Introduction to string theory
Homework 2

Feel free to direct any questions to
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Príklad 1 (Lecture leftovers)**.** Some things appeared in the lecture with the comment that if a person does this like this, then one gets that. Your task will be to actually do these things.

- Using the Polyakov action
$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$
by explicitly varying the metric and using
$$g^{\alpha\beta}g_{\alpha\beta} = 2 \ \Rightarrow\ \delta g^{\alpha\beta}\, g_{\alpha\beta} = -g^{\alpha\beta}\delta g_{\alpha\beta} \ \Rightarrow\ \delta g = -g\, g_{\alpha\beta}\delta g^{\alpha\beta}$$
find the equation of motion for $g_{\alpha\beta}$. Show that using it, this action leads back to the Nambu-Goto action.

- Substitute the solutions $X^\mu_{L/R}(\sigma^\pm)$ into the condition $(\partial_\pm X)^2 = 0$ and derive the conditions for the coefficients $\alpha^\mu_n, \tilde{\alpha}^\mu_i$. Also show what relationship these coefficients must satisfy if the solution X is to be always real.

- From the conditions for L_0 and $\tilde{L}_0$ derive the relationship for the mass of the (excited) string.

Príklad 2 (A couple of motions of a string)**.** In this problem we will look at several different motions of closed strings.

- Let us consider a string whose motion is given by
$$X^0 = R\tau\ ,\ (X^1)^2 + (X^2)^2 \equiv r = R\cos\tau\ ,\ X^i = 0, \forall i > 2\ .$$
Find the corresponding coefficients $\alpha, \tilde{\alpha}$, show that they satisfy the required conditions, and find the mass of such a string.

- Let us consider a string whose motion is given by
$$X^0 = R\tau\ ,\ X^1 = R\cos\sigma\, \cos\tau\ ,\ X^2 = R\cos\sigma\, \sin\tau\ ,\ X^i = 0, \forall i > 2\ .$$
Find the corresponding coefficients $\alpha, \tilde{\alpha}$, show that they satisfy the required conditions, and find the mass of such a string.

- Let us consider a string whose motion is given by
$$X^0 = R\tau\ ,\ X^1 = R\cos\sigma\, \cos\tau\ ,\ X^2 = R\cos 2\sigma\, \sin 2\tau\ ,\ X^i = 0, \forall i > 2\ .$$
Find the corresponding coefficients $\alpha, \tilde{\alpha}$, and show that they do not satisfy the required conditions.

Príklad 3 (Calibration problem.). *I'm not sure how much time these things take up. This is an elective course after all, and the goal is not to kill an awful lot of time with these problems. So I leave it up to you whether you feel you still have time to solve this problem within the credit allowance of the course.*

As we said in the lecture, the string effect can be viewed as a two-dimensional theory of D scalar fields. Find the generalized momenta $P^\mu(\tau,\sigma)$ corresponding to the coordinates/fields X^μ.

The classical field theory is then given by Poisson brackets

$$\begin{aligned}
\left\{P^\mu(\tau,\sigma), X^\mu(\tau,\sigma')\right\} &= \eta^{\mu\nu}\delta(\sigma-\sigma')\ , \\
\left\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\right\} &= 0\ , \\
\left\{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\right\} &= 0\ .
\end{aligned}$$

Show that these conditions lead to the following Poisson brackets for the coefficients $\alpha, \tilde{\alpha}$ and x, p

$$\begin{aligned}
\{p^\mu, x^\mu\} &= \eta^{\mu\nu}\ , \\
\{a^\mu_m, a^\nu_n\} &= i\, m\, \eta^{\mu\nu}\, \delta_{m+n,0}\ , \\
\{\tilde{a}^\mu_m, \tilde{a}^\nu_n\} &= i\, m\, \eta^{\mu\nu}\, \delta_{m+n,0}\ , \\
\{a^\mu_m, \tilde{a}^\nu_n\} &= 0\ .
\end{aligned}$$

Then what are the Poisson brackets of all objects L and $\tilde{L}$?