# Introduction to string theory
**Homework 5**

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*Updated 19. marca 2025* *Submit no later than 26.3.2025*

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**Problem 1** (Primary operators for scalar fields). Derive the propagator for the anti-holomorphic component of the free massless scalar field

$$\bar{X}(\bar{z})\bar{X}(\bar{w}) = -\frac{\alpha'}{2}\log\left(\bar{z}-\bar{w}\right).$$

Show that $\bar{\partial}X$ is primary with weights $h = 0$ and $\tilde{h} = 1$.

Show that in the same case the field $:e^{ikX}:$ is primary with weights $h = \tilde{h} = \alpha' k^2/4$.

**Problem 2** (Casimir energy). Show that a conformal transformation $z \to \tilde{z}(z)$ changes the energy-momentum tensor as follows

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}S(\tilde{z},z)\right]$$

where S is the Schwartzian given by

$$S(\tilde{z},z) = \left(\frac{\partial^3\tilde{z}}{\partial z^3}\right)\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2\tilde{z}}{\partial\partial z^2}\right)\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-2} .$$

Now let us have a conformal transformation between a cylinder and a plane as in the lecture $z = e^{-i(\sigma+i\tau)}$. What does the previous relation lead to in this case? If we have a theory with zero ground state energy in the plane, what energy does it lead to for the theory on a cylinder?

**Problem 3** (Virasor algebra and the central charge). Recall the definition of $L_n$ and $\tilde{L}_n$ as coefficients of the expansion of the energy-momentum tensor in the Laurent series and of $TT$ and $\tilde{T}\tilde{T}$ OPE, respectively. Derive from this knowledge the following commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

and the analogous relation for $\tilde{L}$.

**Hint.** This is done in great detail in many string theory lecture notes. Do your best not to just copy the solution but also to understand why and how things are done. And appreciate where the important steps are.