Introduction to string theory
Homework 4

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Problem 1 (Momentum and angular momentum for a string)**.** As we have seen, the action for a string is invariant under the global transformation

$$X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu \ ,$$

where c^μ is a constant vector and Λ is a Lorentz transformation matrix. Show that the conserved currents are for translations

$$P^\alpha_\mu = T\partial^\alpha X_\mu$$

and for rotations and boosts

$$J^a_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu$$

and verify that they are conserved for solutions satisfying the equations of motion. Find the conserved charges from them and see how they look when expressing the solutions in terms of x, p and the modes $\alpha, \tilde{\alpha}$.

Hint. Consider what an infinitesimal transformation looks like and that the variation of the action must be possible to be written as

$$\delta S = \int d^\sigma \ J^\alpha \partial_\alpha \varepsilon \ ,$$

where ε is a small parameter characterizing the transformation.

Problem 2 (Motion of an open string)**.** In the static gauge $X^0 = R\tau$, show using the gauge constraints and the boundary condition that the ends of the open string move at the speed of light.

Problem 3. Let us have a $p+1$ dimensional brane whose world-volume is embedded in a D dimensional space-time using D functions X^μ of $p+1$ variables $\sigma = (\tau, \sigma^1, \ldots, \sigma^p)$. Consider that a reasonable generalization of the action for such an object is

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\det \, \partial_\alpha X \cdot \partial_\beta X} \ .$$

Show that for a suitable choice of the parameter Λ_p this action is equivalent to the action

$$S = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-\gamma}\, \gamma^{\alpha\beta} \, \partial_\alpha X \cdot \partial_\beta X + \Lambda_p \int d^{p+1}\sigma \sqrt{-\gamma} \ .$$

Deliberate that in a suitable gauge there exists a solution to the equations of motion that has the form

$$X^a = 0 \ , \ a = 0, \ldots, p \quad , \quad X^I = 2\pi\alpha' \phi^I(\sigma) \ , \ I = p+1, \ldots, D-1 \ ,$$

which is a finite straight gate with deviations given by the functions σ^I.

Problem 4 (Energy-momentum tensor for a free scalar field)**.** Let us have the action of a scalar field in two-dimensional Euclidean space

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \partial^\alpha X \ .$$

Find the relation for $T_{\alpha\beta}$ from the definition

$$T_{\alpha\beta} = -\frac{4}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}}.$$

Also find the relation for T in the complex coordinates $z, \bar{z}$ and show that for the solution $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$ the zz component of T is holomorphic and the $\bar{z}\bar{z}$ component is antiholomorphic.

Hint. We did not show this in the lecture, but it is not difficult to see that the equation of motion for the free scalar field in complex coordinates is $\partial \bar{p}artial X = 0$ and thus we obtain the declared form of the solution.