \quad \Rightarrow \quad P = Q = 0 \quad \Rightarrow \quad dp_2 = \frac{eB}{2} dx_1 \quad , \quad dp_1 = -\frac{eB}{2} dx_2. \quad (3.14)$$

Using this

$$d\Pi_1 = \sqrt{eB} dx_2 \quad , \quad d\Pi_2 = \sqrt{eB} dx_1 \quad (3.15)$$

and finally

$$\#states = \frac{eB}{2\pi} dx_1 dx_2. \quad (3.16)$$

This means that the number of states per unit area is

$$a = \frac{eB}{2\pi}. \quad (3.17)$$

³There is an \hbar in the denominator of this expression, which is 1 in our units.

There is one electron state per area this big, since the electrons are fermions. We do not consider the spin degeneration, i.e. the possibility of two electrons in the same state, since we deal with systems at low temperatures, where we can assume total polarization of electron spins in the direction of B . Therefore, in area of size A , the number of electron states of a given energy is

$$\frac{A}{a} = \frac{2\pi}{eB}A. \quad (3.18)$$

Note, that this number does not depend on the value of the energy itself. Now, assuming that n lowest energy levels are filled, charge per unit area is

$$ne\frac{A}{a} = n\frac{e^2B}{2\pi}. \quad (3.19)$$

This is the averaged time component of the 4-current

$$\langle J_0(x) \rangle \equiv \int d^2x \Psi^* J_0 \Psi = n\frac{e^2B(x)}{2\pi}, \quad (3.20)$$

where Ψ is the wave function of the electronic system.

3.1.2 Electric field as a perturbation

Now, we are ready to introduce electric field. Namely, we are interested what happens, if we introduce a perturbation to the potential $(\delta A_0, \delta A_i)$. Even though it is not our case, let us first investigate the response of the system due to the perturbation δA_i . We compute

$$\delta \langle J_0(x) \rangle = \frac{ne^2}{2\pi} \delta B. \quad (3.21)$$

Since $B = \partial_1 A_2 - \partial_2 A_1 = \varepsilon^{ij} \partial_i A_j$, we get

$$\delta \langle J_0(x) \rangle = \frac{ne^2}{2\pi} \varepsilon^{ij} \partial_i \delta A_j(x) = n\frac{e^2}{2\pi} \int d^2y \varepsilon^{ij} \partial_i \delta^{(2)}(x-y) \delta A_j(y), \quad (3.22)$$

where $\delta^{(2)}(x-y)$ is the two dimensional Dirac delta function. On the other hand, since the perturbation Hamiltonian is⁴

$$H_{per} = \int d^2y J_i(y) \delta A_i(y) \quad (3.23)$$

the first order correction to the wave function is

$$\delta \Psi = \int d^2y \frac{\delta A_i J_i}{H - E} \Psi \quad (3.24)$$

⁴Recall the coupling of charges given by a current J_μ to the electromagnetic field of potential A_μ of the form $\int A_\mu J_\mu$.

Using this, we compute

$$\begin{aligned}\delta \langle J_0(x) \rangle &= \int d^2y (\Psi + \delta\Psi)^* J_0(x) (\Psi + \delta\Psi) - \int d^2y \Psi^* J_0(x) \Psi = \\ &= \int d^2y \left\langle J_i(y) \delta A_i(y) \frac{1}{H-E} J_0(x) + J_0(x) J_i(y) \delta A_i(y) \frac{1}{H-E} \right\rangle.\end{aligned}\quad (3.25)$$

This, together with (3.22) yields

$$n \frac{e^2}{2\pi} \varepsilon^{ij} \partial_i \delta^{(2)}(x-y) = \left\langle J_j(y) \frac{1}{H-E} J_0(x) + J_0(x) J_j(y) \frac{1}{H-E} \right\rangle.\quad (3.26)$$

Now, we introduce the true perturbation that is going to be relevant in our case, δA_0 and consequently the change in the spatial current. We compute

$$\begin{aligned}\delta \langle J_i(y) \rangle &= \int d^2x (\Psi + \delta\Psi)^* J_i(x) (\Psi + \delta\Psi) - \int d^2x \Psi^* J_i(x) \Psi = \\ &= \int d^2x \left\langle J_i(y) \frac{1}{H-E} J_0(x) + J_0(x) J_i(y) \frac{1}{H-E} \right\rangle \delta A_0(x)\end{aligned}\quad (3.27)$$

which using formula (3.26) becomes

$$\begin{aligned}\delta \langle J_j(x) \rangle &= \int d^2x \frac{ne^2}{2\pi} \varepsilon^{ij} \partial_i \delta^{(2)}(x-y) \delta A_0(y) = \\ &= - \int d^2x \frac{ne^2}{2\pi} \varepsilon^{ij} \delta^{(2)}(x-y) \partial_i \delta A_0(y) = \frac{ne^2}{2\pi} \varepsilon^{ji} \partial_i \delta A_0(x)\end{aligned}\quad (3.28)$$

Since $\partial_i \delta A_0(x) = \delta E_i(x)$ we get

$$\langle J_i \rangle = \frac{ne^2}{2\pi} \varepsilon^{ij} E_j\quad (3.29)$$

and

$$\sigma^{ij} = \frac{ne^2}{2\pi} \varepsilon^{ij}\quad (3.30)$$

where n is the highest filled level.

So we have shown, that the perturbation of the system gives rise to a electric current perpendicular to the electric field. The conductivity is a step function in the number n , which is the number of the Landau levels filled. How do we see that this number increases discretely with B ?

Imagine a fully filled n -th Landau level. We can imagine it as droplets of electrons in the sample. If the droplet is filled, there is no room to move inside the droplet and the only possible motion is on the edges. As we increase the magnetic field, we raise the degeneracy of this level and we also release electrons from the impurities of the sample, which fill in these new places. However, as we raise and raise the B , the states around the edges of the sample become energetically too demanding and it is easier to fill a higher level, than to place an electron near the edge. n increases by one and σ is a step function of the magnetic field.

3.2 Construction of an electronic wave function for QHE

In this section, we will treat the quantum Hall effect in a little different fashion. We will reproduce the result of previous part by the construction of the wave function of the electronic droplet in the sample. This approach then allows for some discussion of the fractional quantum Hall effect.

Rewriting the Hamiltonian (3.8) in terms of creation and annihilation operators we get

$$H = \frac{eB}{\mu} \left(a^\dagger a + \frac{1}{2} \right) , \quad (3.31)$$

where

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2eB}} \left[p_1 + ip_2 + \frac{eB}{2}(x_2 - ix_1) \right] , \\ a^\dagger &= \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2eB}} \left[p_1 - ip_2 - \frac{eB}{2}(x_2 - ix_1) \right] . \end{aligned} \quad (3.32)$$

We now introduce the usual representation of operators x and p to get for the creation operator

$$a = \frac{1}{\sqrt{2eB}} \left[-i \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) - \frac{ieB}{2} (x_1 + ix_2) \right] . \quad (3.33)$$

If we introduce complex coordinates in the x_1, x_2 -plane, namely

$$z = x_1 + ix_2 \quad , \quad \bar{z} = x_1 - ix_2 \quad , \quad (3.34)$$

the creation operator becomes⁵

$$a = \frac{-i}{\sqrt{2eB}} \left[\frac{\partial}{\partial \bar{z}} + \frac{eB}{2} z \right] . \quad (3.35)$$

Now if we look for the lowest energy level, we need to solve $a\psi = 0$, i.e.

$$\left[\frac{\partial}{\partial \bar{z}} + \frac{eB}{2} z \right] \psi(z, \bar{z}) = 0 \quad , \quad (3.36)$$

which is easy enough to solve and yields

$$\psi(z, \bar{z}) = e^{-\frac{eB}{4}\bar{z}z} f(z) \quad , \quad (3.37)$$

⁵To see, that this combination of $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ is equal to $\frac{\partial}{\partial \bar{z}}$ note for example, that then

$$\frac{\partial \bar{z}}{\partial \bar{z}} = 1 \quad , \quad \frac{\partial z}{\partial \bar{z}} = 0$$

where $f(z)$ is any function of the z variable only. Recall, that operators of magnetic translations Π_1, Π_2 represented the degeneracy of the energy levels. Combining these two into a pair of 'creation and annihilation' operators, we get

$$\begin{aligned} c &= \frac{\Pi_1 - i\Pi_2}{\sqrt{2}} = -\frac{i}{\sqrt{2eB}} \left(\frac{\partial}{\partial \bar{z}} - \frac{eB}{2} z \right) , \\ c^\dagger &= \frac{\Pi_1 + i\Pi_2}{\sqrt{2}} = -\frac{i}{\sqrt{2eB}} \left(\frac{\partial}{\partial z} + \frac{eB}{2} \bar{z} \right) . \end{aligned} \quad (3.38)$$

If we now act by these on the lowest energy level wave function $\psi(z, \bar{z})$, we obtain

$$c\psi(z, \bar{z}) = (-i)e^{-\frac{eB}{4}\bar{z}z} \left[\left(-\frac{1}{2}\bar{z}f + \frac{\partial f}{\partial z} \right) + \frac{1}{2}\bar{z}f \right] = (-i)e^{-\frac{eB}{4}\bar{z}z} \frac{\partial f}{\partial z}. \quad (3.39)$$

Similarly $c^\dagger\psi(z, \bar{z}) = ie^{-\frac{eB}{4}\bar{z}z}zf$. The degeneracy corresponds to adding or removing one power of z from f . So we see, that freedom choosing $f(z)$ in (3.37) represents the degeneracy of the lowest energy level. Each degenerate state corresponds to a choice $f(z) = z^n$ and in principle, there is infinite number of functions $1, z, z^2, \dots$ available. We will see shortly how this number is cut off by the constraint of the spatial extent of the droplet. However, we first need to determine the multi-particle wave function from these single particle ones.

We will use method of Slater determinant to do this. This provides a prescription for a fermionic multi-particle wave function, providing we have the single particle orbitals. Which we do. In general, N -particle Slater determinant is given by

$$\Psi(\vec{r}_1, \dots, \vec{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\vec{r}_1) & \psi_2(\vec{r}_1) & \dots & \psi_N(\vec{r}_1) \\ \psi_1(\vec{r}_2) & \psi_2(\vec{r}_2) & \dots & \psi_N(\vec{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\vec{r}_N) & \psi_2(\vec{r}_N) & \dots & \psi_N(\vec{r}_N) \end{vmatrix}. \quad (3.40)$$

The factor $1/\sqrt{N!}$ takes care of the normalization and the determinant makes sure that the function is anti-symmetric under exchange of any two \vec{r}^s . In our case, this turns into

$$\Psi(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} e^{-\frac{eB}{4}\bar{z}_i z_i} \begin{vmatrix} 1 & z_1 & \dots & z_1^N \\ 1 & z_2 & \dots & z_2^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & \dots & z_N^N \end{vmatrix}. \quad (3.41)$$

Either by direct computation or by induction, one can show that this equals

$$\Psi(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} e^{-\frac{eB}{4}\bar{z}_i z_i} \prod_{i < j} (z_i - z_j). \quad (3.42)$$

Now, we need to figure out the electron density of the electron gas that is described by this wave function. Here, we note that in the probability density $\Psi(z_1, \dots, z_N)$, we will encounter terms like

$$e^{-\frac{eB}{2}\bar{z}z}(\bar{z}z)^k = e^{-\frac{eB}{2}r_i^2}(r_i^2)^k, \quad (3.43)$$

for $k = 0, 1, \dots, N - 1$ and for different r_i . Plotting these functions we see, that they are all peaked around a value, that changes with k and gets larger for larger k . Therefore, we can identify the peak of this function for $k = N - 1$ as the radius of the electron droplet formed by the electrons. To find the maximum we compute

$$\frac{d}{d r^2} \left(e^{-\frac{eB}{2}r^2}(r^2)^{N-1} \right) = -\frac{eB}{2}e^{-\frac{eB}{2}r^2}(r^2)^{N-1} + (N-1)(r^2)^{N-2}e^{-\frac{eB}{2}r^2} \quad (3.44)$$

and set this derivative to zero

$$\begin{aligned} 0 &= -\frac{eB}{2}e^{-\frac{eB}{2}R^2}(R^2)^{N-1} + (N-1)(R^2)^{N-2}e^{-\frac{eB}{2}R^2} \\ &\quad \downarrow \\ N &= 1 + \frac{eB}{2\pi}(\pi R^2) = 1 + \frac{eB}{2\pi}A. \end{aligned} \quad (3.45)$$

Note, that this gives the same result as the formula (3.17), up to a factor of $1/A$, which is small in the limit of a large number of electrons, i.e. large area. So from this point, the discussion goes the same way it did in the previous section, with the same result of quantum Hall effect.

3.3 Fractional QHE

We rewrite the final formula for the Hall conductivity as

$$\sigma_H = \nu \frac{e^2}{2\pi}, \quad (3.46)$$

where ν is a filling number. In the case of the integer QHE this number was just n , the highest filled Landau level. Remember that the step-like function for $\sigma_H(B)$ was caused by the fact, that the electrons could not move within the droplet since there was 'no space to move'.

Now, the Coulomb term we have neglected in the beginning will come into play. Since even if the Landau level is not completely filled, the Coulomb repulsion might prevent the electrons from moving within the droplet and give rise to QHE with fractional filling number.

It is very difficult to see this explicitly. Indeed, the problem with Hamiltonian (3.8) has not been solved to any satisfactory extent yet. A qualitative description was provided by Laughlin⁶,

⁶Laughlin together with Strmer and Tsui received the Nobel Prize in Physics in 1998.

who suggested the following wave function for the electronic system

$$\Psi_{FQHE}(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} e^{-\frac{eB}{4} \sum_i \bar{z}_i z_i} \prod_{i < j} (z_i - z_j)^{2p+1} \quad , \quad (3.47)$$

where p is an integer. The motivation for such wave function is following. First, to preserve anti-symmetry we need to raise the z 's to an odd power. Also, following the derivation of (3.45), we see, that a higher power will introduce a fractional coefficient into the formula for electron density, which in the limit of large area becomes

$$\frac{N}{A} = \frac{1}{2p+1} \left(\frac{eB}{2\pi} \right). \quad (3.48)$$

And this gives rise to quantum Hall effect with a filling number

$$\nu = \frac{1}{2p+1} \quad , \quad p = 1, 2, \dots \quad . \quad (3.49)$$

This means that the number of filled states is less than the number of states available, thus the name filling fraction.

We shall not investigate the properties of these states more. We will only show, that they exhibit some very peculiar properties, namely that they act like particles with fractional charge and that they obey neither bosonic, nor fermionic statistics.

Starting with (3.47), we define a new object

$$\Psi(w) = \left(\prod_i (z_i - w) \right) \frac{1}{\sqrt{N!}} e^{-\frac{eB}{4} \sum_i \bar{z}_i z_i} \prod_{i < j} (z_i - z_j)^{2p+1} \quad , \quad (3.50)$$

where w is a fixed point, which will turn out to be the position of the new particle exhibiting the strange properties. Recall that the product in this expression runs over all the electrons that form the droplet.

3.3.1 Fractional charge

First, let us see what charge does the object described by the wave function (3.50) carry. To do this, we recall result from quantum mechanics, which states that the charge of the particle is related to the phase picked up by the wave function when the particle is carried along a trajectory in magnetic field. Namely the phase change is equal to

$$e^{iq \oint A \cdot dx} \quad , \quad (3.51)$$

where q is the charge of the particle and A is the vector potential of the field. The phase of the wave function is $\log \Psi(w)$ so the phase change of the wave function on a path parametrized as

$w(\tau)$ is

$$\frac{1}{i} \oint d\tau \sum_i \left(\frac{d}{d\tau} \log(z_i - w) \right) = \frac{1}{i} \sum_i \int dz \delta(z - z_i) \oint d \log(z - w). \quad (3.52)$$

We have expressed the sum as an integration over dz with a δ -function factor. The second integral is going to produce a factor of $2\pi i$ for every z_i enclosed by the curve, due to multivalued nature of the complex logarithm. Sum over the δ -functions is just the particle density, which can be expressed as $\langle J_0 \rangle / e$ so the final result we get for the phase change of the wave function is

$$2\pi \frac{\langle J_0 \rangle}{e} \times (\text{area enclosed by the curve}). \quad (3.53)$$

Using the expression for J_0 this becomes

$$e\nu B S = e\nu \int B dS = e\nu \oint A \cdot dx. \quad (3.54)$$

We have used the fact that $B S$ is the magnetic flux through the area enclosed by the curve which we then expressed in terms of the vector potential using the Stokes theorem. So we finally got the expression for the phase change, which shows that the charge of the object described by (3.50) is

$$q = e\nu = \frac{e}{2p+1}. \quad (3.55)$$

3.3.2 Fractional statistics and inequivalent representations of the Heisenberg algebra

Let us now look for Lagrangian, which would describe the dynamics of the excitations (3.50). This means a Lagrangian, which would lead to

$$J_\mu = \frac{e^2}{2\pi k} \varepsilon_{\mu\nu\alpha} \partial_\nu A_\alpha. \quad (3.56)$$

This expression for the current puts (3.29) and (3.20) together and as we have seen leads to the QHE with filling fraction $\nu = k^{-1}$. We are not going to derive the Lagrangian here, but rather prove that the following expression does the job

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\alpha\beta} a_\mu \partial_\alpha a_\beta - a \left(j_\mu + \frac{eB_\mu}{2\pi} \right). \quad (3.57)$$

In this expression a_μ is some yet undetermined dynamical field, $B_\mu = \varepsilon_{\mu\alpha\beta} \partial_\alpha A_\beta$ are components of the electromagnetic field and j_μ is the current for excitations, i.e.

$$j_0 = \sum_n \delta(x - x_n) \quad , \quad j_i = \sum_n \frac{dx_{n,i}}{dt} \delta(x - x_n) \quad , \quad (3.58)$$

where the sum is over the excitations, not the electrons that form the droplet. The equation of motion for a_μ is

$$\begin{aligned} \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu a_\mu)} - \frac{\partial \mathcal{L}}{\partial a_\mu} &= 0 \\ \frac{k}{4\pi} \partial_\nu \varepsilon_{\alpha\nu\mu} a_\alpha - \left[\frac{k}{4\pi} \varepsilon_{\mu\alpha\beta} \partial_\alpha a_\beta - \left(j_\mu + \frac{eB_\mu}{2\pi} \right) \right] &= 0 \\ \frac{k}{2\pi} \varepsilon_{\mu\alpha\beta} \partial_\alpha a_\beta &= j_\mu + \frac{eB_\mu}{2\pi} \end{aligned} \quad (3.59)$$

After adding the standard Maxwell term to our Lagrangian, we obtain the equation of motion for the electromagnetic potential

$$\begin{aligned} \partial_\mu F_{\mu\nu} + \frac{e}{2\pi} \varepsilon_{\mu\nu\alpha} \partial_\alpha a_\mu &= 0 \\ J_\mu &= \frac{e}{2\pi} \varepsilon_{\mu\alpha\beta} \partial_\alpha a_\beta = e \frac{j_\mu}{k} + \frac{e^2}{2\pi k} \varepsilon_{\mu\nu\alpha} \partial_\nu \partial A_\alpha. \end{aligned} \quad (3.60)$$

We have used the fact that $-\partial_\mu F_{\mu\nu}$ is the electromagnetic current. So we see that we have obtained the desired result for J_μ , which leads to QHE for $k = 2p + 1$. J_μ has also a component that corresponds to the current due to the excitations j_μ .

The effective coupling between the excitations and the magnetic field is given by the $a \cdot j$ term in the Lagrangian, so we need to solve the equation of motion for a_μ to be able to write an effective Lagrangian for the excitations only. To do this we recall the complex coordinates and define

$$\begin{aligned} a_z &\equiv a = \frac{a_1 - ia_2}{2} \quad , \\ a_{\bar{z}} &\equiv \bar{a} = \frac{a_1 + ia_2}{2}. \end{aligned} \quad (3.61)$$

Note that the signs in these expressions are not a choice, but are dictated by the signs in the expression for the derivatives

$$\begin{aligned} \frac{\partial}{\partial z} &\equiv \partial = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad , \\ \frac{\partial}{\partial \bar{z}} &\equiv \bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) \quad , \end{aligned} \quad (3.62)$$

since the gauge potentials need to have the same form as derivatives. The equation of motion for a_μ then becomes

$$\begin{aligned} \frac{k}{2\pi} \varepsilon_{0ij} \partial_i a_j &= j_0 \\ \frac{2}{i} \frac{k}{2\pi} (\partial \bar{a} - \bar{\partial} a) &= \sum_n \delta(x - x_n). \end{aligned} \quad (3.63)$$

To solve these, we make use of the following identity

$$\begin{aligned}\frac{\partial}{\partial z} \left(\frac{1}{\bar{z} - \bar{w}} \right) &= \pi \delta(z - w) \\ \frac{\partial}{\partial \bar{z}} \left(\frac{1}{z - w} \right) &= \pi \delta(\bar{z} - \bar{w}).\end{aligned}\quad (3.64)$$

The only place the derivatives can have non-zero value is at the singularity. At the singularity we use Cauchy's theorem

$$\begin{aligned}\oint f(z) \frac{dz}{z - w} &= \int dz d\bar{z} f(z) \frac{\partial}{\partial \bar{z}} \frac{1}{z - w} = 2i \int d^2x f(z) \frac{\partial}{\partial \bar{z}} \frac{1}{z - w} \\ &= 2\pi i f(w)\end{aligned}\quad (3.65)$$

which proves the previous formula. Now it is trivial to write down the solution to (3.63) as

$$\begin{aligned}a &= -\frac{i}{2k} \sum_n \frac{1}{z - z_n}, \\ \bar{a} &= \frac{i}{2k} \sum_n \frac{1}{\bar{z} - \bar{z}_n}.\end{aligned}\quad (3.66)$$

This allows us to write an effective Lagrangian for the excitations

$$\begin{aligned}L &= \sum_n \left(\frac{1}{2} m_n \dot{x}_n^2 \right) - a_\mu j_\mu = \sum_n \left(\frac{1}{2} m_n \dot{x}_n^2 - a \dot{z}_n - \bar{a} \dot{\bar{z}}_n \right) = \\ &= \sum_n \frac{1}{2} m_n \dot{x}_n^2 + \frac{i}{2k} \sum_{n \neq m} \left(\frac{\dot{\bar{z}}_n}{\bar{z}_n - \bar{z}_m} - \frac{\dot{z}_n}{z_n - z_m} \right).\end{aligned}\quad (3.67)$$

To see what the statistics of the particles is, we need to see what happens to the wave function after exchange of the particles. Recall that the boson wave function is symmetric under such change, while fermion wave function is anti-symmetric, i.e. picks up a minus sign. So let us have two excitations and let's take one of them and carry it around the other one back to its original position. The change of phase is given by

$$\oint S = \frac{i}{2k} \oint \left(\frac{d\bar{z}_2}{\bar{z}_2 - \bar{z}_1} - \frac{dz_2}{z_2 - z_1} \right) = \frac{i}{2k} [\log(\bar{z}_2 - \bar{z}_1) - \log(z_2 - z_1)]_0^{2\pi} = \frac{2\pi}{k} \quad (3.68)$$

The particle exchange corresponds to half of this phase change, so the resulting phase change is π/k , or

$$\Psi(w_1, w_2) = e^{i \frac{\pi}{2p+1}} \Psi(w_2, w_1). \quad (3.69)$$

This shows that the excitations do not follow any of the two standard statistics. Moreover, the representation of the Heisenberg algebra which describes them is different from the standard

one. So we are going to encounter the first of the two exceptions of the Stone von-Neuman theorem.

The canonical momenta corresponding to the Lagrangian

$$L = \sum_n \left(\frac{1}{2} m_n \dot{x}_n^2 + a_i x_{n,i} \right) \quad (3.70)$$

are

$$p_{n,i} = m \dot{x}_{n,i} + a_i. \quad (3.71)$$

With the standard Heisenberg algebra commutation relations

$$[p_{n,i}, p_{m,j}] = [x_{n,i}, x_{m,j}] = 0 \quad , \quad [x_{n,i}, p_{m,j}] = i \delta_{ij} \delta_{nm}. \quad (3.72)$$

The corresponding Hamiltonian is

$$H = \sum_n p_n \dot{x}_n - L = \sum_n \frac{(p_n - a(x_n))^2}{2m_n}. \quad (3.73)$$

We notice, that this can be expressed as $H = \sum_n P_n^2 / 2m_n$ for the new momenta P_n . The commutation relations for these become

$$[P_{n,i}, P_{m,j}] = i \left[\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right] \delta_{nm} = i \frac{2\pi}{k} \varepsilon_{ij} \delta(x_n - x_m) \quad , \quad (3.74)$$

where we have used the equation of motion for a_i . If we define the Hilbert space for these operators to be the space of functions that vanish if $x_n = x_m$ for any pair of particles, this commutation relation clearly vanishes. So we obtained a different representation of the Heisenberg algebra, namely one in terms of P, x , opposing to the p, x . We however need to make sure these is not just an unitary transformation of each another.

Chapter 4

Solving the BCS theory of superconductivity and representations of the Heisenberg algebra

The BCS theory, named after Bardeen, Cooper and Schrieffer, is a microscopic theory of superconductivity proposed in 1957. At sufficiently low temperatures, electrons near the Fermi surface become unstable against the formation of Cooper pairs. Cooper showed such binding will occur in the presence of an attractive potential, no matter how weak. In conventional superconductors, an attraction is generally attributed to an electron-lattice interaction. The BCS theory, however, requires only that the potential be attractive, regardless of its origin. In the BCS framework, superconductivity is a macroscopic effect which results from "condensation" of Cooper pairs. These have some bosonic properties, while bosons, at sufficiently low temperature, can form a large Bose-Einstein condensate. Superconductivity was simultaneously explained by Nikolay Bogoliubov, by means of the so-called Bogoliubov transformations.

In many superconductors, the attractive interaction between electrons (necessary for pairing) is brought about indirectly by the interaction between the electrons and the vibrating crystal lattice (the phonons). Roughly speaking the picture is the following.

An electron moving through a conductor will attract nearby positive charges in the lattice. This deformation of the lattice causes another electron, with opposite spin, to move into the region of higher positive charge density. The two electrons then become correlated. There are a lot of such electron pairs in a superconductor, so that they overlap very strongly, forming a highly collective condensate. Breaking of one pair results in changing of energies of remained macroscopic number of pairs. If the required energy is higher than the energy provided by kicks from oscillating atoms in the conductor (which is true at low temperatures), then the electrons

will stay paired and resist all kicks, thus not experiencing resistance. Thus, the collective behavior of condensate is a crucial ingredient of superconductivity.

All three of the BCS trio were awarded the Nobel Prize in Physics for this discovery in 1972. It was the second award for Bardeen, who got the prize in 1956 for research in semi-conductor physics.

4.1 Representations of the Heisenberg algebra and gas of bosons

We will first discuss the case of bosons. This is obviously a different subject than the superconductivity of electron gas. However, we will develop a lot of useful apparatus.

States of spinless particles are labeled by a momentum \vec{k} . Since the particles interact, their momentum changes, which means destroying states of some momentum and creating particles of other momentum. This creation and annihilation of particles is conveniently taken into account using the formalism of creation and annihilation operators. For a fixed value of k , we get a one-degree-of-freedom case with one set of operators a, a^\dagger , which was briefly discussed in the section (1.1). Along the lines of development of this section we start with the dimensional algebra a, a^\dagger with $[a, a^\dagger] = 1$ and we postulate existence of the vacuum state $|0\rangle$, such that

$$a |0\rangle = 0$$

By applying a^\dagger , we get a new state, that we call $|1\rangle = a^\dagger |0\rangle$. We compute

$$a |1\rangle = aa^\dagger |0\rangle = (1 + a^\dagger a) |0\rangle = |0\rangle$$

By acting on this state with the operator a^\dagger , we produce a new, unnormalized state. We check the normalization to be¹

$$\langle 0 | aaa^\dagger a^\dagger |0\rangle = \langle 0 | [a, a^\dagger a^\dagger] |0\rangle = \langle 0 | a(a^\dagger + a^\dagger) |0\rangle = 2 \langle 0 |0\rangle = 2 \quad (4.2)$$

and therefore we define the new state

$$|2\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |0\rangle \quad (4.3)$$

¹We have used the formula

$$[A, BC] = B[A, C] + [A, B]C \quad (4.1)$$

which is properly normalized. This way, we come up with the states, that are result of n actions of a^\dagger as a basis of the vector space, where the operators a, a^\dagger act. With proper normalization we have

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle \quad (4.4)$$

From the construction we get $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and we easily check the other property from (4.10)²

$$a |n\rangle = \frac{1}{\sqrt{n!}}[a, a^{\dagger n}] |0\rangle = \frac{n}{\sqrt{n!}}(a^\dagger)^{n-1} |0\rangle = \sqrt{n} |n-1\rangle \quad (4.6)$$

Therefor, by postulating the commutation relations we got the corresponding vector space spanned by vectors (4.4), where the operators act in such way that they create and destroy particles. This can be also done the other way around. We start with a set of n particle states $|n\rangle$ and define operator, that destroys one particle

$$a |n\rangle = c_n |n-1\rangle \quad (4.7)$$

Then from the definition of the Hermitian conjugate we see, that

$$a^\dagger |n-1\rangle = c_n^* |n\rangle \quad (4.8)$$

and the Hermitian conjugate a^\dagger creates a particle. We also compute $a^\dagger a |n\rangle = c_n^* c_n |n\rangle$ and since there is nothing else than n to characterize the state, $c_n^* c_n$ has to be function of n only. Since $a |0\rangle = c_0 |-1\rangle$ for consistency we need $c_0 = 0$. This all suggests that $a^\dagger a$ is a good candidate for the number operator so we chose $c_n = \sqrt{n}$. Computing the commutator of a, a^\dagger we get, that these two obey

$$[a^\dagger, a] = 1 \quad (4.9)$$

We see, that starting from the algebra we get the multi-particle states and from the multi-particle picture we get the same algebra. Therefor, these two descriptions are equivalent.

For the case of a variable momentum \vec{k} , the story is very similar. We will omit the vector sign in the following. For a general value of momentum k , we have an operator a_k , that destroys a particle with momentum k and an operator a_k^\dagger , that creates such particle. This means, that

²We have used

$$[A, B^n] = n[A, B]B^{n-1} \quad (4.5)$$

if $[A, B]$ is a c -number.

if we have a state of n_{k_i} particles with a momentum k_i , these operators act in the following fashion

$$\begin{aligned} a_{k_i} |n_{k_1}, \dots, n_{k_i}, \dots\rangle &= \sqrt{n_{k_i}} |n_{k_1}, \dots, n_{k_i} - 1, \dots\rangle \\ a_{k_i}^\dagger |n_{k_1}, \dots, n_{k_i}, \dots\rangle &= \sqrt{n_{k_i} + 1} |n_{k_1}, \dots, n_{k_i} + 1, \dots\rangle \end{aligned} \quad (4.10)$$

This yields the the usual commutation relations

$$\begin{aligned} [a_k, a_l^\dagger] &= \delta_{kl} \\ [a_k, a_l] = [a_k^\dagger, a_l^\dagger] &= 0 \end{aligned} \quad (4.11)$$

This means, that states with different momentum are created and destroyed independently of each other. We have postulated the properties of operators (4.10) and got algebra with these commutation relations. As we have seen it can work also the other way around. We postulate algebra (4.11) and a vacuum state $|0\rangle$, for which $a_k |0\rangle = 0$ for every k and obtain the creation and annihilation properties of the algebra operators.

Example of a multi-dimensional Heisenberg algebra is a hydrogen atom, where the states $|n, p, m\rangle$ are given by three numbers. We can study the following transition

$$|2, 1, 1\rangle \rightarrow |1, 0, 0\rangle \quad (4.12)$$

Formally, we have three different kinds of particles. The initial state is

$$|2, 1, 1\rangle = A_{2,1,1}^\dagger |0, 0, 0\rangle \quad (4.13)$$

i.e. two particles of the first kind created, one of the second and third kind. The final state is

$$|1, 0, 0\rangle = a_k^\dagger A_{1,0,0}^\dagger |0, 0, 0\rangle \quad (4.14)$$

i.e. one particle of the first kind created and a photon of momentum \vec{k} created. The transition (4.12) is described by the following term in the Hamiltonian

$$H_{int} = c A_{1,0,0}^\dagger A_{2,1,1} a_k^\dagger \quad (4.15)$$

It is then easy to calculate the spontaneous emission coefficient in this formalism, namely

$$\langle 0, 0, 0 | a_k A_{1,0,0} H_{int} A_{2,1,1}^\dagger | 0, 0, 0 \rangle \quad (4.16)$$

Note, that this result is not present in the books on quantum mechanics, since the situation does not preserve the number of the particles, which is not possible to describe in the standard wave-function language.

Statistics in terms of the algebra

In terms of the creation and annihilation operators, the Hamiltonian of a system of free particles is

$$H = \sum_l \omega_l a_l^\dagger a_l \quad (4.17)$$

We check, that the energy of a particle with momentum k is

$$H a_k^\dagger |0\rangle = \sum_l \omega_l a_l^\dagger a_l |0\rangle = \sum_l \omega_l a_l^\dagger \delta_{kl} |0\rangle = \omega_k (a_k^\dagger |0\rangle) \quad (4.18)$$

$$H |n_{k_1}, \dots\rangle = \left(\sum_i N_{k_i} \omega_{k_i} \right) |n_{k_1}, \dots\rangle \quad (4.19)$$

As before, the number operator $N_k = a_k^\dagger a_k$ has the same eigenvectors as the Hamiltonian, with the eigenvalues the number of particles with corresponding momentum.

We can formulate the statistics of the bosonic gas in terms of the Heisenberg algebra. The gas of such particles is described by the partition function³

$$Z = \text{Tr} (e^{-\beta H}) \quad (4.21)$$

We see that

$$\begin{aligned} \text{Tr} (e^{-\beta \omega_i a_i^\dagger a_i}) &= \sum_{n_i} \langle \dots, n_i, \dots | e^{-\beta \omega_i a_i^\dagger a_i} | \dots, n_i, \dots \rangle = 1 + e^{-\beta \omega_i} + e^{-\beta 2 \omega_i} + \dots = \\ &= \frac{1}{1 - e^{-\beta \omega_i}} \end{aligned} \quad (4.22)$$

and therefore the partition function is

$$\begin{aligned} Z &= \text{Tr} (e^{-\beta \sum_i \omega_i a_i^\dagger a_i}) = \text{Tr} (e^{-\beta \omega_{k_1} a_{k_1}^\dagger a_{k_1}} e^{-\beta \omega_{k_2} a_{k_2}^\dagger a_{k_2}} \dots) = \\ &= \sum_{n_{k_1}, n_{k_2}, \dots} \langle n_{k_1}, n_{k_2}, \dots | e^{-\beta \omega_{k_1} a_{k_1}^\dagger a_{k_1}} e^{-\beta \omega_{k_2} a_{k_2}^\dagger a_{k_2}} \dots | n_{k_1}, n_{k_2}, \dots \rangle = \\ &= \prod_k \frac{1}{1 - e^{-\beta \omega_k}} \end{aligned} \quad (4.23)$$

where the summation in the second formula is over all the possible combinations of occupation numbers n_{k_i} . We calculate the mean energy as

$$\langle H \rangle = \frac{\text{Tr} (e^{-\beta H} H)}{Z} = -\frac{\partial}{\partial \beta} \log Z = \sum_k \frac{\partial}{\partial \beta} \log (1 - e^{-\beta \omega_k}) = \sum_k \frac{1}{e^{\beta \omega_k} - 1} \omega_k \quad (4.24)$$

We see, that the occupation number of the particles with momentum k is $\frac{1}{e^{\beta \omega_k} - 1}$, i.e. the Bose-Einstein distribution, as expected for the bosonic particles.

³This is an analogue of the classical formula

$$Z = \int d(\text{phase space}) e^{-\beta H} \quad (4.20)$$

which sums over all possible states.

4.2 Fermionic gas

The difference here is, that the occupation numbers of all the particles can be only 0 or 1. This is basically the Pauli exclusion principle. Therefore, we need the consecutive action of the creation operator to vanish, i.e.

$$c_k^\dagger c_k^\dagger = 0 \quad (4.25)$$

as an operator, since this holds for any state. This can be achieved by the following commutation rules⁴

$$\begin{aligned} c_k c_l^\dagger + c_l^\dagger c_k &= \delta_{kl} \\ c_k c_l + c_l c_k &= c_k^\dagger c_l^\dagger + c_l^\dagger c_k^\dagger = 0 \end{aligned} \quad (4.26)$$

Algebra of operators c_k, c_k^\dagger with these relations describes the fermionic particles as did the algebra of operators a_k, a_k^\dagger describe the bosonic particles.

Alternatively, for a fixed momentum we could have defined operator such that

$$\langle 0|c|0\rangle = 0 \quad , \quad \langle 1|c|0\rangle = 0 \quad , \quad \langle 0|c|1\rangle = 1 \quad , \quad \langle 1|c|1\rangle = 0 \quad (4.27)$$

From this

$$c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow c^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (4.28)$$

And these obviously yield $c^\dagger c + c c^\dagger = 1$ and $c^\dagger c^\dagger = c c = 0$, as desired by 4.26, when generalized to variable momentum. So we see that the fermionic particles are described by an algebra defined by anti-commutators rather than commutators. This is the 'only' difference, all the other computations proceed along the same lines.

The summation in the computation of partition function is much easier now, since the occupation numbers can be only 0, 1. and we get

$$Z = \text{Tr} \left(e^{-\beta \sum_k \omega_k c_k^\dagger c_k} \right) = \prod_k (1 + e^{-\beta \omega_k}) \quad (4.29)$$

By the same procedure as for the case of bosons we get

$$\langle H \rangle = \sum_k \frac{1}{e^{-\beta \omega_k} + 1} \omega_k \quad (4.30)$$

⁴These are not the commutation rules in the usual sense $AB - BA$, but anti-commutation rules. However, these define a product in the operator algebra, therefore the name commutation relation.

which is the Fermi-Dirac distribution for fermions, as expected. This way, we can describe the electron gas in metal, which is governed by the Hamiltonian

$$H = \sum_k \omega_k c_l^\dagger c_k + \sum_{k,l} f(c_{k+l}^\dagger c_k a_l + h.c.) \quad (4.31)$$

where $h.c.$ denotes hermitian conjugation and a_l is the annihilation operator for the photon of momentum l . The summation over possible electron spins and photon polarizations is not explicit in this formula.

4.3 BCS theory

This section deals with electrons in a "metal", which is basically a system, where the electrons can be considered as an ideal fermionic gas. The interaction of electrons and the ions of the lattice is reflected into the change of the mass of the electron to a effective mass m_* . Moreover, the electrons interact with the lattice excitations in a electron-phonon interaction. This gives rise to a attractive effective electron-electron interaction, which in some cases can overcome the electron repulsion and this effect leads to the superconductivity.

These are actually sufficient conditions for the superconductivity. Namely a free electron picture and attractive electron-electron interaction that leads to formation of electron pairs. These pairs then act like bosons and superconductivity is essentially Bose-Einstein condensation of these "particles".

Due to the Pauli exclusion principle electrons occupy all the states up to a certain energy, which is called the Fermi energy E_F , with the corresponding momentum k_F . This energy is rather high, of order $10^4 K$. Due to this at any reasonable energy any physical process is going to occur only for electrons with energy $\approx E_F$, which is called the Fermi surface. For a nice discussion of this and many other aspects of condensed matter physics we refer the reader to *Callaway-Quantum Theory of the Solid State*.

The following Hamiltonian governs our problem. Note, that the first term is just the regular free electron Hamiltonian and the second term introduces the electron-electron interaction.

$$H = \sum_k \varepsilon_k \left(c_{k\downarrow}^\dagger c_{k\downarrow} + c_{k\uparrow}^\dagger c_{k\uparrow} \right) - V_0 \sum_{k,k'} c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow} c_{k\uparrow} \quad (4.32)$$

where c_k, c_k^\dagger are creation and annihilation operators for the electron with momentum k and corresponding component of spin. Here, $\varepsilon_k = E_k - E_F$ is the energy difference from the Fermi surface. Note, that this can be negative. To simplify the notation, we introduce two sets of

operators⁴

$$\begin{aligned} c_{-k\downarrow} &= c_{-k} \\ c_{k\uparrow} &= b_k \end{aligned} \quad (4.33)$$

for the spin up and down operators. The Hamiltonian becomes

$$H = \sum_k \varepsilon (b_k^\dagger b_k + c_{-k}^\dagger c_{-k}) - V_0 \sum_{k,k'} b_{k'}^\dagger c_{-k'}^\dagger c_{-k} b_k \quad (4.34)$$

In this sum, we sum over both positive and negative momenta k . The negative sign in the second terms indicates an attractive force and we see, that the interaction can occur only for electrons with opposite momenta⁵. b 's and c 's form the usual fermion version of Heisenberg algebra described by the commutation relations

$$\begin{aligned} c_{-k} c_{-l}^\dagger + c_{-l}^\dagger c_{-k} &= \delta_{kl} \\ b_k b_l^\dagger + b_l^\dagger b_k &= \delta_{kl} \\ c_{-k} c_{-k} + c_{-l} c_{-k} &= c_{-k}^\dagger c_{-l}^\dagger + c_{-l}^\dagger c_{-k}^\dagger = 0 \\ b_k b_l + b_l b_k &= b_k^\dagger b_l^\dagger + b_l^\dagger b_k^\dagger = 0 \\ b_k c_{-l} + c_{-l} b_k &= b_k c_{-l}^\dagger + c_{-l}^\dagger b_k = b_k^\dagger c_{-l} + c_{-l} b_k^\dagger = b_k^\dagger c_{-l}^\dagger + c_{-l}^\dagger b_k^\dagger = 0 \end{aligned} \quad (4.35)$$

To find the ground state of the Hamiltonian, we need to diagonalize it. In order to do so, we introduce a new set of operators⁶

$$\begin{aligned} A_k &= \alpha_k c_{-k} - \beta_k b_k^\dagger \\ A_k^\dagger &= \alpha_k c_{-k}^\dagger - \beta_k b_k \end{aligned} \quad (4.36)$$

where α, β are real. Similarly, we introduce

$$\begin{aligned} B_{-k} &= \alpha_k b_k + \beta_k c_{-k}^\dagger \\ B_{-k}^\dagger &= \alpha_k b_k^\dagger + \beta_k c_{-k} \end{aligned} \quad (4.37)$$

the choice of the sign of α 's and β 's will become clear in a moment. Before we go further note, that these operators are rather strange, since they combine creation and annihilation operators.

Computing the commutation relation between these we get

$$\begin{aligned} A_k A_l^\dagger + A_l^\dagger A_k &= (\alpha_k^2 + \beta_k^2) \delta_{kl} \\ B_{-k} B_{-l}^\dagger + B_{-l}^\dagger B_{-k} &= (\alpha_k^2 + \beta_k^2) \delta_{kl} \end{aligned} \quad (4.38)$$

⁵The term annihilates two electrons in the $k, -k$ states and creates two electrons at the $k', -k'$ states. Since the total momentum vanishes, the new state can have any momentum k' .

⁶This is called a Bogoljubov transformation. It is a very neat trick used widely in physics, where we redefine our operators in such a way, that we do not spoil the commutation relations, but the Hamiltonian is simpler when expressed in new operators.

so if A 's and B 's are to form a representation of the Heisenberg algebra, we need to have $\alpha_k^2 + \beta_k^2 = 1$. We can also check that all the other commutation relations vanish, thanks to the choice of the sign for parameters α, β . Since A 's and B 's form a proper representation of the Heisenberg algebra, we can express the Hamiltonian in the terms of these operators. We invert the definitions (4.36) and (4.37) to get

$$\begin{aligned} b_k &= \alpha_k B_{-k} - \beta_k A_k^\dagger \\ b_k^\dagger &= \alpha_k B_{-k}^\dagger - \beta_k A_k \\ c_{-k} &= \alpha_k A_k + \beta_k B_{-k}^\dagger \\ c_{-k}^\dagger &= \alpha_k A_k^\dagger + \beta_k B_{-k} \end{aligned} \quad (4.39)$$

We insert these into the Hamiltonian (4.34) and obtain for the free particle part

$$H_0 = \sum_k \varepsilon_k (\alpha_k^2 - \beta_k^2) (\mathcal{N}_k + \mathcal{N}_{-k}) + 2 \sum_k \varepsilon_k \beta_k^2 - 2 \sum_k \alpha_k \beta_k \varepsilon_k (B_{-k}^\dagger A_k^\dagger + A_k B_{-k}) \quad (4.40)$$

where $\mathcal{N}_k = A_k^\dagger A_k$ and $\mathcal{N}_{-k} = B_{-k}^\dagger B_{-k}$. We see, that this part is no longer diagonal. For the interaction part we get

$$\begin{aligned} H_{int} &= V_0 \sum_{k,k'} 2\alpha_{k'} \beta_{k'} \alpha_k \beta_k (\mathcal{N}_k + \mathcal{N}_{-k}) - V_0 \sum_{k,k'} \alpha_{k'} \beta_{k'} \alpha_k \beta_k \\ &+ V_0 \sum_{k,k'} \alpha_{k'} \beta_{k'} (\alpha_k^2 - \beta_k^2) (A_k B_{-k} + B_{-k}^\dagger A_k^\dagger) + \dots \end{aligned} \quad (4.41)$$

where $+\dots$ denotes a number of terms we have neglected, since we are interested only in the terms up to second order in A 's and B 's. The off-diagonal part of the Hamiltonian $H_0 + H_{int}$ is then

$$\sum_{k,k'} [V_0(\alpha_k^2 - \beta_k^2)\alpha_{k'}\beta_{k'} - 2\varepsilon_k\alpha_k\beta_k] (A_k B_{-k} + B_{-k}^\dagger A_k^\dagger) \quad (4.42)$$

We want this expression vanish, which means that every single coefficient must vanish, since for different k , operators A_k, B_{-k} are different. Thus we get the following constraint

$$2\varepsilon\alpha_k\beta_k - V_0(\alpha_k^2 - \beta_k^2) \sum_{k'} \alpha_{k'}\beta_{k'} = 0 \quad (4.43)$$

together with $\alpha_k^2 + \beta_k^2 = 1$. The latter one can be solved by introduction of a new parameter θ_k such that

$$\begin{aligned} \alpha_k &= \sin \theta_k \\ \beta_k &= \cos \theta_k \end{aligned} \quad (4.44)$$

Inserting this into the former constraint and introducing yet another parameter $\Delta = \frac{1}{2}V_0 \sum_k \sin 2\theta_k$ we get

$$\begin{aligned} -\varepsilon_k \sin 2\theta_k - \Delta \cos 2\theta_k &= 0 \\ \tan 2\theta_k &= -\frac{\Delta}{\varepsilon_k} \end{aligned} \quad (4.45)$$

This is solved by

$$\begin{aligned} \sin 2\theta_k &= \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}} \\ \cos 2\theta_k &= \frac{-\varepsilon_k}{\sqrt{\varepsilon_k^2 + \Delta^2}} \end{aligned} \quad (4.46)$$

Finally, inserting this into the definition of Δ we obtain

$$\Delta = \frac{1}{2}V_0 \sum_k \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}} \quad (4.47)$$

The parameter Δ is called the gap and this equation the gap equation. This name is going to become clear in a while, when we will realize that Δ is the lowest energy excitation energy for the electrons in the superconducting phase. We solve this equation by changing the sum to integral over quasi-continuous spectrum

$$\Delta = \frac{1}{2}V_0 \int_{-\omega_D}^{\omega_D} d\varepsilon G(\varepsilon) \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}} \quad (4.48)$$

where we have already changed the integration to be over the energies using $\int_{\text{angles}} d^3k/(2\pi)^3 = G(\varepsilon)d\varepsilon$, with $G(\varepsilon)$ being the degeneracy of the corresponding energy level. The zero energy level was taken at the Fermi level. ω_D is the Debye frequency, which is the highest possible frequency of lattice excitations. We assume that G does not change over the small interval of energies around the Fermi level and evaluating the integral we get

$$\Delta \left[1 - V_0 G_0 \sinh^{-1} \left(\frac{\omega_D}{\Delta} \right) \right] = 0 \quad (4.49)$$

with two solutions

$$\Delta = 0 \quad , \quad \Delta = \frac{\omega_D}{\sinh \frac{1}{V_0 G_0}} \quad (4.50)$$

with $G_0 \equiv G(0)$ is the degeneracy of the Fermi energy. In the case of $V_0 G_0 \ll 1$, which is true for most of the materials, we get the final formula for the gap

$$\Delta = 2\omega_D e^{-\frac{1}{V_0 G_0}} \quad (4.51)$$

We see, that this expression has an essential singularity at point $V_0 = 0$, where it does not allow Taylor expansion and thus this is result, that can not be obtained by a perturbation theory.⁷

The value of Debye frequency ω_D depends on the mass of the atoms in the lattice of the material. So if we measure the gap for materials that consist of different isotopes of the same element, we can test our theory.⁸ Obviously, this has been done and nicely fits the theory. The gap is going give the transition temperature to the superconducting state, which explains why these are so small.

To see, which value of parameter Δ is preferred, we need to compute the ground state energy for each case. This can be done without the explicit construction of the ground state, which is defined by

$$A_k |G\rangle = B_{-k} |G\rangle = 0 \quad (4.52)$$

A general state is constructed by

$$A_{k_1}^\dagger \dots A_{k_n}^\dagger B_{-l_1}^\dagger \dots B_{-l_m}^\dagger |G\rangle = |\Psi_{m,n}\rangle \quad (4.53)$$

with particle numbers

$$\mathcal{N}_k |\Psi\rangle = \sum_{i=1}^n \delta_{k,k_i} |\Psi\rangle \quad , \quad \mathcal{N}_{-l} |\Psi\rangle = \sum_{i=1}^m \delta_{l,l_i} |\Psi\rangle \quad (4.54)$$

Since $\mathcal{N} |G\rangle = 0$, we get⁹

$$H |G\rangle = \left(2 \sum_k \varepsilon_k \beta_k^2 - V_0 \sum_k \alpha_k \beta_k \alpha'_k \beta'_k \right) |G\rangle = \sum_k \left(\varepsilon_k \cos 2\theta_k - \frac{1}{2} \Delta \sin 2\theta_k \right) |G\rangle \quad (4.55)$$

and the ground state energy as a function of Δ is

$$E_\Delta = \sum_k \left(\varepsilon_k \cos 2\theta_k - \frac{1}{2} \Delta \sin 2\theta_k \right) \quad (4.56)$$

now using the same trick as before with the change of summation to integration we find out that the energy is

$$E_\Delta = - \frac{G_0}{2} \varepsilon \sqrt{\varepsilon^2 + \Delta^2} \Big|_{-\omega_D}^{\omega_D} \quad (4.57)$$

⁷Consider function that is zero for $x \leq 0$ and $e^{-1/x}$ for $x > 0$. Such function is continuous for all x . Moreover, the derivative of any order exists at $x = 0$, but vanishes. So the function admits a Taylor series at $x = 0$, but this series is zero for any $x > 0$. This explains the singularity and also illustrates, that infinite differentiability does not yield analyticity, i.e. $C^\infty \neq C^\omega$.

⁸Different isotopes have different mass but the same electronic structure.

⁹We drop a term, that vanishes due to $\sum \varepsilon_k = 0$.

Here we can assume that $\Delta \ll 1$, since $V_0 \ll 1$ and we get

$$E_\Delta = -\frac{G_0}{2}\Delta^2 \quad (4.58)$$

Therefore a non-zero Δ is energetically preferred. The state $|G\rangle$ is superconducting state. Here, the Hamiltonian becomes

$$H = \sum_k \sqrt{\varepsilon_k^2 + \Delta^2} (\mathcal{N}_k + \mathcal{N}_{-k}) \quad (4.59)$$

for the energies around the Fermi level, i.e. $\varepsilon_k \approx 0$, changing the number particle costs a non-zero energy due to the $\Delta^2 > 0$. For small thermal fluctuations, electrons can not jump to higher energy levels, since there is not enough energy to overcome the gap created by Δ , which leads to no dissipation of the flow of electrons and superconductivity is present. On the contrary, with $\Delta = 0$, there are energy levels arbitrarily close to the Fermi level, which leads to a dissipation.

Explicit construction of the ground state

The ground state is given by (4.52). Since any consecutive action of A_k and B_{-k} vanishes, a good guess for the ground state is

$$|G\rangle = \prod_k f_k B_{-k} A_k |0\rangle \quad (4.60)$$

where $|0\rangle$ is state of no electrons $b_k |0\rangle = c_k |0\rangle = 0$. This is

$$|G\rangle = \prod_k f_k \left(\alpha_k b_k + \beta_k c_{-k}^\dagger \right) \left(\alpha_k c_{-k} - \beta_k b_k^\dagger \right) |0\rangle = \prod_k (-f_k) \left(\alpha_k \beta_k + \beta_k^2 c_{-k}^\dagger b_k^\dagger \right) |0\rangle \quad (4.61)$$

We see, that such state contains a huge number of electron pairs of opposite momenta and spins (Cooper pairs), that are created by action of $c_{-k}^\dagger b_k^\dagger$. This means that $|G\rangle$ is not an eigenstate of the number operator. To get f 's, we compute the normalization. First, we write the ground state as $\prod_k Q_k |0\rangle$, which yields

$$\langle G|G\rangle = \langle 0| \dots Q_2^\dagger Q_1^\dagger Q_1 Q_2 \dots |0\rangle \quad (4.62)$$

Here, we can move the operators Q_i^\dagger and Q_i together, since we always jump over an even number of other operators, giving an even power of -1 as result. Thus

$$\begin{aligned} \langle G|G\rangle &= \langle 0| \prod_k Q_k^\dagger Q_k |0\rangle = \prod_k f_k^2 \langle 0| \alpha_k \beta_k \alpha_k \beta_k + \beta_k^4 b_k c_{-k} c_{-k}^\dagger b_k^\dagger = |0\rangle \\ &= \prod_k f_k^2 (\alpha_k^2 \beta_k^2 + \beta_k^4) = 1 \end{aligned} \quad (4.63)$$

which yields $f_k = \beta_k^{-1}$ and

$$|G\rangle = \prod_k (\alpha_k + \beta_k c_{-k}^\dagger b_k^\dagger) |0\rangle \quad (4.64)$$

To see what the overlap of the vacuum state $|0\rangle$ and the new ground state $|G\rangle$ is, we compute $\langle 0|G\rangle$

$$\langle 0|G\rangle = \langle 0| \prod_k [\alpha_k + \beta_k c_{-k}^\dagger b_k^\dagger] |0\rangle = \prod_k \langle 0| \alpha_k + \beta_k c_{-k}^\dagger b_k^\dagger |0\rangle = \prod_k \alpha_k \quad (4.65)$$

Since $\alpha_k < 1$, for an infinite number of possible momentum values k , we get this overlap to vanish and states $|0\rangle$ and $|G\rangle$ are orthogonal. Moreover, for any finite-particle state

$$|\psi\rangle = b^\dagger \dots c^\dagger \dots |0\rangle \quad (4.66)$$

the overlap with $|G\rangle$ is just a *number* $\times \langle 0|G\rangle = 0$ and thus any bounded operator \mathcal{O} has a zero matrix element between $|0\rangle$ and $|G\rangle$, $\langle 0|\mathcal{O}|G\rangle = 0$. Therefore there is no unitary operator that connects $|0\rangle$ and $|G\rangle$, since such operator would have all the elements vanishing

$$|G\rangle = U|0\rangle \Rightarrow \langle \psi|G\rangle = 0 = \langle \psi|U|0\rangle = U_{\psi,0} \quad (4.67)$$

From the group theory point of view, b, c 's and A, B as defined by (4.35) and (4.36)(4.37) form two representations of the fermionic Heisenberg algebra, acting in two different vector spaces, spanned by $(b_k^\dagger)^n |0\rangle, (c_{-k}^\dagger)^n |0\rangle$ and $(A_k^\dagger)^n |G\rangle, (B_{-k}^\dagger)^n |G\rangle$ respectively. These, in principle, could be equivalent. However, since for infinite dimensional algebra the states $|0\rangle$ are $|G\rangle$ orthogonal and there is no unitary operator to map one on the other, these representations are not equivalent and represent different physical phenomena. This is one possible scenario, under which Stone-Von Neuman theorem (1.2.1) fails.

4.4 Recapitulation

We started with a Heisenberg algebra of operators

$$\begin{aligned} [x_i, p_j] &= i\delta_{ij} \\ [x_i, x_j] = [p_i, p_j] &= 0 \end{aligned} \quad (4.68)$$

from this, we constructed creation and annihilation operators as $x \pm ip$, which obey the standard commutation rules, according to the nature of particles we wish to describe. If the number of the operators is finite, we can construct only one irreducible representation, giving only one possible physics. However for infinite number number of operators, we can get different, non-equivalent representations, yielding different behavior of the particles. More precisely, different

phase of the system corresponds to a different irreducible representation of the algebra and the commutation rules. Different representation of the commutation rules gives then a different representation of algebra of observables, which are constructed from these operators.

Another possible case, where we can get non-equivalent representations is representing an algebra on a non-connected vector space, which is the case of fractional Hall effect.

At the very end, just as a comment, let us mention that the superconductivity is an effect of the symmetry breaking. The symmetry that is being broken is the $U(1)$ symmetry of the electromagnetic interactions. The original vacuum $|0\rangle$ has this symmetry, however the new vacuum $|G\rangle$ does not. The field responsible for the breaking is the Cooper pair field $\phi = c_k^\dagger b_k^\dagger + h.c.$, which had vanishing expectation value in the original vacuum but picks up an expectation in the new one, since $\langle G | c_k^\dagger b_k^\dagger | G \rangle \neq 0$.

Chapter 5

Quarks and the spectroscopy of hadrons

In this section, we will discuss application of the group theory methods to derive mass spectrum of the bound states of quarks, the hadrons. We will do this without any knowledge of the strong interactions, which are responsible for this binding and we will use just symmetry arguments. Historically, this analysis was done long before the current theory of strong interactions, the QCD, was established and nothing was known about quarks at that time. Illustrating the power of symmetry considerations, this is one of the most brilliant thought and exciting stories in the development of the particle physics theory.

This section heavily relies on the representation theory developed in the appendix A, especially A.6. Reader should make sure that this material is familiar before proceeding any further.

5.1 Products of irreducible representations

In the section (A.6), we have classified the irreducible representations of the $SU(3)$ group. We have also stated, that all the product representations of these representations can be reduced to a sum of these irreducible ones. Here, we are going to give some explicit examples how this is done.

First, let's take the simplest case, the product $\mathbf{3} \otimes \mathbf{3}^*$. This is the representation in the space of tensors $\phi_i \chi^j$, which transforms as $\phi_i \chi^j \rightarrow g_{ik} g^{jl} \phi_k \chi^l$. We know, that the reduction is done by the invariant tensors of the group, which are $\delta_j^i, \varepsilon^{ijk} \equiv \varepsilon_{ijk}$ in the case of $SU(3)$. First

, we contract by δ_j^i to obtain

$$\delta_j^i \phi_i \chi^j = \phi_i \chi^i \rightarrow g_{ik} (g^*)^{il} \phi_k \chi^l = (g^\dagger g)_k^l \phi_k \chi^l = \phi_k \chi^k \quad (5.1)$$

This quantity thus does not change under the action of the group and transforms according to the trivial representation of $SU(3)$, which we denote as $\mathbf{1}$. Denoting $\phi_i \chi^i = (\phi \cdot \chi)$, we can write $\phi_i \chi^j$ as

$$\phi_i \chi^j = \frac{(\phi \cdot \chi)}{3} \delta_i^j + \Phi_i^j \quad (5.2)$$

where Φ is traceless $\Phi_i^i = 0$. It has 8 independent components and thus transforms according to the eight dimensional representation $\mathbf{8}$. In this case, there are not enough indexes to contract with the ε tensor, so we are done. This way, we got $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}$.

The next case is the product $\mathbf{3} \otimes \mathbf{3}$, i.e. tensors $\phi_i \psi_j$. Symmetrizing and anti-symmetrizing we get

$$\phi_i \psi_j = \frac{\phi_i \psi_j + \phi_j \psi_i}{2} + \frac{\phi_i \psi_j - \phi_j \psi_i}{2} \quad (5.3)$$

we see, that the first term is a symmetric tensor of rank $(2, 0)$, i.e. the six dimensional irreducible representation we have denoted $\mathbf{6}$. In the second term, we can contract the two antisymmetric indexes with the ε^{kij} tensor to get a object with one upper index, i.e. $\mathbf{3}^*$ representation. This way, we got $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}^*$.

The same way we get $\mathbf{3}^* \otimes \mathbf{3}^* = \mathbf{6}^* \oplus \mathbf{3}$. We see, that this is just a conjugation of the previous case, as expected.

Finally, let us do a more complicated case $\mathbf{8} \otimes \mathbf{8}$, the tensors of the form $T_j^i S_l^k$. First we try to contract with the δ tensor. The contraction by $\delta_j^i \delta_l^k$ gives zero. The contraction by $\delta_i^i \delta_j^k$ gives $T_k^i S_i^k$, which can be checked to transform trivially and thus is a trivial representation denoted as $\mathbf{1}$. So far we have $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \dots$

The completely symmetric part of the tensor, symmetric in all the indexes, is $(2, 2)$ representation, which is 27 dimensional.

Anti-symmetrization in indexes i, k gives a possibility to contract these two with ε_{mik} , yielding three lower indexes j, l, m . Completely symmetric part of this tensor gives $(3, 0)$ irreducible representation $\mathbf{10}$. Anti-symmetrization in two of these, e.g. j, l gives again a possibility of contraction, giving an upper index. We have ended with a quantity with one lower and one upper index, i.e. the $\mathbf{8}$ representation. The same procedure for lower indexes gives representations $\mathbf{10}^*$ and a copy of $\mathbf{8}$ again. Thus, we have $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8} \oplus \mathbf{8}$. The following diagram describes the story. $[a, b]$ means complete anti-symmetrization in the corresponding

indexes, (a, b) means complete symmetrization in the corresponding indexes.

$$\begin{aligned}
8 \otimes 8 = T_j^i S_l^k &\xrightarrow{\delta} T_k^i S_i^k \rightarrow 1 \\
&\rightarrow T_{(j}^{(i} S_{l)}^k \rightarrow 27 \\
&\rightarrow T_j^{[i} S_l^{k]} \xrightarrow{\varepsilon} U_{jlm} \rightarrow U_{(jlm)} \rightarrow 10 \\
&\qquad\qquad\qquad \rightarrow U_{[jl]m} \xrightarrow{\varepsilon} V_m^n \rightarrow 8 \\
&\rightarrow T_{[j}^i S_{l]}^k \xrightarrow{\varepsilon} U^{jlp} \rightarrow U^{(jlp)} \rightarrow 10^* \\
&\qquad\qquad\qquad \rightarrow U^{[jl]q} \xrightarrow{\varepsilon} V_r^q \rightarrow 8
\end{aligned} \tag{5.4}$$

We see, that $(8 \otimes 8)^* = 8 \otimes 8$, thus it is a real representation. Therefor if it contains the irreducible representation 10, it must also contain the irreducible representation 10^* .

5.2 Mesons and Baryons

Mesons and baryons are particles, that form the known matter around us. These are formed by quarks, that come in six different types (flavors), namely the up, down, strange, charm, top and bottom quarks. Here, we will deal only with the first three of these flavors, as the other three are heavy enough to neglect from low energy considerations. The quarks of the same flavor also come in different kinds, what we call different colors. We therefore have u_a, d_a, s_a with $a = 1, 2, 3$. We should keep in the back of our minds that u, d, s are spinors and therefor are four component quantities and carry an implicit spinor index.

The quark-quark interaction is governed by the Dirac Lagrangian for fermionic fields, with a gauged $SU(3)$ color symmetry $q \rightarrow Uq$ with $q = u, d, s$, that mixes the different colors of the same quark. The Lagrangian is

$$L_{Dirac} = i\bar{q}^b \gamma^\mu (\partial_\mu \delta_b^a + (A_\mu)_b^a) q_a. \tag{5.5}$$

Therefor the important part of the Hamiltonian of the theory is

$$H = \dots + \bar{u}^a A_a^b u_b + \bar{d}^a A_a^b d_b + \bar{s}^a A_a^b s_b + m_u \bar{u}^a u_a + m_d \bar{d}^a d_a + m_s \bar{s}^a s_a + \dots \tag{5.6}$$

where the matrices describing the interactions are $A_a^b = \gamma^\mu \mathcal{A}_{\mu a}^b$, and $\mathcal{A}_{\mu a}^b$ are gluons and γ^μ are the Gell-Mann matrices. Note, that in the original Dirac Lagrangian there is no mass term for the quarks. Indeed the quarks enter the theory massless and the mass term in the Hamiltonian is due to the weak symmetry breaking.

This Hamiltonian has the $SU(3)$ -color symmetry of the Lagrangian. This theory is the local gauge theory which yields the strong interaction mediated by gluons and is responsible

for quark dynamics. As mentioned before, we will not consider any details of this part of the Lagrangian here.

Moreover, we write $Q^a = (u_a, d_a, s_a)^T$ and now Q_α^a with $\alpha = 1, 2, 3$ represents the up, down and strange quark respectively. Now, we can write the Hamiltonian as

$$H = \dots + \bar{Q}_\alpha^a A_a^b Q_{b\alpha} + m_u \bar{u}^a u_a + m_d \bar{d}^a d_a + m_s \bar{s}^a s_a + \dots \quad (5.7)$$

The first term has an explicit $SU(3)$ symmetry distinct from the original color symmetry. If we take $Q_\alpha^a \rightarrow g_{\alpha\beta} Q_\beta^a$, which yields $\bar{Q}_\alpha^a \rightarrow g_{\beta\alpha}^* \bar{Q}_\beta^a$, we get

$$\bar{Q}_\alpha^a A_a^b Q_{b\alpha} \rightarrow \underbrace{g_{\lambda\alpha}^* g_{\alpha\mu}}_{\delta_{\lambda\mu}} \bar{Q}_\lambda^a A_a^b Q_{b\mu} = \bar{Q}_\alpha^a A_a^b Q_{b\alpha} \quad (5.8)$$

This symmetry mixes different kinds of quarks. These are flavors of the quarks and therefore this symmetry is called the $SU(3)$ -flavor symmetry. However, since the mass terms are not flavor symmetric, this is not a complete symmetry. The mass term of the Hamiltonian can be written as $\bar{Q}_\alpha M_{\alpha\beta} Q_\beta$, with the mass matrix

$$M_{\alpha\beta} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad (5.9)$$

The masses of the quarks are $m_u = 3 \text{ MeV}$, $m_d = 4 \text{ MeV}$, $m_s = 150 \text{ MeV}$. We are comparing them with the masses of hadrons, which are $\sim 1000 \text{ MeV}$, so as the first approximation, we can say that $m_u = m_d = m_s = M_Q \ll m_{hadron}$, and see what happens.¹ Now we have a complete $SU(3)$ -flavor symmetric Hamiltonian

$$H' = \dots + \bar{Q}_\alpha^a A_a^b Q_{b\alpha} + M \bar{Q}_\alpha Q_\alpha + \dots \quad (5.10)$$

Therefore, as discussed in the first chapter, we should get degenerate states of the energy, which in this case is the mass. Moreover, these degenerate states should form an irreducible representations of the symmetry group, which is the flavor- $SU(3)$. Also

Mesons

Physical states are singlets of the $SU(3)$ -color symmetry. This means that the states have no $SU(3)$ -color indexes, which are denoted a, b in our case. One way to for such a singlet is

¹This looks like and obvious oversimplification. 3 is clearly no way close to 150. Or is it? As mentioned, we are comparing it with masses of hadrons, which are roughly at least ten times larger. At this scale, 3 and 150 become a little more similar than before. Also, it even though not really exact, this approximation can give us some very useful insight into what to expect from the theory. And then we can try to make better approximations or even try to solve the theory. This is what we mean by 'see what happens'.

$$\bar{Q}^{a\alpha} Q_{b\beta} \delta_b^a = \Psi_\beta^\alpha. \quad (5.11)$$

So these particles, called mesons, are constituting of a quark and an anti-quark and transform as $\mathbf{3} \otimes \mathbf{3}^*$ representation. As we have seen, this representation splits into the $\mathbf{1}$ and the $\mathbf{8}$ irreducible representation. We therefor expect the mesons to come in one singlet and one octet, i.e. groups of one and eight, that have a degenerate energy, i.e. the same mass. Remember that we neglected a difference of mass 150 MeV , so anything within this difference is acceptable in this approximation.

Looking in to the particle tables, we find these two groups of spin-0 particles

η'	958 MeV	(5.12)
π^0	135 MeV	
π^\pm	139 MeV	
η	547 MeV	
K^\pm	494 MeV	
K^0, \bar{K}^0	498 MeV	

and two groups of spin-1 particles

ϕ	1020 MeV	(5.13)
ρ^0, ρ^\pm	770 MeV	
$K^{*\pm}$	892 MeV	
K^{*0}, \bar{K}^{*0}	890 MeV	
ω	782 MeV	

As we see the groups do not have perfectly equal mass, since the symmetry is not perfect. We will need to make corrections to the assumption of equal masses of the quarks. Fortunately (or as expected), as we will see this can be done and we can correct for these differences. Moreover this procedure will yield some non-trivial, testable predictions.

Baryons

Another way to construct a $SU(3)$ -color singlet is

$$Q_{a\alpha} Q_{b\beta} Q_{c\gamma} \varepsilon^{abc} = \Psi_{\alpha\beta\gamma}. \quad (5.14)$$

These particles consist of three quarks and are baryons. Writing down the $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ representation as $\mathbf{3} \otimes (\mathbf{6} \oplus \mathbf{3}^*) = (\mathbf{3} \otimes \mathbf{6}) \oplus (\mathbf{3} \otimes \mathbf{3}^*)$, recalling that $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}$ and reducing

² $\mathbf{3} \otimes \mathbf{6}$ into $\mathbf{8} \oplus \mathbf{10}$ we obtain

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}. \quad (5.15)$$

Again checking the particle tables we find a group of eight baryons of spin- $\frac{1}{2}$

p	938 MeV	(5.16)
n	939 MeV	
Σ^\pm, Σ^0	1189, 1192 MeV	
Ξ^0	1315 MeV	
Ξ^-	1321 MeV	
Λ	1116 MeV	

and a group of ten baryons of spin- $\frac{3}{2}$

$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$	~1232 MeV	(5.17)
$\Sigma^{*\pm}, \Sigma^{*0}$	~1385 MeV	
Ξ^{*0}, Ξ^{*-}	~1530 MeV	
Ω^-	~1672 MeV	

The singlet and the other octuplet are not compatible with Pauli exclusion principle and thus are not realized as physical states. Considering the $\mathbf{3}^* \otimes \mathbf{3}^* \otimes \mathbf{3}^*$ reduction we would find the anti-particles for these baryons.

We see that the tables indicate the correct groups of particles, however we need to correct for the different masses of the quarks.

5.3 Mass Correction

Baryon octet

We will show the procedure in detail for the octet of baryons.

Since the masses of the up and down quarks are very similar, compared to the mass of the strange quark, the second approximation to the masses of the quarks is $m_u = m_d \neq m_s$, and we write the mass term as

$$\bar{\mathbf{Q}} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \mathbf{Q} = \bar{\mathbf{Q}} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda' \end{pmatrix} \mathbf{Q} \quad (5.18)$$

²We get this by the same procedure as before. In $\phi_i T_{jk}$ we do a complete symmetrization to obtain $(3, 0)$, thus the $\mathbf{10}$ representation. The anti-symmetric part in ij gives a new upper index and thus a $(1, 1)$ representation, which is the $\mathbf{8}$ representation. Remember that the indexes jk are already symmetric, since T is an irreducible tensor.

The Hamiltonian therefor still has a smaller symmetry group $SU(2)$, that mixes the upper two entries of Q . So we expect the states to fall into irreducible representations of this group. It is the $SU(2)$ -isospin symmetry. So the original $\mathbf{3}$ representation of $SU(3)$ is now a $\mathbf{2} \oplus \mathbf{1}$ representation of $SU(2)$, where the top two entries transform as a doublet and the last one as a singles. This leads to

$$\mathbf{3}^* \otimes \mathbf{3} = (\mathbf{2} \otimes \mathbf{2}) \oplus (\mathbf{2} \otimes \mathbf{1}) \oplus (\mathbf{2} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}) = \underbrace{\mathbf{3} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2}}_{\text{octet}} \oplus \underbrace{\mathbf{1}}_{\text{singlet}} \quad (5.19)$$

where we have used the result $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$ for $SU(2)$. It should be clear, which representations are understood to be of the $SU(3)$ and which of $SU(2)$.

Looking at the masses of the mesons, we easily find corresponding groups of particles, that fall into these groups.

Also the octuplet of the baryons does have a triplet of isospin one particles, two douplets of isospin half particles and one isospin singlet. These are triplet Σ^0, Σ^\pm , two douplets p, n and Ξ^-, Ξ^0 and Λ forms a singlet. It seems that we are doing something right, but can we get also the numbers?

For this, we need to write the Hamiltonian in terms of the composite baryons. To do this, we need to take couple of steps back.

The baryon octet is and adjoint representation ϕ_j^i of $SU(3)$, i.e.

$$\phi \rightarrow g\phi g^\dagger \quad (5.20)$$

So the eight baryons can be rearranged into a 3×3 matrix. The ϕ is

$$\phi = \phi^a \frac{\lambda^a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\phi^3 + \frac{1}{\sqrt{6}}\phi^8 & \frac{1}{\sqrt{2}}\phi^1 - \frac{i}{\sqrt{2}}\phi^2 & \frac{1}{\sqrt{2}}\phi^4 - \frac{i}{\sqrt{2}}\phi^5 \\ \frac{1}{\sqrt{2}}\phi^1 + \frac{i}{\sqrt{2}}\phi^2 & -\frac{1}{\sqrt{2}}\phi^3 + \frac{1}{\sqrt{6}}\phi^8 & \frac{1}{\sqrt{2}}\phi^6 - \frac{i}{\sqrt{2}}\phi^7 \\ \frac{1}{\sqrt{2}}\phi^4 + \frac{i}{\sqrt{2}}\phi^5 & \frac{1}{\sqrt{2}}\phi^6 + \frac{i}{\sqrt{2}}\phi^7 & -\frac{2}{\sqrt{6}}\phi^8 \end{pmatrix} \quad (5.21)$$

where we just explicitly wrote out the Gell-Man matrices (1.43). So we need to correctly identify particles (5.16) with the components of ϕ . This takes some phenomenology and trial and error, but eventually leads to

$$\phi = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\Lambda \end{pmatrix} \quad (5.22)$$

We see, that $Tr(\bar{\phi}\phi) = Tr(g\bar{\phi}g^{-1}g\phi g^{-1}) = Tr(\bar{\phi}\phi)$ is invariant due to the cyclicity of the trace. Therefore a reasonable ansatz for the symmetric mass term is $\lambda Tr(\bar{\phi})$. However the symmetry

is broken and we need to have a symmetry breaking mass term. Also, this term should have a residual $SU(2)$ -isospin symmetry as discussed above.

We introduce a matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.23)$$

and the mass term (5.18) becomes $\lambda\bar{Q}Q + (\lambda' - \lambda)\bar{Q}AQ$. Term $Tr(\bar{B}AB)$ does not have the $SU(3)$ symmetry and introduces the symmetry breaking, as well as the term $Tr(\bar{B}BA)$ does. So our final ansatz for the mass term of the Hamiltonian is

$$aTr(\bar{B}B) + bTr(\bar{B}AB) + cTr(\bar{B}BA) \quad (5.24)$$

Computing

$$\begin{aligned} Tr(\bar{B}B) &= \bar{\Sigma}^0\Sigma^0 + \bar{\Sigma}^+\Sigma^+ + \bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \bar{p}p + \bar{n}n + \bar{\Sigma}^0\Sigma^0 + \bar{\Lambda}\Lambda + \bar{\Sigma}^-\Sigma^- \\ Tr(\bar{B}AB) &= \bar{B}_{\alpha 3}B_{3\alpha} = \bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \frac{2}{3}\bar{\Lambda}\Lambda \\ Tr(\bar{B}BA) &= \bar{B}_{3\alpha}B_{\alpha 3} = \bar{p}p + \bar{n}n + \frac{2}{3}\bar{\Lambda}\Lambda \end{aligned} \quad (5.25)$$

we get $m_p = m_n = a + c$, $m_\Sigma = a$, $m_\Xi = a + b$, $m_\Lambda = a + \frac{2}{3}(b + c)$. We have 3 parameters and 4 experimental values of masses, therefore we get one constraint as a test of our theory. After some algebra we get

$$2(m_p + m_\Xi) = m_\Sigma + 3m_\Lambda \quad (5.26)$$

Using the values from the table (5.16) we find out, that this expression holds up to a very small 0.7% difference.

Meson octet

The same procedure would give the same condition also for the octet of the scalar mesons, however now with the masses squared, since the corresponding scalar mass term is $m^2\phi^2$ opposing to $m\bar{\phi}\phi$. As one can check, condition

$$2(m_{K^\pm}^2 + m_{K^0}^2) = m_\pi^2 + 3m_\eta^2 \quad (5.27)$$

is again satisfied to a great precision.

Baryon decuplet

A similar approach will lead also to the corrections of the masses of the decuplet baryons (5.17). As before, we write the new mass ansatz as

$$\phi^{ijk} M_i^l M_j^M M_k^n \phi_{lmn} \quad (5.28)$$

where

$$M_j^i = a (\delta_j^i + b \delta^{i3} \delta_{j3}) \quad (5.29)$$

Therefor the mass of a particle is $a^3(1+b\#3)$, where $\#3$ is number of 3-indexes in the irreducible representation corresponding to the particle. This means, that particles should form groups with equally separated mass, according to the number of 3-indexes.

Looking at the decuplet, we expect

$$10 = 4 \oplus 3 \oplus 2 \oplus 1 \quad (5.30)$$

Where the 4 corresponds to the representation of $SU(2)$ where no index ϕ_{ijk} is 3, 3 corresponds to one index being 3, i.e. ϕ_{ij3} and similarly for 2 and 1. Moreover the first particles should have isospin $\frac{3}{2}$, next ones isospin 1, etc. This means that the first group are the Δ particle, second the Σ particles, third Ξ particles and finally Ω particle.

Looking at the table (5.17), we see, that the differences are 153 MeV , 145 MeV , 142 MeV , which is again very accurate. Note, that the Ω particle was predicted this way, as a missing decuplet particle, with a mass one step larger than the Ξ particles. The later discovery of this particle was a huge argument in favor of the quark model.

Particle interactions

As it turns out, group theory can tell us something also about the particle interactions. If we denote the matrix describing the baryon octet B and the scalar meson octet M , the term

$$\lambda_1 \text{Tr}(\bar{B} B M) + \lambda_2 \text{Tr}(B \bar{B} M) \quad (5.31)$$

describes interaction of two baryons and a meson. For example if we extract the part that governs the interaction of p, n with η, π we find the corresponding terms in the Hamiltonian

$$\lambda_2 \left[(\bar{p}p - \bar{n}n) \frac{\pi^0}{2} + \bar{n}p\pi^+ + \bar{p}n\pi^- \right] \quad (5.32)$$

and

$$\left[-\frac{2}{\sqrt{6}}\lambda_1 + \lambda_2 \frac{1}{\sqrt{6}} \right] (\bar{p}p + \bar{n}n)\eta \quad (5.33)$$

From these two terms, we could get the branching ratios for different reactions or production rates based purely on the group theory.

Chapter 6

Conformal symmetries, isometries, phase transitions, etc.

The physical motivation of studying conformal symmetries is vast. In two dimensional statistical mechanics, all the phase transitions have conformal symmetry at the transition point and also other way around, all the possible phase transitions in the 2D models can be characterized by the conformal symmetries. In string theories one dimensional string sweeps a two dimensional world-sheet in the space-time. Requirement of conformal symmetry on this two dimensional surface gives rise to a variety of physical phenomena, which seem to describe all the known physics. We will shortly see, why two dimensional problems play such an important role in the conformal symmetric theories.

6.1 Description of conformal transformations

Let us begin with an n -dimensional space, in which we have a metric¹

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + \dots \quad \mu, \nu = 1, 2, \dots, n \quad (6.1)$$

which gives distance between two points and is a generalization of the usual distance in \mathbb{R}^n given by Pythagoras' theorem. We now impose an infinitesimal coordinate transformation, given by a vector function ξ

$$x^\mu \rightarrow x^\mu + \xi^\mu(x) \quad (6.2)$$

¹Couple of examples of metrics which might be familiar to the reader are

$$dx_1^2 + dx_2^2 + dx_3^2, \quad dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad dt^2 - (dx^2 + dy^2 + dz^2), \quad \left(1 - \frac{2GM}{r}\right) - \frac{1}{1 - \frac{2GM}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$$

Symmetries of the space, or isometries (meaning 'the same metric'), are then transformations, that leave the metric unchanged $ds^2(x + \xi) = ds^2(x)$. We compute

$$\begin{aligned} ds^2(x + \xi) &= g_{\mu\nu}(x + \xi)d(x + \xi)^\mu d(x + \xi)^\nu = \\ &= g_{\mu\nu}dx^\mu dx^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\mu dx^\nu \xi^\alpha + g_{\mu\nu}dx^\alpha dx^\nu \frac{\partial \xi^\mu}{\partial x^\alpha} + g_{\mu\nu}dx^\alpha dx^\mu \frac{\partial \xi^\nu}{\partial x^\alpha} \end{aligned} \quad (6.3)$$

where we have dropped terms of higher order in ξ . Therefore if ξ represents an isometry, it must obey the following condition

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\alpha} \frac{\partial \xi^\mu}{\partial x^\alpha} + g_{\alpha\nu} \frac{\partial \xi^\nu}{\partial x^\alpha} = 0 \quad (6.4)$$

This equation is called the Killing equation² and the solutions are called the Killing vectors³. This equation characterizes all the possible continuous symmetries of the space, it can however not give discrete symmetries, as these are not composition of infinitesimal ones.

For the Minkowsky metric $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ we get an equation

$$\frac{\partial \xi_\nu}{\partial x^\mu} + \frac{\partial \xi_\mu}{\partial x^\nu} = 0 \quad (6.7)$$

which has the following general solution

$$\xi_\mu = a_\mu + \omega_{\mu\nu} x^\nu \quad (6.8)$$

and

$$x^\mu \rightarrow x^\mu + a^\mu + \omega^\mu_\nu x^\nu \quad (6.9)$$

Here a_μ are translations in the space-time and $\omega_{\mu\nu}$ is anti-symmetric.

Space part ω_{ij} is in the most general form $\varepsilon_{ijk}\theta^k$, thus $\varepsilon_{ijk}\theta^k x^j$ is a rotation. The time components ω_{i0} represent the boosts. Note, that if ω_{i0} is non-zero, we get $\delta x^i = \omega_{i0}x^0$. However, the anti-symmetry of ω yields that also ω_{0i} is non-zero and x^0 will change too. Time and space

²This equation is covariantly invariant and thus although giving results in particular coordinates, these are not coordinate dependent. The invariance is obvious from the form

$$g_{\mu\alpha} \nabla_\nu \xi^\alpha + g_{\nu\alpha} \nabla_\mu \xi^\alpha = 0 \quad (6.5)$$

Just use

$$\nabla_\nu \xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\nu} + \Gamma^\alpha_{\mu\nu} \xi^\mu \quad (6.6)$$

and see that it is the same.

³Obviously, the dramatic potential of the Killing vectors will be recognized by the Hollywood produces very shortly. Killing vector, that went wrong and by changing the metric conditions tears its victims apart is a certain block-buster.

are interconnected and transform together. This is the essence of the special relativity and all the theory can be build up just from the group considerations of invariance of Minkowsky space-time.

The same result holds for the Euclidean space, where $g_{\mu\nu} = \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$. Namely, equation (6.9) describes the most general isometry of this space. However raising the space index in ω does not produce a minus sign.

To illustrate this, consider the only nonzero element to be $\omega_{01} = \omega$. The the transformation is

$$\begin{aligned} x^0 &\rightarrow x^0 + \omega_1^0 x^1 \\ x^1 &\rightarrow x^1 + \omega_0^1 x^0 \end{aligned} \quad (6.10)$$

in both cases. However for Euclidean case we obtain

$$\begin{aligned} x^0 &\rightarrow x^0 + \omega x^1 \\ x^1 &\rightarrow x^1 + \omega_0^1 x^0 = x^1 + \omega_{10} x^0 = x^1 - \omega x^0 \end{aligned} \quad (6.11)$$

while for Minkowsky case again

$$\begin{aligned} x^0 &\rightarrow x^0 + \omega x^1 \\ x^1 &\rightarrow x^1 + \omega_0^1 x^0 = x^1 - \omega_{10} x^0 = x^1 + \omega x^0 \end{aligned} \quad (6.12)$$

So the time component mixes with the spatial ones in a different way.

Conformal transformations

Before, we have constrained the transformations by condition $\delta ds^2 = 0$, which leaves the space completely intact. However, we can ask less constraining question and seek transformations that do change the metric, but by a constant, i.e. just scale all the distances by a factor.⁴ These are conformal transformations.

Similarly as before, this condition leads to the equation⁵

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} + g_{\mu\nu} \frac{\partial \xi^\nu}{\partial x^\alpha} = \lambda g_{\mu\nu} \quad (6.13)$$

We can determine the constant λ by multiplying this expression by $g^{\mu\nu}$ and taking the sum

$$\lambda n = \left(\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) g^{\mu\nu} + 2 \frac{\partial \xi^\alpha}{\partial x^\alpha} = \xi^\alpha \frac{\partial}{\partial x^\alpha} \log(\det g) + 2 \frac{\partial \xi^\alpha}{\partial x^\alpha} \quad (6.14)$$

⁴It is not too difficult to see, that such transformation preserves angles.

⁵Note, that this transformation scales the metric by $1 + \lambda$ not λ , since $\delta g_{\mu\nu} = \lambda g_{\mu\nu}$.

We have used the following formula

$$\text{Tr}(g^{-1}\partial_\alpha g) = \partial_\alpha \text{Tr}(\log g) = \partial_\alpha \log(\det g) \quad (6.15)$$

which follows from the definition of the variation of a matrix $\delta \log M = M^{-1}\delta M$. After some algebra we can express λ from (6.14) as

$$\lambda = \frac{1}{\sqrt{\det g}} \frac{2}{n} \frac{\partial}{\partial x_\alpha} \left(\xi_\alpha \sqrt{\det g} \right) \quad (6.16)$$

yielding the conformal Killing equation

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} + g_{\mu\nu} \frac{\partial \xi^\nu}{\partial x^\alpha} = \frac{1}{\sqrt{\det g}} \frac{2}{n} \frac{\partial}{\partial x_\alpha} \left(\xi_\alpha \sqrt{\det g} \right) g_{\mu\nu} \quad (6.17)$$

6.2 Conformal transformations of the plane

For the flat plane, we have $g_{\mu\nu} = \delta_{\mu\nu}$ where $\mu, \nu = 1, 2$, which turns conformal Killing equation (6.17) into

$$\frac{\partial \xi_\nu}{\partial x^\mu} + \frac{\partial \xi_\mu}{\partial x^\nu} - \delta_{\mu\nu} \frac{\partial \xi^\alpha}{\partial x^\alpha} = 0 \quad (6.18)$$

which gives two independent equations

$$\begin{aligned} (\mu, \nu) &= (1, 2), (2, 1) & \frac{\partial \xi_2}{\partial x^1} &= -\frac{\partial \xi_1}{\partial x^2} \\ (\mu, \nu) &= (1, 1), (2, 2) & \frac{\partial \xi_1}{\partial x^1} &= \frac{\partial \xi_2}{\partial x^2} \end{aligned} \quad (6.19)$$

This set of equations is just Cauchy-Riemann equations for the real and imaginary part of an analytic complex function. Therefor a complex function ξ will always give a conformal transformation of the plane, by taking $\xi_1 = \text{Re} \xi$, $\xi_2 = \text{Im} \xi$, where all these function are functions of $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$. One can check, that due to (6.19)⁶

$$\frac{\partial \xi}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) (\xi_1 + i \xi_2) = 0 \quad (6.20)$$

which means that ξ is function of z only.⁷ We know, that if we demand ξ to be defined on the whole complex plane, the only possible solution is $\xi \equiv \text{const}$, so we need to allow some

⁶To see, that this combination of $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ is equal to $\frac{\partial}{\partial \bar{z}}$ note for example, that then

$$\frac{\partial \bar{z}}{\partial z} = 1 \quad , \quad \frac{\partial z}{\partial \bar{z}} = 0$$

⁷And consequently $\bar{\xi}$ function of \bar{z} only.

singularities. The next most simple possibility is to have ξ singular for $z = 0$ and $z \rightarrow \infty$. This corresponds to imposing some boundary conditions on our physical states. Expanding this function into Laurent expansion, we get

$$\xi(z) = \sum_{n=-\infty}^{\infty} a_n z^{n+1} \quad (6.21)$$

where a_n are arbitrary constants, yielding infinite number of conformal transformations of the plane.⁸

To obtain the generators of these transformations, we write the transformation as

$$z \rightarrow z + \xi(z) \quad (6.22)$$

under which a function $f(z)$ changes as

$$\delta f = f(z + \xi) - f(z) = \xi \frac{\partial f}{\partial z} = \sum a_n z^{n+1} \frac{\partial}{\partial z} f \quad (6.23)$$

From here, we can identify the generators as⁹

$$L_n = -z^{n+1} \frac{\partial}{\partial z} \quad n = 0, \pm 1, \pm 2, \dots \quad (6.24)$$

To obtain the commutation rules, we act on a general function f

$$[L_n, L_m] f = z^{n+1} \frac{\partial}{\partial z} \left(z^{m+1} \frac{\partial f}{\partial z} \right) - z^{m+1} \frac{\partial}{\partial z} \left(z^{n+1} \frac{\partial f}{\partial z} \right) = (m - n) z^{m+n+1} \frac{\partial f}{\partial z} \quad (6.25)$$

which yields

$$[L_n, L_m] = (n - m) L_{m+n} \quad (6.26)$$

Algebra of operators with such commutation rule is called Virasoro algebra.

6.3 Quantum corrected Virasoro algebra and phase transitions

We have the generators of the conformal transformations as acting on the functions $f(z)$, namely (6.24). However, for a quantum theory, we need more than that. In this theory, we have a Hilbert space \mathcal{H} of states, and all reasonable operators are to be operators on this Hilbert space.

⁸This is not true for higher dimensions and n -dimensional space R^n has for $n > 2$ conformal symmetry group $SO(n + 2)$.

⁹The minus sign is a convention.

Therefor we need to express the Virasoro generators in the terms of some fundamental operators. Recalculating the commutation relations gives the following correction to the commutation rule (6.26)

$$[L_n, L_m] = (n - m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (6.27)$$

We got an extra central term, that is non-zero for $m = -n$. There are few notes to be made about this expression

- An obvious thing to do is to try to redefine the operators $L_n \rightarrow \tilde{L}_n$, such that the new operators obey the old commutation rule. One can see, that this is not possible. Let $\tilde{L}_n = L_n + l_n$ and let $[L_n, L_m] = (n - m)L_{n+m} + K(m, n)$. Then

$$K = [L_n, l_m] + [l_n, L_m] + [l_n, l_m] - (n - m)l_{m+n} \quad (6.28)$$

However one can see that this contradicts the Jacobi identity.

- For a better idea how this procedure work, let us give an example how these quantum generators are found for the Galileian boost. This is transformation to a moving frame, i.e. transformation $x \rightarrow x - vt$, where v is the speed of the frame. Functions change like

$$f(x) \rightarrow f(x - vt) \approx f(x) - vt \frac{\partial}{\partial x} f(x) \quad (6.29)$$

and thus the generator is $-t \frac{\partial}{\partial x}$. To realize it on a general Hilbert space, we need to find an operator G , that generates the corresponding transformation U . Under such transformation, operators \hat{x} and \hat{p} change in the following fashion

$$\begin{aligned} \hat{x} &\rightarrow \hat{x} - vt \\ \hat{p} &\rightarrow \hat{p} - v \end{aligned} \quad (6.30)$$

Operator \mathcal{O} transforms as $U^\dagger \mathcal{O} U$, thus the change of the operator is

$$\delta \mathcal{O} = i[\mathcal{O}, \hat{G}] \quad (6.31)$$

yielding $[\hat{x}, \hat{G}] = ivt$, thus $\hat{G} = vtp + q(\hat{x})$, where q is a function to be determined. For $\mathcal{O} = \hat{p}$ we get $[\hat{p}, \hat{G}] = iv$ giving $q(\hat{x}) = -\hat{x}v$. Together, we got for the generator of the Galileian boost

$$\hat{G} = v(t\hat{p} - \hat{x}) \quad (6.32)$$

- Let us give a glimpse of how this is done for the case of Virasoro algebra. For a massless scalar field Φ , the energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\alpha \Phi)(\partial^\alpha \Phi) \quad (6.33)$$

where the energy density is $H = \int T_{00}$. This way, we can identify the generators (6.24) with the components of the energy-momentum tensor and after recalculating the commutation rules obtain the quantum correction in (6.27)

- The factor $\frac{1}{12}$ in the correction in (6.27) comes from the regularizing the sum $\sum_n n$. Using the analytical continuation of the Riemann ζ -function $\zeta(z) = \sum_{n=0}^{\infty} n^{-z}$, which can be defined uniquely, gives $\zeta(-1) = -\frac{1}{12}$.

Parameter c is called the central charge and characterizes the theory. All unitary representations of this algebra give all the possible conformal theories in 2D. As we will later see, the unitarity condition does not allow all the possible values of c . These are some examples for some values of c

- $c = 1$ gives so called X4 model in statistical mechanics
- for $c < 1$ we have a discrete set of c due to the unitarity, these include the Insing model, tri-critical Insing model, 3-state Potts model, RSOS model, etc.
- $c = 26$ is the bosonic string theory
- $c = 10$ is the super-string theory
- for a Lie group with structural constants $f^{abc} f^{nbc} = C_A \delta^{an}$ and k an integer we get $c = \frac{k \dim G}{k + C_A}$, which give WZW models, for the case of $SU(2)$ this is spin $\frac{1}{2}$ Heisenberg model in two dimensions
- the case $c = -2$ is believed to describe turbulence in hydrodynamics

However cases of $c < 0$ are not unitary.

Phase transitions

We will distinguish two kinds of phase transitions. First and second order. Sometimes, second order transitions are distinguished further into higher order ones. The first order transitions are points with non-zero latent heat, i.e. it takes some energy to change the state.

The second order transitions are more subtle. When computing the correlation function, we get

$$\langle \Phi(x) \Phi(y) \rangle \propto e^{-|x-y|/\xi} \frac{1}{|x-y|^\eta} \quad (6.34)$$

where ξ is the correlation length. For example for the field theory given by the Hamiltonian $H = \frac{1}{2} \int (\partial\Phi)^2 + m^2 \Phi^2$ we get $\xi = m^{-1}$ and in statistical mechanics $m^2 \propto T$. The second order

transitions are characterized by $\xi = \infty$, i.e. $m = 0$ at the point of transition $T = T_C$. Thus, at this point

$$\langle \Phi(x)\Phi(y) \rangle \propto \frac{1}{|x-y|^\eta} \quad (6.35)$$

We see, that under the scaling transformation $x, y \rightarrow \lambda x, \lambda y$ this quantity changes as

$$\langle \Phi(x)\Phi(y) \rangle \rightarrow \frac{1}{\lambda^\eta} \frac{1}{|x-y|^\eta} \quad (6.36)$$

Rescaling the fields $\Phi \rightarrow \lambda^{\eta/2}\Phi$ gives a complete scaling symmetry of the theory. This scaling is done by the L_0 generator, since

$$(1 + \varepsilon L_0)z = (1 + \varepsilon z\partial_z)z = (1 + \varepsilon)z = \lambda z \quad (6.37)$$

The following theorem yields the conformal symmetry.

Theorem 6.3.1. *If a (field) theory has local interactions, then scaling invariance implies full conformal invariance.*

Meaning. If the Hilbert space carries an unitary representation of the L_0 operator (the theory is scaling invariant), then it must carry an unitary representation of the whole Virasoro algebra (the theory has the conformal symmetry).

At the transition point, we can fully describe the system in the terms of this algebra.

6.4 The Ising Model

This model describes magnetic phase transitions, where at high temperatures the system is disordered, however for lower temperatures some order occurs. Heisenberg model gives the following Hamiltonian for the spins on a lattice

$$H = - \sum J_{ij} \vec{S}_i \cdot \vec{S}_j \quad (6.38)$$

This expression already includes the electrostatic energy and the exclusion principle. J_{ij} is a coupling between two spins at sites i and j , for $J > 0$ the aligned position is preferred, for $J < 0$ the anti-parallel position of spins is preferred. In the following, we will assume, that all the spins are aligned in one direction and thus $S \rightarrow \sigma = \pm 1$, however all that follows can be generalized for a general direction of the spin. For the nearest neighbor approximation (6.38) changes into¹⁰

$$H = -J \sum \sigma_i \sigma_{i+1} \quad , \quad \sigma_i = \pm 1 \quad (6.39)$$

¹⁰This sum goes over four sites $i + 1$ for every site i .

Coupling with the external magnetic field is introduced by an extra term $\sum_i g\sigma_i B$.

One dimensional Heisenberg chain was solved by Bethe in 1931, two dimensional problem was solved by Onsager in 1941. The phase transition point is defined by $\langle\sigma\rangle \neq 0$ under and $\langle\sigma\rangle = 0$ above the T_C . This can be done by explicit calculation of

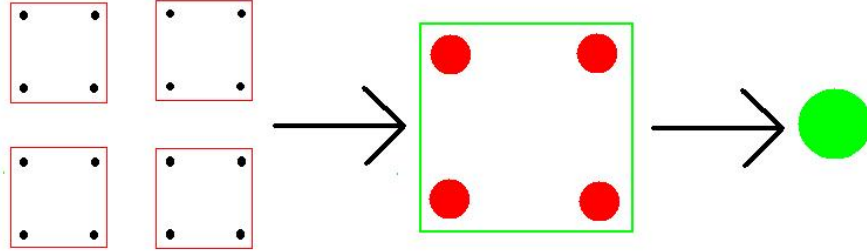
$$\langle\sigma\rangle = \frac{\text{Tr}(\sigma e^{-\beta H})}{\text{Tr}(e^{-\beta H})} \quad (6.40)$$

Using the conformal symmetry of the problem, we can get Kadanof scaling relations, giving relations between several parameters of the theory near the transition temperature T_C . We define a new temperature parameter ε as

$$\varepsilon = \frac{T - T_C}{T_C} \quad (6.41)$$

The mean magnetization is then function of ε and B .

We will also introduce a procedure, called block spinning. Here, the summation in $\text{Tr}(e^{-\beta H})$ is not done site by site, by by the block of several spins. This is equivalent to first doing the calculations over a square of $L \times L$ spins and than considering this square as new object in a different Insing model. Now we ask, how do the properties of the model scale under block spinning.



Let us denote parameters of the new model by tilde. Temperature scales as distance, so $\tilde{\varepsilon} = L\varepsilon$. Magnetic field scales as an unknown power of the distance $\tilde{B} = L^x B$ and there are L^d sites in one block in the d -dimensional space. To obtain the mean magnetization, we define the free energy $F = -\frac{1}{\beta} \log Z$. We compute

$$\frac{\partial F}{\partial B} = -\frac{1}{\beta} \frac{\partial \log Z}{\partial B} = -\frac{1}{\beta Z} \frac{\partial Z}{\partial B} = g \frac{\text{Tr}(\sigma e^{-\beta H})}{\text{Tr}(e^{-\beta H})} = g\langle\sigma\rangle \quad (6.42)$$

so the mean magnetization can be computed as variation of the free energy. Since the free energy is an extensive quantity and the free energy of L^d sites is L^d times the free energy of one site, we get

$$L^d \delta F = \delta \tilde{F} = g\langle\tilde{\sigma}\rangle \delta \tilde{B} = L^d \underbrace{\langle\sigma\rangle}_{\delta F} \delta B \quad (6.43)$$

which yields

$$\langle \tilde{\sigma} \rangle = L^{d-x} \langle \sigma \rangle \quad (6.44)$$

Since magnetization is a function of ε and B , and moreover the same function for both block spinned and the original system, we get

$$f(L\varepsilon, L^x B) = L^{d-x} f(\varepsilon, B) \quad (6.45)$$

where $\langle \sigma \rangle = f(\varepsilon, B)$. Realizing, that this function f can not be function of ε and B independently and that $\tilde{\varepsilon}/\tilde{B}^{1/x} = \varepsilon/B$ we can see that f must be in the form

$$f(\varepsilon, B) = \varepsilon^{d-x} f\left(\frac{\varepsilon}{B^{1/x}}\right) \quad (6.46)$$

We know, that below the transition point, i.e. $\varepsilon < 0$, there is a non-zero magnetization as $B \rightarrow 0$, yielding $\langle \sigma \rangle = \varepsilon^{d-x} f(-\infty)$ and thus

$$\langle \sigma \rangle \propto (T - T_C)^{d-x} \quad \text{as } T \rightarrow T_C \text{ from below} \quad (6.47)$$

Similarly, for temperatures above the transition temperature, the net magnetization is proportional to the external field $\langle \sigma \rangle = \chi B$, where χ is the susceptibility of the system. Therefore we need $f \propto B$ which yields

$$f\left(\frac{\varepsilon}{B^{1/x}}\right) = \left(\frac{\varepsilon}{B^{1/x}}\right)^x \quad (6.48)$$

and

$$\chi(T) \propto (T - T_C)^{d-2x} \quad \text{as } T \rightarrow T_C \text{ from above} \quad (6.49)$$

So if we can determine x , we can get measurable predictions for the temperature behavior of $\langle \sigma \rangle$ and χ near the transition point. Since the two are connected, measuring one gives a prediction for the other one. However, we can go a step further and get one more condition.

If the site length is a , then the two-point correlation functions are

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle &= \frac{1}{a^\eta |i - j|^\eta} \\ \langle \tilde{\sigma}_i \tilde{\sigma}_j \rangle &= \frac{1}{L^\eta a^\eta |i - j|^\eta} \end{aligned} \quad (6.50)$$

since the site length of the block spinned system is La . Using (6.44), we get

$$\langle \tilde{\sigma}_i \tilde{\sigma}_j \rangle = L^{2d-2x} \langle \sigma_i \sigma_j \rangle \quad (6.51)$$

This yields $\eta = 2x - 2d$. This parameter can be computed from the Virasoro algebra.

An Aside : Block spinning and renormalization group

The idea of block spinning is closely connected to renormalization group techniques used in the quantum field theories. The formulation of this approach that we are going to present is due to Wilson.

We start by defining our theory with the Hamiltonian H and the partition function

$$Z = \sum e^{-\beta H} \quad (6.52)$$

The trick is to introduce a momentum cutoff μ_1 and split the sum in the partition function into momenta smaller and larger than μ_1

$$Z = \sum_{k < \mu_1} \sum_{k > \mu_1} e^{-\beta H} \quad (6.53)$$

We can perform the sum over the momenta larger than μ_1 and call the result

$$e^{-\beta H_{eff}^{(1)}(\mu_1)} \quad (6.54)$$

for some function $H_{eff}(\mu_1)$. We have made explicit the dependence of this function on the cutoff momentum we choose. So we have a new theory, that is given by the new effective Hamiltonian. This is like block spinning since $k \sim 1/L$. Now we can repeat the same procedure for a different cutoff $\mu_2 < \mu_1$

$$Z = \sum_{k < \mu_2} \sum_{\mu_2 < k < \mu_1} e^{-\beta H_{eff}^{(1)}} \quad (6.55)$$

to obtain a new effective Hamiltonian $H_{eff}^{(2)}(\mu_1, \mu_2)$. Repeating this procedure, we get set of effective Hamiltonians

$$H_{eff}^{(1)}(\mu_1), H_{eff}^{(2)}(\mu_1, \mu_2), H_{eff}^{(3)}(\mu_1, \mu_2, \mu_3), \dots \quad (6.56)$$

If we take μ 's fine enough, we can consider a smooth function $H_{eff}(\mu)$ and

$$Z = \sum_{k < \mu} e^{-\beta H_{eff}(\mu)} \quad (6.57)$$

At the point of a phase transition, the theory looks the same on any scale, or for any momentum, and thus

$$\frac{\partial H_{eff}}{\partial \mu} = 0 \quad (6.58)$$

The derivative of the effective Hamiltonian is called the Beta function and we look for its zeros μ_0 . $H_{eff}(\mu_0)$ then describes the phase transition theory.

6.5 The unitary representations of Virasoro algebra

At this point, we will construct the unitary representations of the Virasoro algebra.

But first, let us recall how this was done for the $su(2)$. Starting from the highest state $|j\rangle$, for which $J_3 |j\rangle = j |j\rangle$ and $J_+ |j\rangle = 0$, we construct lower states by successive action L_- , until we come to state $|-j\rangle$, which is annihilated by J_- . This way, we get $2j + 1$ dimensional vector space spanned by all the states $|j\rangle, \dots, |-j\rangle$.

For the Virasoro algebra, we observe, that the operators L_0, L_1, L_{-1} close under the commutation rule (6.27), since

$$[L_1, L_{-1}] = 2L_0 \quad [L_0, L_1] = -L_1 \quad [L_0, L_{-1}] = L_{-1} \quad (6.59)$$

These commutation rules very similar to the rules of $su(2)$ algebra and form $sl(2, C)$ algebra. However, if we try to find a closed subalgebra for higher n 's, we fail. For example for $n = 2$, we get

$$[L_2, L_{-2}] = 4L_0 + \frac{c}{2} \quad [L_0, L_2] = -2L_2 \quad [L_1, L_2] = -L_3 \quad \dots \quad (6.60)$$

so beyond $n = 1, 0, -1$ we need to include the whole algebra and there is no closed subalgebra.

Now, we will construct the unitary representations. Let us have a state $|h\rangle$, which is the eigenstate of L_0 , i.e. $L_0 |h\rangle = h |h\rangle$, and is normalized, i.e. $\langle h|h\rangle = 1$. Observing the commutation relations we see, that L_n for $n > 0$ act like creations operators and for $n < 0$ like annihilation operators. Thus, we postulate

$$L_n |h\rangle = 0 \quad n \geq 1 \quad (6.61)$$

Then we construct all the possible states

$$\begin{aligned} &L_{-1} |h\rangle \quad , \quad L_{-2} |h\rangle \quad , \quad \dots \\ &L_{-1}^2 |h\rangle \quad , \quad L_{-2}^2 |h\rangle \quad , \quad \dots \\ &\vdots \end{aligned} \quad (6.62)$$

Since $L_{-n}^\dagger = L_n$, computing the norm of the state $L_{-1} |h\rangle$, we get

$$\langle h | L_1 L_{-1} |h\rangle = \langle h | \underbrace{[L_1, L_{-1}]}_{2L_0} |h\rangle = 2h \geq 0 \quad (6.63)$$

where the last conditions comes from the fact, that the norm needs to be non-negative. Therefore, we got $h \geq 0$. For states $|\alpha\rangle = L_{-1}^2 |h\rangle, |\beta\rangle = L_{-2} |h\rangle$ we get

$$\begin{aligned} \langle \alpha | \alpha \rangle &= 2h(4h + 1) \\ \langle \beta | \beta \rangle &= 4h + \frac{c}{2} \\ \langle \alpha | \beta \rangle &= 6h \end{aligned} \quad (6.64)$$

Since these two states are not orthogonal, for the bases we would have to find an orthogonal linear combination of the two. However here, we just need a condition on the norms, which can be obtained from the matrix

$$M = \begin{pmatrix} \langle \alpha | \alpha \rangle & \langle \alpha | \beta \rangle \\ \langle \alpha | \beta \rangle & \langle \beta | \beta \rangle \end{pmatrix} \quad (6.65)$$

namely by the condition $\det M \geq 0$. The eigenvalues of this matrix m_1, m_2 are the norms of the orthogonal combination states, which need to have both non-negative norm. This yields

$$\begin{aligned} (8h^2 + 2h)(4h + c/2) - 36h^2 &\geq 0 \\ 32h^2 + (4c - 28)h + c &\geq 0 \end{aligned} \quad (6.66)$$

This condition allows only some h 's for a given c . Similar analysis can be done systematically for all the states, yielding the possible values of h for a given c . For such states, representation derived from $|h\rangle$ by the described method yields an unitary representation of the Virasoro algebra.

For $c \geq 1$, all the values of $h \geq 0$ are allowed. For $c < 1$, we get a 'minimal series'

$$c = 1 - \frac{6}{m(m+1)} \quad m = 2, 3, \dots \quad (6.67)$$

This is a discrete set of numbers c , that allow a unitary representation. For a given m , we get

$$h_{p,q} = \frac{((m+1)p - mq)^2}{4m(m+1)} \quad \begin{array}{l} p = 1, 2, \dots, m-1 \\ q = 1, \dots, p \end{array} \quad (6.68)$$

As for the case of $su(2)$, where the number, characterizing the representation could be associated with some physical quantities, the numbers $h_{(p,q)}$ determine the physics of the problem.

For the case $m = 2$ we get $h = 0$, which is a trivial one dimensional representation. For $m = 3$, we get $c = \frac{1}{2}$ and possible values of h for (p, q)

$$\begin{aligned} h_{(1,1)} &= 0 \\ h_{(2,1)} &= \frac{1}{2} \\ h_{(2,2)} &= \frac{1}{16} \end{aligned} \quad (6.69)$$

Since $L_{-1} = -d_z$, we can understand the consecutive action of L_{-1} on $(h_{(1,1)}, h_{(2,1)}, h_{(2,2)})$ as spreading the triplet over the whole space. As mentioned before, the case $m = 3$ is the case of the Insing model. Here, $h_{(2,2)}$ can be associated with the spin density, thus the spin operator will scale with the factor $\frac{1}{16}$. Therefor

$$\langle \sigma(z, \bar{z}) \sigma(z', \bar{z}') \rangle = \frac{1}{(z - z')^{1/8} (\bar{z} - \bar{z}')^{1/8}} = \frac{1}{|x - x'|^{1/4}} \quad (6.70)$$

giving value

$$\eta = \frac{1}{4} \quad \Rightarrow \quad x = \frac{1}{8} + d$$

in (6.50) and giving predictive power to the scaling relations.

A similar analysis will work for higher m 's. At the end, let us note, that there are operators, that do not change the number c of the theory. Such operators are called irrelevant, since their addition does not change any properties of the theory.

Appendix A

Notes on the theory of Lie group, Lie algebras and their representations

In this section we review some basic group theory and develop the theory of representations of Lie groups. As mentioned before, first couple of sections do not require this material, however starting with the section 5 it becomes quite important.

The discussion is far from complete and we summarize only a couple of results that are relevant for the topics covered in the main text.

A.1 Groups

The importance of groups in physics is hard to overestimate. This follows from the fact that transformations of objects form a group and therefore anytime we want to describe transformations in physics we encounter group theory.

Definition. Group is the set of elements $G = \{g_1, \dots\}$ with a composition rule $\circ : G \times G \rightarrow G$ and these properties

- a) it is closed, i.e. $\forall g_1, g_2 \in G, g_1 \circ g_2 \in G$,
- b) associativity, i.e. $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$,
- c) there is an identity element, i.e. $\exists e \in G, \forall g \in G, e \circ g = g \circ e = g$,
- d) there is an inverse element, i.e. $\forall g \in G \exists \tilde{g} \in G, g \circ \tilde{g} = \tilde{g} \circ g = e$,

Note that the group is not required to be commutative, i.e. in general $g_1 \circ g_2 \neq g_2 \circ g_1$. From now on, we will omit the composition sign and simply denote $g_1 \circ g_2 \equiv g_1 g_2$.

Examples of groups.

- Z_N - cyclic group of order N , with elements

$$1, \omega, \dots, \omega^{N-1}, \quad (\text{A.1})$$

where $\omega = e^{\frac{i2\pi}{N}}$ and the composition rule being the usual multiplication of complex numbers. We can see that this is essentially rotating around a circle, where multiplication by ω^k turns the circle around by $\frac{2\pi k}{N}$. A different realization of this group is the set of natural numbers $0, 1, \dots, N-1$ with the composition rule being the sum modulo N .

- Permutation group - permutations of N elements set with the composition rule being a consecutive permutation. Obviously this group has $N!$ elements.
- Rotations in three dimensions - we have a space R^2 described by two coordinates x_1, x_2 and the rotation by angle θ is given by

$$\vec{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R(\theta)\vec{x} \quad (\text{A.2})$$

Explicit calculation would show, that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$, so we obtain closure. Identity is clearly given by $R(0)$ and inverse by $R^{-1}(\theta) = R(-\theta)$, so this follows the definition of group. The number of elements of this group is infinite and the elements are parametrized by θ from interval $[0, 2\pi)$, which is a circle. Note that because of this all the cyclic groups are subsets of this group. Let us now define this notion better.

Definition. We call a subset H of a group G , $H \subset G$, a subgroup, if H is a group under the same composition rule as G .

Obviously, every subgroup must contain the identity e .

Examples.

- In $Z_4 = \{1, i, -1, -i\}$ the set $\{1, -1\}$ forms a subgroup, which is Z_2 . Generally Z_n is a subgroup in Z_{2n} .
- The rotations of the three dimensional space form a group. Rotations around a fixed axis in the space form a subgroup of this group.

A very nice review and development of group theory relevant for quantum mechanics and subsequently chemistry is Cotton - Chemical Applications of Group Theory.

A.2 (Matrix) Representations of groups

Definition. n -dimensional matrix representation of a group G is a mapping $\rho : G \rightarrow M_n$, which maps elements of group G onto the $n \times n$ matrices and respects the composition rule, i.e.

$$g_1 \xrightarrow{\rho} G_1, g_2 \xrightarrow{\rho} G_2, g_3 \xrightarrow{\rho} G_3, g_1 g_2 = g_3 \Rightarrow G_1 G_2 = G_3. \quad (\text{A.3})$$

Matrix representation is then homomorphism of the group G and the set of $n \times n$ matrices M_n .

This for example implies that

$$e \xrightarrow{\rho} I_n, \quad \tilde{g} \xrightarrow{\rho} G^{-1}. \quad (\text{A.4})$$

However, two elements of the group G can be mapped onto the same matrix.

Matrix representations are very special case of group representations, however they are the most important ones, since in physics we deal usually with Hilbert and other linear spaces, where introduction of a basis turns every representation into a matrix representation.

Example. We already had $Z_4 = \{1, i, -1, -i\}$, which is a one dimensional representation of Z_4 . One can see, that also matrices

$$\{I_2, i\sigma_2, -I_2, -i\sigma_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \quad (\text{A.5})$$

satisfy the same composition rules and thus this is a two dimensional representation of Z_4 . Representation

$$\{I_n, -I_n, I_n, -I_n\} \quad (\text{A.6})$$

is n -dimensional with two elements of Z_4 being mapped onto the same matrix.

Equivalent, reducible and irreducible representations

If we have a representation ρ of a group G , we can generate a new representations by a similarity transform, i.e.

$$\rho_S : g \rightarrow S^{-1} G S \quad (\text{A.7})$$

with S a fixed invertible $n \times n$ matrix. However, these representations are in no sense different. The change $\rho \rightarrow \rho_S$ is just a change of a basis in the linear space, on which the matrices act and brings no new information.

Definition. Representations ρ_1, ρ_2 , for which

$$G^{(1)} = S^{-1}G^{(2)}S \quad \forall g \in G \quad (\text{A.8})$$

for some fixed invertible matrix S are called equivalent.

Such representations have the same properties and do not need to be treated separately.

We got two different representations of group Z_4 , one dimension representation (A.6) and two dimensional representation (A.5). We can now construct a three dimensional representation of this group by simply putting these two matrices into one

$$\begin{pmatrix} G^{(1)} & 0 \\ 0 & G^{(2)} \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \quad (\text{A.9})$$

We see that neither this is a new representation, just two different representations living next to each other. We call such representation reducible, since it reduces into different, less dimensional representations. We could see this by the block-diagonal form of matrices (A.10). However, using the similarity transform (A.8), we could mess things up and un-diagonalize these matrices.

Definition. If all matrices G can be brought to a block diagonal form by a single similarity transformation, the representation is reducible.

Thus, reducible transformation is equivalent to a block diagonal one and the linear space where the matrices act has an invariant subspaces, which do not mix among themselves, or

$$gv = \begin{pmatrix} G^{(1)} & 0 & \dots \\ 0 & G^{(2)} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \\ \vdots \end{pmatrix} = \begin{pmatrix} G^{(1)}v^{(1)} \\ G^{(2)}v^{(2)} \\ \vdots \end{pmatrix}. \quad (\text{A.10})$$

If the representation can not be brought to the block-diagonal form, it is irreducible. If we manage to describe all the irreducible representations of the particular group, we have described all the possible representations, since the rest are just build up from these.¹

A.3 Lie groups and Lie algebras

We have mentioned the group of rotations in the 2D and 3D space. This is a continuous group, that has an uncountably infinite amount of elements. This is a special case of what we call the

¹This a rather non-trivial property of the group G that is called complete reducibility. However it is not true for all the groups. Here, we will just state that all the representations we will come across are completely reducible.

Lie group. Elements of such groups are described by a continuous index and the dimension of the group is the number of indexes needed to describe particular element.

We will denote G the Lie group and \bar{G} it's corresponding Lie algebra. For specific cases, we will use upper case letters for groups and lower case letters for algebras.

Definition. Lie group is a group with continuous indexes describing the elements of the group and in which the composition rule satisfy

- a) in $g_{\theta_1}g_{\theta_2} = g_{\Theta(\theta_1, \theta_2)}$, the function Θ is analytical in both arguments,
- b) in $g_{\theta}\tilde{g}_{\alpha(\theta)} = e$, the function α is analytical.

Analyticity of the functions allows us to work with power series expansions. Without loss of generality, we will take the unity of the group to correspond to $\theta = 0$. An element of the group very close to an element $g(\theta)$ can be written as

$$g(\theta^i + \delta\theta^i) = g(\theta^i)g(\delta\theta^i) = g[\Theta^i(\theta, \delta\theta)] = g\left[\Theta^i(\theta, 0) + \delta\theta^i \frac{\partial\Theta^i(a, b)}{\partial b}\Big|_{a=\theta, b=0}\right]. \quad (\text{A.11})$$

In this expression, i labels the different directions in the parameter space.

Now consider a function on the group. This could be thought of as a function on the θ -space, i.e.

$$f(g) = f[g(\theta)] = f(\theta). \quad (\text{A.12})$$

Strictly speaking the first and the last f are a different function, but we hope that it is clear which function we mean. Now using the previous formula, we get

$$f[g(\theta^i + \delta\theta^i)] = f(\theta^i + \delta\theta^k N_k^i) = f(\theta) + \delta\theta^k N_k^i \frac{\partial f}{\partial \theta^i}, \quad (\text{A.13})$$

where we have denoted

$$N_k = \frac{\partial\Theta^k(a, b)}{\partial b}\Big|_{a=\theta, b=0}. \quad (\text{A.14})$$

We define²

$$X_k = N_k(\theta)_k^i \frac{\partial}{\partial \theta^i}. \quad (\text{A.15})$$

These operators provide translation in the space of functions on the group. They themselves form a vector space. When we try to compute the commutators of these operators, we get a very messy formula

$$[X_k, X_l]f = \left(N_k^i \frac{\partial N_l^j}{\partial \theta^i} - N_l^i \frac{\partial N_k^j}{\partial \theta^i}\right) \frac{\partial f}{\partial \theta^j} = \left(N_k^i \frac{\partial N_l^j}{\partial \theta^i} - N_l^i \frac{\partial N_k^j}{\partial \theta^i}\right) (N^{-1})_j^a X_a f, \quad (\text{A.16})$$

²In this formula we stress for the first and the last time that N depends on the θ .

where we have used that the inverse of N exists due to the group properties. However, Lie proved things are not so bad.

Theorem A.3.1 (Lie (first) theorem). *For the commutators of X 's*

$$[X_a, X_b] = C_{ab}^c X_c \quad (\text{A.17})$$

the parameters C are independent of θ .

Meaning. We can do the analysis at any point of the group, results do not depend on this. Namely, we can do the analysis at the unity of the group.

We now make a step in what seems a different direction, but as we will shortly see, it is not. Recall the definition of algebra as a linear vector space with a bilinear product, that is not necessarily associative. This product is usually denoted as³

$$[V_a, V_b] = f_{ab}^c V_c \quad (\text{A.18})$$

where V_a is a basis in the algebra. If this product is antisymmetric

$$[V_a, V_b] = -[V_b, V_a] \quad \Rightarrow \quad f_{ab}^c = -f_{ba}^c \quad (\text{A.19})$$

and obeys Jacobi identity

$$\begin{aligned} [[V_a, V_b], V_c] + [[V_b, V_c], V_a] + [[V_c, V_a], V_b] &= 0 \\ f_{ab}^i f_{ci}^d + f_{bc}^i f_{ai}^d + f_{ca}^i f_{bi}^d &= 0 \quad , \end{aligned} \quad (\text{A.20})$$

we call this algebra a Lie algebra. Note, that the Jacobi identity is trivial if we assume associativity.

Theorem A.3.2 (Lie (second) theorem). *For every Lie group, there is a corresponding Lie algebra \bar{G} , which is isomorphic to the commutator algebra of the translation operators (A.17).*

Meaning. Multiple Lie groups can correspond to the same Lie algebra. However algebra corresponds to only one group \tilde{G} . All the other groups can be obtained from the covering group by the identification of points. In this way, the Lie algebra is sensitive only to local behavior of the group, and not to its global geometry. As an example, groups $SU(2)$ and $SO(3)$ have the same Lie algebra.

A very nice reference for much more details about this topic is Gilmore - Lie Groups, Lie Algebras and Some Applications.

³The notation is due to the fact, that any associative algebra is a Lie algebra with the "new" product defined as a commutator $[a, b] := ab - ba$.

Exponential mapping

We start at the identity of the group. We set identity to correspond to $\theta = 0$, i.e. $e = g_0$. For small values of the group parameter we get $g_\varepsilon = e + \varepsilon_k x_k$. Large steps on the group can be done by successive infinitesimal steps. Let us consider only one dimensional walk. For N steps of length ε we get

$$g_\theta = (1 + \varepsilon x) \dots (1 + \varepsilon x) = (1 + \varepsilon x)^N = \left(1 + \frac{\theta}{N} x\right)^N, \quad (\text{A.21})$$

we now take limit of $N \rightarrow \infty, \varepsilon \rightarrow 0$ with a fixed θ to obtain

$$g_\theta = e^{\theta x}. \quad (\text{A.22})$$

This way we can get to any $g_\theta \in G$ as a composite of walks in all the group directions and

$$g_\theta = e^{\theta_a x_a}. \quad (\text{A.23})$$

The usual procedure is as follows. We are given the group G in a defining representation. Then, using this representation we obtain the corresponding algebra in terms of the coefficients c_{bc}^a . From this, we construct different representations of the algebra and by (A.23) different representations of the original group.

Classification of Lie algebras

From (A.19) we get that the parameters C in (A.17) are anti-symmetric in a, b . The Jacobi identity (A.20) becomes an algebraic equation for C 's and solving it gives a complete classification of all possible Lie algebras. This classification was done by Killing and Cartan. They have found out, that there are four different sets of Lie algebras

$$\begin{aligned} A_n &= su(n+1) \\ B_n &= so(2n+1) \\ C_n &= sp(2n) \\ D_n &= so(2n) \end{aligned} \quad (\text{A.24})$$

for every $n = 1, 2, \dots$ and set of five anomalous algebras⁴

$$E_6, E_7, E_8, G_2, F_4. \quad (\text{A.25})$$

All of these have huge application in physics, $sp(2n)$ algebras are connected to the canonical transformations in classical and quantum mechanics, $su(n+1)$ as unitary transformations in quantum mechanics, $so(n)$ as rotations of space and as Lorentz transformations. The anomalous algebras found their use in more exotic theories, such as string theories.

⁴The fact, that we have exactly five such algebras is closely connected to the fact, that there are five Platonic bodies.

A.4 The most important Lie groups

Orthogonal groups $O(n)$

These groups are the sets of all real orthogonal transformations in n dimensions. As matrices, these are given by $n \times n$ matrices, that obey

$$R^T R = R R^T = I_n \quad (\text{A.26})$$

i.e. $R^T = R^{-1}$. If also $\det R = 1$, we call these special orthogonal matrices⁵. To see what these transformations represent, we compute the length of a vector

$$|x' - y'|^2 = (x' - y')^T \cdot (x' - y') = (x - y)^T R^T R (x - y) = |x - y|^2. \quad (\text{A.27})$$

This means such transformations leave the length of a vector unchanged, thus are the rotations in n dimensions.

We could consider a different scalar product in this equation. For example in the special theory of relativity, length of the 4-vector is given by $x_1^2 + x_2^2 + x_3^2 - t^2$. If we are looking for the transformations, that leave such length unchanged, we seek matrices X , for which

$$X^T \eta X = \eta \quad , \quad (\text{A.28})$$

with $\eta = \text{diag}(1, 1, 1, -1)$. Such transformations are Lorentz transformations and form $O(3, 1)$ group.

All scalar products are given by signature (k, l) and the matrix η is given by

$$\eta = \text{diag}(\underbrace{1, \dots, 1}_k, \underbrace{-1, \dots, -1}_l). \quad (\text{A.29})$$

Transformations, that preserve this scalar product form the group $O(k, l)$. For example conformal transformations in 3 dimensions form group $O(4, 2)$. If we deal with propagation of light, we need only conformal symmetry, since for light $ds^2 = 0$.

Unitary groups $U(n)$

This groups are formed by unitary transformations in n dimensional complex space, i.e. by $n \times n$ complex matrices, for which

$$U^\dagger U = U U^\dagger = I_n \quad \Rightarrow \quad U^\dagger = U^{-1} \quad , \quad (\text{A.30})$$

⁵The previous formula implies $\det R = \pm 1$ and the group splits into two disconnected parts and we clearly need the one containing the identity. The two parts are connected by the reflections $x_i \rightarrow -x_i$ for some i .

where $U^\dagger = (U^*)^T$. Again, if $\det U = 1$, we call it special unitary group $SU(n)$. In physics, these group occur whenever unitarity is important. $SU(2)$ is the group of rotations and spins in quantum mechanics, also iso-spin group that rotates nucleon states, $SU(3)$ is group of the quark model, $SU(3) \otimes SU(2) \otimes U(1)$ is the symmetry group of the standard model.

***Symplectic groups* $Sp(2n)$**

Important matrices in both classical and quantum mechanics are $2n$ -dimensional matrices in the block diagonal form $\Omega = \text{diag}(\underbrace{\omega, \dots, \omega}_n)$, with

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.31})$$

Naturally, the set of transformations, that preserve such matrix are going to play a huge role. Such $2n \times 2n$ matrices S , for which $S^T \Omega S = \Omega$, form a symplectic group. Observing the Poisson brackets or commutator of classical and quantum mechanics, we see, that Ω represents the canonical structure and thus $Sp(2n)$ are canonical transformations, which preserve such structure.

A.5 The explicit construction of the representations of $SU(2)$

We will illustrate the properties of the Lie algebras on this particular example.

The $SU(2)$ group is formed by 2×2 unitary matrices, for which $U^\dagger U = I_2$ and $\det U = 1$. Since in the defining representation g is an unitary matrix, we can write it as

$$g = e^{ih} \quad , \quad (\text{A.32})$$

where h is some hermitian matrix $h^\dagger = h$. Using $\det e^A = e^{\text{Tr}A}$ we get that trace of h vanishes. Thus

$$h = \begin{pmatrix} \alpha & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & -\alpha \end{pmatrix} = \alpha\sigma_3 + \beta_1\sigma_1 + \beta_2\sigma_2 \quad , \quad (\text{A.33})$$

where α, β_1, β_2 are real numbers and σ_i are usual Pauli matrices. Thus, any element of the $SU(2)$ group can be written as

$$g = e^{i\frac{1}{2}\theta_i\sigma_i}. \quad (\text{A.34})$$

For infinitesimal θ 's this turns into

$$g = I_2 + i\theta_k x_k \quad , \quad (\text{A.35})$$

with $x_k = \frac{1}{2}\sigma_k$ being the generators, i.e. the basis in the Lie algebra $su(2)$ in the defining representation. The commutation relations for Pauli matrices can be computed explicitly as

$$[x_a, x_b] = i\varepsilon_{abc}x_c \quad , \quad (\text{A.36})$$

which is now the defining relation for $su(2)$ algebra. If we construct any three operators, that obey this commutation rule, we have found a representation of the algebra. These operators then need not to be two dimensional.

A different way to obtain $su(2)$ algebra is to consider any general 2×2 complex matrix

$$M = a_0 I_2 + ia_k \sigma_k = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \quad (\text{A.37})$$

with a 's real. Constraint $\det M = 1$ gives $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$, i.e. a 3-sphere in 4 dimensional space. Thus, the group $SU(2)$ is isomorphic to S^3 , or a unit sphere in the space of quaternions.

Now, let us act by these matrices on some two dimensional vector space $\phi = (\phi_1, \phi_2)$ and $\phi' = g\phi$ or $\phi'_i = g_{ij}\phi_j$. And let us have a representation on the product of two vector spaces, with elements $\phi_i\chi_j$, where the group acts as

$$\phi'_i\chi'_j = g_{ik}g_{jl}\phi_k\chi_l. \quad (\text{A.38})$$

We easily check, that this is a representation, since

$$\phi''_i\chi''_j = g_{ik}^{(2)}g_{jl}^{(2)}g_{kn}^{(1)}g_{lm}^{(1)}\phi_n\chi_m = (g^{(2)}g^{(1)})_{in}(g^{(2)}g^{(1)})_{jm}\phi_n\chi_m. \quad (\text{A.39})$$

This quantity transforms as a 2-tensor. In general, tensors are defined by their transformation rule under the action of the group. Tensor $\phi_{i\dots j}$ is a quantity, that transforms in the following way

$$\phi'_{i\dots j} = g_{ik}\dots g_{jl}\phi_{k\dots l}. \quad (\text{A.40})$$

A special kind of tensors turn out to be very important, because of the following theorem.

Theorem A.5.1. *The reduction of tensor representations to the irreducible representations can be done using the invariant tensors and for any compact group, any unitary representation can be obtained by such reduction of products of the defining representation.*

All the tensors are obtained as a product of ϕ_i 's and are generally reducible.

We can straightforwardly check, that the antisymmetric tensor ε_{ij} is invariant under the action of $SU(2)$.

$$\varepsilon'_{ij} = g_{ik}g_{jl}\varepsilon_{kl} = g_{11}g_{22} - g_{12}g_{21} = \varepsilon_{ij} \det g = \varepsilon_{ij} \quad (\text{A.41})$$

Following theorem stops us from looking further.

Definition. The rank of the Lie algebra is the maximum number of mutually commuting linearly independent elements.

The rank of $su(2)$ is 1, since no two σ_i commute.

Theorem A.5.2. *The number of the invariant tensors of the Lie algebra and corresponding Lie group is equal to the rank of the algebra.*

If we have $c'_{ij} = g_{ik}g_{jl}c_{kl}$, we can define a new quantity

$$c = \varepsilon^{ij}c_{ij}. \quad (\text{A.42})$$

We can easily check, that this is a scalar quantity⁶

$$c' = \varepsilon^{ij}c'_{ij} = \varepsilon^{ij}g_{ik}g_{jl}c_{kl} = \det(g)\varepsilon^{kl}c_{kl} = c \quad (\text{A.43})$$

and thus deserves no indexes. c then forms a trivial representation, when all the elements are mapped onto the identity matrix $g \rightarrow I_n$. We have so far the trivial one dimensional representation c and the defining two dimensional representation ϕ_i . Also, as we have seen, the tensors c_{ij} form a representation, which is 4 dimensional. In general, it is described by 4×4 matrix M and we have

$$\begin{pmatrix} c'_{11} \\ c'_{12} \\ c'_{21} \\ c'_{22} \end{pmatrix} = M \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix}. \quad (\text{A.44})$$

However we can see that

$$c_{12} - c_{21} = \varepsilon^{ij}c_{ij} = c \quad , \quad (\text{A.45})$$

⁶This means it does not change under the action of the group, i.e. in transforms according to the trivial representation.

so if we make a linear transformation such that some element becomes $c_{12} - c_{21}$, then this element is going to transform trivially. That way, we get

$$\begin{pmatrix} c_{12} - c_{21} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \times 3 \end{pmatrix} \begin{pmatrix} c_{12} - c_{21} \\ \vdots \end{pmatrix}. \quad (\text{A.46})$$

The 4 dimensional representation is a reducible and reduces into one and three dimensional representations. As we will see, these are irreducible. We write $c_{ij} = (c_{ij} - \frac{1}{2}\varepsilon_{ij}c) + \frac{1}{2}\varepsilon_{ij}c$, where we have not changed the c and its transformation rule. Evaluating this we find out, that c_{ij} splits into

$$c_{ij} = c_{ij}^A + c_{ij}^S, \quad c_{ij}^A = \frac{1}{2}\varepsilon_{ij}c, \quad c_{ij}^S = \frac{c_{ij} + c_{ji}}{2} \quad (\text{A.47})$$

i.e. the symmetric and the antisymmetric part. We already know that c_{ij}^A transforms under a trivial representation, we can see that c_{ij}^S

$$c_{ij}^{\prime S} = \frac{1}{2}(g_{ik}g_{jl}c_{kl} + g_{jk}g_{il}c_{kl}) = g_{ik}g_{jl}\frac{c_{kl} + c_{lk}}{2} = g_{ik}g_{jl}c_{kl}^{\prime S}, \quad (\text{A.48})$$

is transformed also under some representation of $SU(2)$. Thus we need to transform

$$\begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} \rightarrow \begin{pmatrix} c_{12} - c_{21} \\ c_{11} \\ \frac{c_{21} + c_{12}}{2} \\ c_{22} \end{pmatrix}, \quad (\text{A.49})$$

which is done by matrix

$$S = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.50})$$

Therefore this matrix is the similarity transform, that brings the matrix M into block diagonal form. This could be seen also after block-diagonalizing the matrix M , however our approach was much easier to do. The three dimensional representation is irreducible, since we have no more invariant tensors to contract with.

We found out, that the product representation $\mathbf{2} \otimes \mathbf{2}$ reduces into sum of two representations $\mathbf{1} \oplus \mathbf{3}$. We will denote the representations by a bold face number. This is the quantum mechanical case of the composition of two spins $\frac{1}{2}$, where we get either spin 0, or spin 1. Coefficients in the matrix S are the corresponding Clebsch-Gordan coefficients⁷.

⁷Due to a different normalization, the usual Clebsch-Gordan coefficients are a square root of these.

We can apply the same analysis to higher order tensors. Let us have a tensor of the order k

$$\Phi_{i_1 \dots i_k} \quad (\text{A.51})$$

and let us contract indexes i_α, i_β by the ε tensor to obtain tensor of the order $k - 2$

$$\varepsilon^{i_\alpha i_\beta} \Phi_{i_1 \dots i_\alpha \dots i_\beta \dots i_k} \quad (\text{A.52})$$

This enables us to deal only with the symmetric tensors, since the non-symmetric part can be contracted to obtain a lower order symmetric tensor.⁸ Symmetric tensor of order k has $k + 1$ independent components⁹. Thus $SU(2)$ can have any dimensional representation. As we will see, this is not the case in general.

To find out, how the algebra represents, we express $g_{ij} = \delta_{ij} + i\theta_a \left(\frac{\sigma_a}{2}\right)_{ij}$ for infinitesimal θ and find

$$\begin{aligned} \Phi'_{i_1 \dots i_k} &= g_{i_1 j_1} \dots g_{i_k j_k} \Phi_{j_1 \dots j_k} = \\ &= \Phi_{i_1 \dots i_k} + i\theta_a \left(\frac{\sigma_a}{2}\right)_{i_1 j_1} \Phi_{j_1 i_2 \dots i_k} + \dots + i\theta_a \left(\frac{\sigma_a}{2}\right)_{i_k j_k} \Phi_{i_1 \dots i_{k-1} j_k}. \end{aligned} \quad (\text{A.53})$$

We also should get

$$\Phi'_{i_1 \dots i_k} = (1 + iJ_a \theta_a) \Phi_{i_1 \dots i_k} \quad (\text{A.54})$$

for some generators J_a of the representation, which yields

$$J_a \Phi_{i_1 \dots i_k} = \left(\frac{\sigma_a}{2}\right)_{i_1 j_1} \Phi_{j_1 i_2 \dots i_k} + \dots + \left(\frac{\sigma_a}{2}\right)_{i_k j_k} \Phi_{i_1 \dots i_{k-1} j_k}. \quad (\text{A.55})$$

Namely for J_3 the possible values of $\sigma_3/2$ acting on the tensor are $\pm \frac{1}{2}$, giving the possible resulting eigenvalues of J_3 from $-k/2$ to $k/2$. If we denote this number j , we get for the dimension of the representation $2j + 1$. This is the usual angular momentum representation of $SU(2)$.

We will now introduce the Casimir operators, which will turn out to be very useful to analyze the representations. The Lie algebra $su(2)$ is a vector space spanned by the generators $\sigma_a/2 = t_a$. The definition and the properties of Casimir operators are however independent for any representation of any Lie algebra. Using these generators, we can get more matrices

$$I_{2j+1}, t_{a_1}, t_{a_1} t_{a_2}, t_{a_1} t_{a_2} t_{a_3}, \dots, t_{a_1} \dots t_{a_{2j}}. \quad (\text{A.56})$$

⁸Remember how this worked for the case of 2-tensors. Using the ε tensor, we got the anti-symmetric part c^A , leaving the symmetric part c^S .

⁹Remember that the indexes can have values 1, 2 and due to the symmetry, all ones can be brought to the left side.

We know, that for a product of more than k generators we get a matrix, that is a linear combination of previous ones. The matrices (A.56) span a new algebra, which is in general different than the starting Lie algebra, since the products need not to be its elements. This algebra is called the enveloping algebra.

Definition. Elements of the enveloping algebra, that commute with all the generators t_a , and thus with all the elements of the Lie algebra, are called Casimir operators, i.e.

$$[C_\alpha, t_a] = 0 \quad \forall t_a \quad (\text{A.57})$$

We stress once again that in general, these operators are not elements of the Lie algebra.

Theorem A.5.3. *The number of linearly independent Casimir operators is given by the rank of the algebra.*

For $su(2)$, we have one Casimir operator, namely $J^2 = J_1J_1 + J_2J_2 + J_3J_3$. We know, that there is no other such operator.

As a further example let us show the reduction of the $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ product representation reduces. First we observe that $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = (\mathbf{1} \oplus \mathbf{3}) \times \mathbf{2} = \mathbf{2} \oplus (\mathbf{3} \otimes \mathbf{2})$. The element of $\mathbf{3} \otimes \mathbf{2}$ is $\phi_{ij,k} = \phi_{ji,k}$. According to our theorem, we should contract with the invariant tensor, which is ε_{ij} . There is now point in contracting with first two indexes, since they are symmetric and this contraction would vanish anyway. So we $\phi_{ij,k}\varepsilon_{jk}$ has only one index and transforms as $\mathbf{2}$ representation. What is left then is the symmetric part, which has three symmetric indexes and therefore is an irreducible representation $\mathbf{4}$. Finally

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4}. \quad (\text{A.58})$$

As expected we got the formula for composition of three spins.

A.6 The explicit construction of the representations of $SU(3)$

The defining representation of $SU(3)$ is formed by 3×3 unitary matrices with determinant equal to 1

$$g^\dagger g = I_3 \quad , \quad \det g = 1 \quad (\text{A.59})$$

Since g is unitary, it can be written as e^{ih} with h hermitian and traceless

$$h^\dagger = h \quad , \quad \text{Tr } h = 0. \quad (\text{A.60})$$

Space of these matrices is therefor 8 dimensional, as is the group itself. A general element of the Lie algebra $su(3)$ can be written as $h = \theta_a \frac{\lambda_a}{2}$, with $a = 1, \dots, 8$ and λ_a the Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , & \lambda_8 &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .
\end{aligned} \tag{A.61}$$

The defining commutation relation can be computed to be

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = i f^{abc} \frac{\lambda_c}{2}. \tag{A.62}$$

The explicit form of f^{abc} is rarely needed. It can also be seen that one can pick up to two mutually commuting Gell-Mann matrices, which yields 2 as the rank of the algebra. Therefore, this algebra has two invariant tensors and two Casimir operators.

In the defining representation of the $SU(3)$ we have $T^a = \lambda^a/2$. It is three dimensional and given by

$$\phi'_i = g_{ij} \phi_j \quad , \tag{A.63}$$

with $i, j = 1, 2, 3$. There is however one more three dimensional representation, given by

$$T_a = -\frac{\lambda_a^\dagger}{2}. \tag{A.64}$$

These are easily checked to follow the commutation relation of $su(3)$ algebra (A.62). We see, that at the group level

$$e^{iT_a \theta_a} = e^{-i \frac{\lambda_a^\dagger}{2} \theta_a} = g^* \quad , \tag{A.65}$$

thus this representation is given by $g \rightarrow g^*$.¹⁰ We should ask the question of (ir)reducibility of this representation. This is answered affirmatively.

¹⁰In this expression, the first g is understood as an abstract element of the abstract $SU(3)$ group, the second g as a matrix in the defining representation.

However for $SU(2)$ we do not get a truly new representation. We compute get

$$\begin{aligned} g &= e^{i\theta_a \frac{\sigma_a}{2}} = e^{i\theta_1 \frac{\sigma_1}{2} + i\theta_2 \frac{\sigma_2}{2} + i\theta_3 \frac{\sigma_3}{2}} \quad , \\ g^* &= e^{-i\theta_a \frac{\sigma_a^\dagger}{2}} = e^{-i\theta_1 \frac{\sigma_1}{2} + i\theta_2 \frac{\sigma_2}{2} - i\theta_3 \frac{\sigma_3}{2}}. \end{aligned} \quad (\text{A.66})$$

Now using $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, $\sigma_3\sigma_2 = -\sigma_2\sigma_3$ we get

$$g^* = \sigma_2 e^{-i\theta_1 \frac{\sigma_1}{2} + i\theta_2 \frac{\sigma_2}{2} - i\theta_3 \frac{\sigma_3}{2}} \sigma_2 = \sigma_2^\dagger g \sigma_2 \quad , \quad (\text{A.67})$$

which is a similarity transformation and therefor g and g^* representations of $SU(2)$ are equivalent. This property is called pseudo-reality, since an element and its complex conjugate are not the same, but are equivalent.¹¹

However for $SU(3)$, the $g \rightarrow g^*$ can not be expressed as a linear transformation and we get different representations, which we denote

$$\begin{aligned} \mathbf{3} \quad \phi'_i &= \mathbf{g}_{ij} \phi_j \\ \mathbf{3}^* \quad \chi'^i &= \mathbf{g}^{ij} \chi^j \end{aligned} \quad (\text{A.68})$$

and we will distinguish between the parts transforming according one or another representation by the upper/lower position of the corresponding index. Also we have denoted $g^{ij} \equiv (g^*)_{ij}$. This way a general tensor of rank (p, q) is a quantity, which transforms under the action of the group it the following fashion

$$C'^{j_1 \dots j_p}_{i_1 \dots i_p} = g^{j_1 k_1} \dots g^{j_p k_p} g_{j_1 l_1} \dots g_{i_q l_q} C^{j_1 \dots j_p}_{i_1 \dots i_p}. \quad (\text{A.69})$$

One can easily check, that two consecutive translation give the expected result $C' = (g^{(1)} g^{(2)}) C$.

We now need to look for the invariant tensors. We know, that there are going to be two such tensors. By inspection we find that δ_i^j is invariant

$$\delta_i'^j = g^{jk} g_{il} \delta_l^k = g_{ik} (g^\dagger)^{kj} = (g g^\dagger)_i^j = \delta_i^j \quad (\text{A.70})$$

and the complete anti-symmetric Levi-Chivita tensor ε_{ijk} or ε^{ijk} is invariant

$$\varepsilon'^{ijk} = g_{il} g_{jm} g_{in} \varepsilon^{kmn} = \det g \varepsilon^{ijk} = \varepsilon^{ijk}. \quad (\text{A.71})$$

Note, that the first fact is the consequence of the unitarity of the group and the second of the specialty of the group. We need not to look any further and we know we have found all. By the theorem A.5.1 we also know, that we can get all the irreducible representation from the reduction of tensor representations by contractions using these invariant tensors.

¹¹This fact can be traced back to the anti-commutation rules for Pauli matrices and to the fact, that $su(2)$ forms a Clifford algebra with the anti-commutator $\{\sigma^i, \sigma^j\} = \delta^{ij}$.

We are now going to do this and characterize the irreducible representations of the $SU(3)$.

Starting with two index object C_{ij} . We contract with ε to obtain $C_{ij}\varepsilon^{ijk}$ and inspect the transformation of this object.

$$(C_{ij}\varepsilon^{ijk})' = \varepsilon^{ijk} g_{ip}g_{jq}C_{pq} = g_{kn}^* (C_{pq}\varepsilon^{pqn}) \quad , \quad (\text{A.72})$$

where we have used $\varepsilon_{ijk}g_{ip}g_{jq} = \varepsilon_{pqn}g_{kn}^* \equiv \varepsilon_{pqn}g^{kn}$ which follows from $\varepsilon_{ijk}g_{ip}g_{jq}g_{kn} = \varepsilon_{pqn}$ and the unitarity of g . The quantity $C_{ij}\varepsilon^{ijk}$ therefor transforms according to the $\mathbf{3}^*$ representation and therefore deserves an upper index. So it is convenient to write

$$\phi_{ij}\varepsilon^{ijk} = \chi^k. \quad (\text{A.73})$$

Here we can see why δ_i^j was not among invariant tensors in the case of $SU(2)$, as it is for case of any higher rank special unitary group, there was nothing to contract with only lower indexes.

We therefor take C_i^j and contract with δ , which gives for $c = C_i^j\delta_i^j$

$$c' = g^{jm}g_{in}C_n^m\delta_i^j = (g^\dagger g)_n^m C_n^m = c \quad , \quad (\text{A.74})$$

which transforms as a scalar, and therefor rightfully got no indexes. Using the δ tensor, we can always bring tensor of rang (p, q) to tensor of rank $(p - 1, q - 1)$.

Therefore the irreducible tensors¹² give zero after a contraction between any two upper and lower indexes. They are also completely symmetric in all upper and in all lower indexes, since any anti-symmetry can be contracted with ε tensor to a new lower or upper index respectively. Any symmetric pair of indexes does not allow for contraction with ε , since this contraction would be zero.

So in general, the irreducible tensors are those, which give zero in any contraction with all the invariant tensors.

What is the dimension of space of completely symmetric traceless tensors? The symmetric part has dimension¹³

$$\frac{(p+1)(p+2)}{2} \times \frac{(q+1)(q+2)}{2} \quad (\text{A.76})$$

¹²Tensors, that transform according to an irreducible representation.

¹³All the ones can be put to the left, all the threes to the right, so we need to count all the possibilities of

$$\underbrace{1 \dots 1 | 2 \dots 2 | 3 \dots 3}_p \quad (\text{A.75})$$

which is as picking a place, where ones become twos and where twos become threes, i.e. picking two of $p + 1$ elements (the spaces between indexes).

however we need to subtract the dimension of traceless tensors, which is the same as dimension of $(p-1, q-1)$ symmetric, thus $(p+1)p(q+1)p/4$. The dimension of (p, q) representation of $SU(3)$ is therefor

$$\frac{1}{2}(p+1)(q+1)(p+q+2). \quad (\text{A.77})$$

Some simple cases

- First we consider the case $(p, q) = (0, 0)$. Using (A.75) this representation turns out to be one dimensional, i.e. a trivial representation $g \rightarrow id$ with transformation $\phi \rightarrow \phi$. For $(p, q) = (0, 1)$ we get the 3 dimensional representation $\mathbf{3}$ with transformation $\phi'_i = g_{ij}\phi_j$ and thus the defining representation. For $(p, q) = (1, 0)$ we get also a 3 dimensional representation, with transformation rule $\chi^i = g^{ij}\chi^j$, which is the $\mathbf{3}^*$ representation.
- The first new case comes with $(p, q) = (1, 1)$. The dimension of such representation is 8 and we call it the adjoint representation. Here vectors ϕ_j^i are by construction 3×3 traceless matrices and therefor are a linear combination of Gell-Mann matrices. The transformation rule becomes

$$\begin{aligned} \phi_j^i &= g_{jl}\phi_l^k (g^\dagger)^{ki} \\ \phi' &= g\phi g^\dagger \end{aligned} \quad (\text{A.78})$$

and under the group action the matrix ϕ undergoes the conjugation. We see, that this is a representation of the group on itself, since here ϕ 's are elements of the group.

Any matrix ϕ can be written as

$$\phi = \sqrt{2}\phi^a \frac{\lambda^a}{\sqrt{2}} \quad , \quad (\text{A.79})$$

where we have chosen the factor $\sqrt{2}$ to get a normalization $Tr(\phi\phi) = \phi^a\phi^a$. In (A.78), we write the group elements as $e^{i\theta^a T^a}$ and take an infinitesimal θ 's to obtain

$$\begin{aligned} \sqrt{2}\phi'^a \frac{\lambda^a}{\sqrt{2}} &= \sqrt{2}\phi^a \frac{\lambda^a}{\sqrt{2}} - i\theta^b \underbrace{\left[\frac{\lambda^b}{\sqrt{2}}, \frac{\lambda^c}{\sqrt{2}} \right]}_{if^{abc} \frac{\lambda^a}{\sqrt{2}}} \sqrt{2}\phi^c \\ \phi'^a &= \phi^a + f^{abc}\theta^b\phi^c \end{aligned} \quad (\text{A.80})$$

The quantity $f^{abc}\theta^b$ is a 8×8 matrix¹⁴ and previous equation can be written as

$$\phi' = (I_8 + iT^b\theta^b)\phi \quad , \quad (\text{A.81})$$

¹⁴Having two indexes both ranging from 1 to 8.

with matrix ϕ understood as a column vector of its independent components and eight matrices T defined as $(T^b)_{ac} = -if^{abc}$. This, however was only an infinitesimal transformation, the finite transformation given by

$$\phi' = e^{iT^b\theta^b} \phi. \quad (\text{A.82})$$

We can see that this representation is defined purely by the structure constants, that have defined the algebra. The dimension of T 's is given by the number of the indexes in f^{abc} and therefor the adjoint representation has the same dimension as the group itself. To check, that T 's form the 8 dimensional representation, we need to check that

$$[T^a, T^b] = if^{abc}T^c. \quad (\text{A.83})$$

When we write down the commutator as $T^aT^b - T^bT^a$, use the definition of T 's involving f^{abc} , we find out, that this is a consequence of the Jacobi identity for the original Lie algebra, which states

$$[[t^a, t^b], t^c] + [[t^b, t^c], t^a] + [[t^c, t^a], t^b] = 0. \quad (\text{A.84})$$

One can check, that the same procedure for $SU(2)$ yields the infinitesimal transformations $\sigma'_a/2 = R_c^a(\theta)\sigma^c/2$, where R is the infinitesimal rotation matrix (2.3). Thus, the adjoint representation of the $SU(2)$ group are the usual 3D rotations.

Note, that every group has a representation like this, since we can always define $g\phi g^{-1}$.

- For further irreducible representations we get $(p, q) = (2, 0)$, $(p, q) = (0, 2)$, which are both 6 dimensional and are labeled as **6** and **6***. Similarly $(p, q) = (3, 0)$, $(p, q) = (0, 3)$ are **10** and **10***. For $(2, 2)$ we get **27**. And so on.
- Note that (p, q) and (q, p) representations are related. They have the same dimension and

Appendix B

Problems and Exercises

This section includes the problems, that have been assigned as homework problems or have been a part of (take home) final exam, when the material was taught as a course. Some of the problems are ment as excerscises to gain deeper understanding of the topics discusses in the main text, some of the problems are more or less independent of the text and give a glipse of some other interesting topics that could not have been discussed.

Order of the problems follows the order in which the material is presented in the text, with additional topics following afterwards.

Problem 1. We consider the shell model of the nucleus described as an isotropic three-dimensional harmonic oscillator. Obtain the degeneracies of the lowest 4 states. By the exclusion principle, we can put two protons (because the proton has spin $\frac{1}{2}$) in each orbital state. Similarly, we can put two neutrons for each orbital state. Obtain the atomic numbers and atomic weights of nucleii with filled shells for the lowest four cases. These should give the lowest "magic numbers" for which there is exceptional stability for the nucleus; the higher ones require considerations of spin-orbit interaction in the nucleus, which we shall not discuss here.

Problem 2. The Hamiltonian for a charged particle (of electric charge e) moving in the field of a magnetic monopole of magnetic charge g (located at the origin) can be written as

$$H = \frac{\Pi_i \Pi_i}{2m} \quad (\text{B.1})$$

where $\Pi_i = p_i - eA_i$. The basic commutation rules in terms of these variables are

$$\begin{aligned} [x_i, \Pi_j] &= i \delta_{ij} \\ [\Pi_i, \Pi_j] &= i eg \epsilon_{ijk} \frac{x_k}{r^3} \end{aligned} \quad (\text{B.2})$$

We define the angular momentum operator

$$J_i = \epsilon_{ijk} x_j \Pi_k - c \frac{x_i}{r} \quad (\text{B.3})$$

Determine c from the required commutation rules

$$\begin{aligned} [J_i, x_j] &= i\epsilon_{ijk}x_k \\ [J_i, \Pi_j] &= i\epsilon_{ijk}\Pi_k \end{aligned} \tag{B.4}$$

What is the condition on eg if we require $\hat{x} \cdot J$ to be quantized in units of half-an-integer? (This condition is the famous Dirac quantization condition for magnetic charges.)

Problem 3. The Landau problem on the sphere of radius r refers to the motion of a charged particle on a sphere with a constant radial magnetic field B . (We may think of a magnetic monopole sitting at the center of the sphere.) Since any motion on a sphere is an angular rotation, we can discuss this in terms of two sets of angular momentum operators L_i, R_i , $i = 1, 2, 3$, with $[L_i, R_j] = 0$. R_i correspond to the usual $p - eA$'s, the L_i correspond to the magnetic translations. The Hamiltonian is given by

$$H = \frac{R_1^2 + R_2^2}{2mr^2}$$

The magnetic field enters in the rule that the eigenvalue of R_3 must be $n = 2eBr^2$. The other constraint is that we must only consider states with $L^2 = R^2$.

- a) Calculate the spectrum of H including the degeneracy.
- b) In the limit of large spheres, $r \rightarrow \infty$, what is the limit of the energy eigenvalues and the degeneracy per unit area? Compare your results with those on the plane.

Problem 4. As I have discussed in class, the $SU(3)$ -algebra can be given in terms of the operators

$$T^\alpha = a_i^\dagger t_{ij}^\alpha a_j$$

where $\alpha = 1, 2, \dots, 8$ and $i, j = 1, 2, 3$, and there is summation over i, j in the above formula. The quadratic Casimir operator, the analogue of J^2 for angular momentum, is given by $C = \sum_\alpha T^\alpha T^\alpha$. Calculate C in terms of $n = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3$. I have given explicitly the (3×3) -matrices t^α , but you do not need that; you can use the completeness relation for (3×3) -matrices,

$$\sum_\alpha t_{ij}^\alpha t_{mn}^\alpha = \frac{1}{2}\delta_{in}\delta_{jm} - \frac{1}{6}\delta_{ij}\delta_{mn}$$

The above completeness relation uses the normalization $\text{Tr}(t^\alpha t^\beta) = \delta^{\alpha\beta}$.

Problem 5. We define the operators

$$\Lambda_1 = \frac{r}{2}(\vec{p} \cdot \vec{p} - 1), \quad \Lambda_2 = \frac{r}{2}(\vec{p} \cdot \vec{p} + 1), \quad \Lambda_3 = \vec{x} \cdot \vec{p} - i$$

Show that these operators obey the algebra

$$[\Lambda_i, \Lambda_j] = -i\epsilon_{ijk}\Lambda_k$$

(This is the $SO(2,1)$ -algebra.) The Hamiltonian for the Hydrogen atom (or Kepler problem) can be written as

$$H = \frac{\vec{p} \cdot \vec{p}}{2m} - \frac{\kappa}{r}$$

Express this in terms of the Λ_i .

Problem 6. Consider the λ^a matrices of $SU(3)$. Show that the matrices $\frac{1}{2}\lambda^a$, $a = 1, 2, 3$, form an $SU(2)$ subalgebra. What is the j -value of the representation defined by these? Also show that $\lambda^2, -\lambda^5, \lambda^7$ form an $SO(3)$ subalgebra. What is the spin of the representation of $SO(3)$ defined by these matrices? The definition of these matrices is given below.

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

Problem 7. Use tensor methods to reduce to a sum of irreducible representations the following products for representations of $SU(3)$: $\mathbf{6} \otimes \mathbf{3}^*$, $\mathbf{6}^* \otimes \mathbf{3}^*$ and $\mathbf{8} \otimes \mathbf{3}$.

Problem 8. The $SU(2)$ Kac-Moody algebra is given by a set of operators J_n^a , $a = 1, 2, 3$, $n = 0, \pm 1, \pm 2, \dots$ which obey the commutation rules

$$[J_m^a, J_n^b] = i\epsilon^{abc}J_{m+n}^c + K m \delta^{ab}\delta_{m+n,0} \quad (\text{B.5})$$

m, n are integers, K is a constant. From these rules, notice that J_{-n} , $n > 0$ behave a bit like creation operators and J_n , $n > 0$ behave like annihilation operators. So we define a vacuum state which obeys $J_n^a|0\rangle = 0$, $n \geq 0$. Define a current operator

$$J^a(z) = \sum_{n=-\infty}^{\infty} \frac{J_n^a}{z^{n+1}} \quad (\text{B.6})$$

Calculate the two-point function $\langle 0|J^a(z)J^b(w)|0\rangle$. (This problem arises in conformal field theory and classification of second order phase transitions.)

Problem 9. Consider a three-dimensional current algebra Hamiltonian given by

$$H = \frac{1}{2} \int d^3x [J_0(x)J_0(x) + J_i(x)J_i(x)] \quad (\text{B.7})$$

where the currents obey

$$\begin{aligned} [J_0(\vec{x}, t), J_0(\vec{y}, t)] &= 0 \\ [J_0(\vec{x}, t), J_i(\vec{y}, t)] &= i \frac{\partial}{\partial x^i} \delta^{(3)}(x - y) \\ [J_i(\vec{x}, t), J_j(\vec{y}, t)] &= 0 \end{aligned}$$

The J 's generate some quasiparticle excitations; if A^\dagger is the creation operator for such a quasiparticle, it should obey the 'eigenvalue' equation

$$[H, A^\dagger] = E A^\dagger \quad (\text{B.8})$$

where E is the energy. Assume A^\dagger is of the form

$$A^\dagger = \int d^3x f(x)J_0(x) + \xi_i(x)J_i(x) \quad (\text{B.9})$$

Use this in the eigenvalue equation to find possible solutions for $f(x)$, $\xi_i(x)$ and obtain the eigenvalues as well. What is the mass of the excitation? (This problem occurs in problems involving spontaneous symmetry breaking and Goldstone bosons, e.g., the current algebra treatment of pi-mesons.)

Problem 10. We consider the density $\rho(\vec{x}, t)$ and the velocity field $v_i(\vec{x}, t)$. The Hamiltonian is given by

$$H = \int d^3x \left[\frac{1}{2} \rho v_i v_i + U(\rho) \right]$$

where $U(\rho)$ is some potential energy which depends on the density. If the commutation rules are taken as

$$\begin{aligned} [\rho(\vec{x}, t), \rho(\vec{y}, t)] &= 0 \\ [v_i(\vec{x}, t), \rho(\vec{y}, t)] &= i \frac{\partial}{\partial x^i} \delta^{(3)}(x - y) \\ [v_i(\vec{x}, t), v_j(\vec{y}, t)] &= \frac{i}{\rho(\vec{x})} (\partial_i v_j - \partial_j v_i)(\vec{x}, t) \delta^{(3)}(x - y) \end{aligned}$$

one can calculate the equations of motion using the Heisenberg equation

$$i \frac{\partial A}{\partial t} = [A, H]$$

Show that one can reproduce the equations of motion for a perfect fluid. Obtain also the commutator of $\int d^3x \rho(\vec{x}, t)$ with density and velocity.

Problem 11. Consider functions $f(x, p)$ and $h(x, p)$ which are functions of the usual classical variables x^i and p^i , $i = 1, 2, 3$. These are not operators. Define a new kind of product, called the star product which depends on a parameter \hbar , of two functions by

$$(f * h)(x, p) = f \left(x^i + \frac{i\hbar}{2} \frac{\partial}{\partial p^i}, p^i - \frac{i\hbar}{2} \frac{\partial}{\partial x^i} \right) h(x, p)$$

This can be evaluated by expanding the first term on the right in a Taylor series around x, p . From this definition, obtain $x^i * x^j$, $x^i * p^j$, $p^i * p^j$, all to first order in \hbar . Calculate also $x^i * x^j - x^j * x^i$, $x^i * p^j - p^j * x^i$, $p^i * p^j - p^j * p^i$ and show that the star algebra reproduces the Heisenberg commutation rules. (This is the basis of the so-called deformation quantization.)

Problem 12. Consider the operators

$$\begin{aligned} J_3 &= z \frac{\partial}{\partial z} - \frac{n}{2} \\ J_+ &= -z^2 \frac{\partial}{\partial z} + nz \\ J_- &= \frac{\partial}{\partial z} \end{aligned}$$

where z is a complex variable and n is a positive integer. Show that these obey the commutation rules of angular momentum. The states can be taken as analytic functions of z . The operators J_+ and J_- do not seem to be adjoints of each other as given. If f and h are two functions corresponding to two states, assume that the scalar product is of the form

$$\langle h|f \rangle = \int dzd\bar{z} \rho(z, \bar{z}) \bar{h}f$$

(Keep in mind that f depends only on z and \bar{h} depends only on \bar{z} .) $\rho(z, \bar{z})$ is a weight function for the scalar product. Require the self-adjointness conditions

$$\begin{aligned} \int dzd\bar{z} \rho(z, \bar{z}) \overline{J_+ h} f &= \int dzd\bar{z} \rho(z, \bar{z}) \bar{h} (J_- f) \\ \int dzd\bar{z} \rho(z, \bar{z}) \overline{J_3 h} f &= \int dzd\bar{z} \rho(z, \bar{z}) \bar{h} (J_3 f) \end{aligned}$$

Use this to calculate $\rho(z, \bar{z})$, up to a multiplicative constant. What is the highest power of z allowed in a function f of finite norm? (This problem is related to the coherent state representation of the angular momentum algebra.)

Problem 13. Consider an infinitesimal Lorentz transformation along the x^1 -direction defined by

$$x'_1 = x_1 + vx_0, \quad x'_0 = x_0 + vx_1, \quad x'_2 = x_2, \quad x'_3 = x_3$$

Write this as $x'_\mu = x_\mu + v(I_1)_{\mu\nu}x_\nu$, where $\mu, \nu = 0, 1, 2, 3$. Identify I_1 . Calculate the matrix $\exp(\theta I_1)$ by expanding the exponential. Show that

$$x'_\mu = [\exp(\theta I_1)]_{\mu\nu} x_\nu$$

reproduces a finite Lorentz transformation if $\tanh \theta = v$. Identify similarly I_2 and I_3 . Obtain the commutator $[I_i, I_j]$ and show that it generates a spatial rotation. (This is crucial in understanding the Thomas precession of spin which in turn is important for the gyromagnetic ratio and spin-orbit interaction for relativistic electrons.)

Problem 14. The canonical commutation rules for a scalar field are

$$\begin{aligned}\phi(\vec{x}, t) \phi(\vec{y}, t) - \phi(\vec{y}, t) \phi(\vec{x}, t) &= 0 \\ \phi(\vec{x}, t) \pi(\vec{y}, t) - \pi(\vec{y}, t) \phi(\vec{x}, t) &= i \delta^{(3)}(x - y) \\ \pi(\vec{x}, t) \pi(\vec{y}, t) - \pi(\vec{y}, t) \pi(\vec{x}, t) &= 0\end{aligned}$$

(You do not need to know more field theory for this problem.) The Hamiltonian is given by

$$H = \int d^3x \frac{1}{2} [\pi^2 + (\nabla\phi)^2 + m^2\phi^2]$$

(All operators in the brackets are at the same point \vec{x} .) Let

$$A^\dagger = \int d^3x [u(\vec{x}) \phi(\vec{x}) + v(\vec{x}) \pi(\vec{x})]$$

Impose the equation

$$[H, A^\dagger] = E A^\dagger \tag{B.10}$$

for some constant “eigenvalue” E . Solve for u, v and identify the eigenvalues E . Notice that if we have (B.10), and $|0\rangle$ is the vacuum state with $H|0\rangle = 0$, then $A^\dagger|0\rangle$ is an eigenstate of the Hamiltonian.

Problem 15. Define four γ -matrices by

$$\begin{aligned}\gamma_1 &= (a + a^\dagger), & \gamma_2 &= i(a - a^\dagger) \\ \gamma_3 &= (b + b^\dagger), & \gamma_4 &= i(b - b^\dagger)\end{aligned}$$

where a^\dagger, a and b^\dagger, b are two independent sets of fermionic creation and annihilation operators.

a) Show that these obey the Clifford algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu}$$

(This is the same set of conditions as the algebra of γ -matrices of the Dirac equation.)

- b) There are only four states possible for this set, namely, $|0\rangle$ (defined by $a|0\rangle = b|0\rangle = 0$), $a^\dagger|0\rangle$, $b^\dagger|0\rangle$ and $a^\dagger b^\dagger|0\rangle$. Obtain γ_μ as matrices acting on these states. This will give you one solution for the Dirac matrices.

Problem 16. The Hamiltonian for a charged particle (of electric charge e) moving in the field of a magnetic monopole of magnetic charge g (located at the origin) can be written as

$$H = \frac{\Pi_i \Pi_i}{2m} + \frac{\lambda}{2r^2} \quad (\text{B.11})$$

where $\Pi_i = p_i - eA_i$, and we have also added a separate potential proportional to r^{-2} . The basic commutation rules in terms of these variables are

$$[x_i, \Pi_j] = i \delta_{ij}, \quad [\Pi_i, \Pi_j] = i eg \epsilon_{ijk} \frac{x_k}{r^3} \quad (\text{B.12})$$

(This part is common with an earlier assigned problem.) Define the operators

$$D = tH - \frac{1}{4}(x \cdot \Pi + \Pi \cdot x), \quad K = -t^2 H + 2tD + \frac{m}{2} r^2 \quad (\text{B.13})$$

- a) Obtain the closed algebra of H , K and D .

- b) Define

$$M_1 = D, \quad M_2 = \frac{1}{2}(H - K), \quad M_3 = \frac{1}{2}(H + K) \quad (\text{B.14})$$

Rewrite the algebra of H , K , D you found in part a) using these.

- c) Show that $M_1^2 + M_2^2 - M_3^2$ commutes with all M_i . (In other words, it is a Casimir operator.)

Problem 17. In class I obtained the ground state of the BCS superconductor as

$$|G\rangle = \prod_k (\alpha_k + \beta_k c_{-k}^\dagger b_k^\dagger |0\rangle)$$

where $\alpha_k = \sin \theta_k$ and $\beta_k = \cos \theta_k$. I have also given the formulae relating these to the gap Δ . We define the electric charge operator as

$$Q = -e \sum_k (c_{-k}^\dagger c_{-k} + b_k^\dagger b_k)$$

We also define the Ginzburg-Landau field (or the field representing the Cooper pair) as

$$\Phi^* = c_{-k}^\dagger b_k^\dagger$$

Calculate the expectation values $\langle G|Q|G\rangle$ and $\langle G|\Phi^*|G\rangle$. (These will establish that the state $|G\rangle$ spontaneously breaks the $U(1)$ invariance of electromagnetic interactions.)

Problem 18. To a very good approximation, the nuclear forces binding protons and neutrons into nuclei are independent of spin and isospin. (Isospin is the symmetry between protons and neutrons for nuclear forces.) This means that we can take the four states $|p \uparrow\rangle$, $|p \downarrow\rangle$, $|n \uparrow\rangle$ and $|n \downarrow\rangle$ to transform as the fundamental 4-dimensional representation of the group $SU(4)$. (This enlarged symmetry idea is due to Wigner and is called the Wigner $SU(4)$.) By the Pauli principle, the states you can obtain corresponding to the same orbital state in the nuclear potential must be antisymmetric. Consider nuclei with 3 nucleons (either protons or neutrons). Identify the antisymmetric representation obtained by reducing $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4}$ of $SU(4)$. Physically what are these nuclei? (This will illustrate the point. The real use of this symmetry is for nuclei of much higher atomic weights, for which this method can be used to classify all the nuclear isobars.)

Problem 19. Consider a charged particle which is in an isotropic harmonic oscillator potential in three dimensions, so that the symmetry group classifying the states is $SU(3)$. In this case, the states must all be symmetric, so that the lowest states transform as the $\mathbf{1}$, $\mathbf{3}$, $\mathbf{6}$ and $\mathbf{10}$ of $SU(3)$. An electric field is now applied in the z -direction, so that the symmetry of the Hamiltonian is reduced to the $SU(2)$ subgroup corresponding to the x and y directions. Without doing dynamical calculations, obtain the splitting structure of energy levels under this perturbation, by decomposing the above four $SU(3)$ representations into $SU(2)$ irreducible representations.

Problem 20. The low energy dynamics of pseudoscalar mesons can be described by the Lagrangian

$$L = \frac{1}{2} \text{Tr} [\partial_\mu \Phi \partial_\mu \Phi] - \frac{1}{2} \text{Tr} [\Phi^2 M] \quad (\text{B.15})$$

where M is of the form

$$M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad (\text{B.16})$$

This form incorporates the fact that $SU(3)$ is only an approximate symmetry. The matrix Φ was given in class as

$$\Phi = \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{bmatrix} \quad (\text{B.17})$$

where each letter represents the field corresponding to the particle of the same name. Obtain the possible mass-squared relations among the mesons. (Notice that M represents the square of the mass.)

Problem 21. The Laplace operator on a two-sphere (S^2) is given by L^2 where L_i is the angular momentum operator. This is seen by writing $-\nabla^2$ in spherical coordinates and then setting

the radius r to be a constant, say R . This means that the eigenvalues of the Laplacian on the two-sphere are $l(l+1)$ with degeneracy $2l+1$. A similar analysis can be done for many other spaces, giving a purely algebraic way of solving the Laplace equation.

For this problem, we will consider the three-sphere (S^3) given by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$. This has a symmetry given by $SU(2) \times SU(2)$, i.e., two independent $SU(2)$'s, with generators M_i and N_i respectively. The derivative operator can be taken as $K_i = \frac{1}{2}(M_i - N_i)$ and the spin of the particle on which the Laplacian acts is $S_i = M_i + N_i$. (Spin makes a difference to the eigenvalues of the Laplacian because the curvature of S^3 couples to the spin, something you may have learnt in the course on General Relativity.)

- a) Obtain the commutators $[K_i, K_j]$, $[S_i, K_j]$ and $[S_i, S_j]$.
- b) Taking the Laplacian as $K^2 = K_i K_i$, obtain the eigenvalues and degeneracy of the Laplacian acting on a field of spin $s = 0, \frac{1}{2}$ (While this can be done for all fields of all spins, for this problem it is sufficient to consider spins zero (i.e., a scalar field) and half (i.e., a spin- $\frac{1}{2}$ field like the electron).) This can be done by starting with two independent representations of the two $SU(2)$'s and then combining them to form states corresponding to s .

Problem 22. The Korteweg-de Vries (KdV) equation describes a variety of phenomena in physics including shallow water waves traveling in one dimension. It has soliton solutions and historically was the first example of a soliton. The equation is given by

$$\frac{\partial}{\partial t} U = -\frac{\partial^3}{\partial x^3} U + 6U \frac{\partial U}{\partial x} \quad (\text{B.18})$$

where $U(x, t)$ is the height of the wave. We consider two distinct sets of commutation rules.

Commutation rule A:

$$[U(x, t), U(y, t)] = 3U(y, t) \frac{\partial}{\partial x} \delta(x - y) - \frac{\partial^3}{\partial x^3} \delta(x - y) \quad (\text{B.19})$$

Commutation rule B:

$$[U(x, t), U(y, t)] = \frac{\partial}{\partial x} \delta(x - y) \quad (\text{B.20})$$

- a) Find the Hamiltonians H_A and H_B to be used with the commutation rules (A) and (B) respectively, so as to get the KdV equation as the Heisenberg equation of motion. (The fact that there are two CR's and two Hamiltonians is related to the integrability of the theory.)
- b) Show that the quantity $\int dx U$ is conserved, in addition to the two Hamiltonians.

c) Show that the KdV can be written also as

$$\begin{aligned}\frac{\partial L}{\partial t} &= [L, B] \\ L &= -D^2 + U(x, t) \\ B &= 4D^3 - 3U D - 3D U\end{aligned}\tag{B.21}$$

where D is the operator $\partial/\partial x = ip$. Notice that L is a Schrödinger operator with a time-dependent potential $U(x, t)$.

d) From the above result, show that the spectrum of L is unchanged when the potential $U(x, t)$ changes in time according to the KdV.