Lagrangian relations, half-densities and quantum L_∞ algebras

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based on joint work with Branislav Jurčo and Martin Zika [JPZ25] [2401.06110]

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1 Motivation

This work is describing a combination of two ideas: the *geometric Batalin-Vilkovisky (BV) formalism* and the *symplectic category*. I will first explain these two ideas and their relationship to physics. Then, I will state our result: a mathematically rigorous definition of a *generalized Lagrangian relation*. The talk will therefore start more informally, and then switch to mathematics.

1.1 Idea 1: Geometric BV formalism

following [BV81], [Sch93], [Khu04] and [Šev04]

Batalin-Vilkovisky formalism is a formal method to manipulate the Feynman path integral, especially relevant for theories with (higher) gauge symmetries (see e.g. [KSV25] for a recent application). The basic ingredients are

1. The BV space of fields (i.e. the space of all possible configurations of fields and their antifields on the spacetime), denoted $\mathcal{M}_{\mathrm{BV}}$.

   For a real scalar field theory, $\mathcal{M}_{\mathrm{BV}} = C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R})[-1]$*, where* M *is the spacetime; an element of* $\mathcal{M}_{\mathrm{BV}}$ *is a pair* (ϕ,ϕ^*) *of two functions on* M*, the* scalar field *and its* antifield*. The symbol* $[-1]$ *means that the antifield has ghost*[1] *degree* 1*, and consequently it has Fermi statistics (i.e. it is anticommuting).*

   For Chern-Simons theory, $\mathcal{M}_{\mathrm{BV}} = \Omega^\bullet(M,\mathfrak{g})[1]$*, with a field* $\mathbb{A}$ *being an inhomogeneous form* $\mathbb{A} = c + A + A^* + c^*$ *valued in the Lie algebra* $\mathfrak{g}$*, and the spacetime* M *3-dimensional.*

2. A (-1)-shifted symplectic form ω on $\mathcal{M}_{\mathrm{BV}}$: saying it is (-1)-shifted means that it is nonzero only on vectors of total degree 1.

[1] It is a common convention for a shift $V[n]$ to mean that the elements of V are placed in degree $-n$; the coordinates (elements of the dual of $V[n]$) are then of degree n. Regarding motivation for ghosts number: in gauge theories, it is often useful to extend the space of usual physical fields to include new fields, and moreover it is often possible to include a consistent grading of these fields by integers, their ghost number. In this talk, the statistics of the field is determined by the parity of this ghost number.

For the scalar field theory, we need to choose a volume form on the spacetime M; the symplectic form is then given by[2]

$$\omega(\phi, \phi^*) = \int_M \phi(x)\phi^*(x)\mathrm{dvol}, \quad \omega(\phi_1, \phi_2) = \omega(\phi_1^*, \phi_2^*) = 0.$$

while for Chern-Simons theory

$$\omega(\mathbb{A}_1, \mathbb{A}_2) = \frac{1}{2}\int_M \mathrm{tr}(\mathbb{A}_1 \wedge \mathbb{A}_2) = \int_M \mathrm{tr}(c \wedge c^* + A \wedge A^*)$$

where we see the Lie algebra as a matrix algebra; the $\wedge$ involves the product of matrices (with entries being smooth forms) and the trace of this product gives a 3-form one can integrate over M

3. An action functional S on $\mathcal{M}_{\mathrm{BV}}$ satisfying the *quantum master equation*[3]

$$\int_M \underbrace{\frac{\delta^2}{\delta\varphi^i(x)\delta\varphi_i^*(x)}}_{\Delta} e^{iS/\hbar} = 0$$

   where $\varphi^i(x)$ are all the fields of the theory and $\varphi_i^*(x)$ are their antifields.

   For the scalar theory, one can choose $S[\phi, \phi^*]$ to be an arbitrary functional of ϕ; dependence on ϕ^* is not possible due to degree reasons; it satisfies the quantum master equation since S does not depend on the antifield. For Chern-Simons theory, we have

$$S[\mathbb{A}] = \int_M \frac{1}{2}\mathrm{tr}(\mathbb{A} \wedge d\mathbb{A}) + \frac{1}{6}\mathrm{tr}(\mathbb{A} \wedge [\mathbb{A}, \mathbb{A}]).$$

   The quantum master equation follows from the gauge invariance of the Chern-Simons action.

   Theories written in the BRST formalism can be easily encoded in BV formalism (by adding antifields to all fields, ghosts etc.). The BV actions obtained in this way are *linear* in antifields.

4. Finally, a *Lagrangian* submanifold $\mathcal{L} \subset \mathcal{M}_{\mathrm{BV}}$. Lagrangian means that the symplectic form vanishes on this submanifold and it has "half" of the dimension on $\mathcal{M}_{\mathrm{BV}}$ (equivalently, one can require that it is maximal among submanifolds on which the symplectic form vanishes, i.e. not contained in any other bigger such submanifold.)

   For scalar field theory, the only reasonable possibility is $\mathcal{L} = \{\phi(x) \text{ arbitrary}, \phi^*(x) = 0\}$. For Chern-Simons theory, the choice of the Lagrangian submanifold corresponds to the choice of the propagator; an example of the Lorentz gauge is given by the condition $d * \mathbb{A} = 0$. See the book of Mnëv [Mne19, Sec. 4.9, 5.2] for more details

With these ingredients in place, the BV path integral for the partition function is written as[4]

$$Z = \int_{\mathcal{L} \subset \mathcal{M}_{\mathrm{BV}}} e^{iS/\hbar} \tag{1}$$

[2]Since both our examples of $\mathcal{M}_{\mathrm{BV}}$ are vector spaces, I identify the tangent space at any point with $\mathcal{M}_{\mathrm{BV}}$ itself. In general, the BV symplectic form is a smooth closed non-degenerate two-form.

[3]For infinite-dimensional $\mathcal{M}_{\mathrm{BV}}$ the *BV operator* Δ is typically singular; a consistent definition involves the discussion of renormalization and was done by Costello [Cos11].

[4]In reality, it turns out that the exponential factor $e^{iS/\hbar}$ should be accompanied with a choice of *half-density*, since half-densities are the natural geometric object one can integrate over Lagrangian submanifolds. I will not emphasize this difference too much.

The main result of BV formalism is that, if the quantum master equation is satisfied, then the partition function Z does not change if we deform $\mathcal{L}$. The advantage of BV formalism for gauge theories is that the BV action functional S can be written fully covariantly, without choosing a gauge; the different choices of gauge fixings give different Lagrangians[5] $\mathcal{L}$. The gauge invariance of the path integral then follows from the main result of BV formalism. Usually, the Lagrangian is chosen in such a way that we can compute the partition functions (or expectation values, if wished) using convenient perturbative expansions.

In the case of the scalar field theory, the BV integral reduces to the usual path integral over $\phi \in C^\infty(M, \mathbb{R})$. In Chern-Simons, we recover the path integral using the propagator related to $\mathcal{L}$.

Finally, we can state the first main idea of our work:

As pointed out by Pavol Ševera [Šev04] that one should see both the Lagrangian $\mathcal{L}$ and the integrand $e^{iS/\hbar}$ as a objects of the same kind: distributional half-densities; with the former being a Dirac-like $\delta_{\mathcal{L}}$ with support on $\mathcal{L}$. Alternatively, we can see $e^{iS/\hbar}$ as a *generalized Lagrangian*, obtained by smearing. We use the the term generalized Lagrangian in this talk and in [JPZ25]. With this in mind, we rewrite the BV integral as

$$Z = \int_{\mathcal{L} \subset \mathcal{M}_{\mathrm{BV}}} e^{iS/\hbar} =: \langle \delta_{\mathcal{L}} | e^{iS/\hbar} \rangle$$

1.1.1 Analogy between complex integrals and Batalin-Vilkovisky formalism

following Domenico Fiorenza [Fio]

Let me finish this section by mentioning that a similar philosophy to computing integrals is taught in standard complex analysis:

complex analysis	**BV formalism**
definite integrals of $f(x)$ over $\mathbb{R}$	the usual path integral of ${}^{iS_{\text{not BV}}/\hbar}$ over the space of fields $\mathcal{M}$
replacing $\mathbb{R}$ with $\mathbb{C}$ (adding imaginary direction)	replacing $\mathcal{M}$ with $\mathcal{M}_{\mathrm{BV}}$ (adding anti-fields)
extending the function $f(x)$ to $f(z)$	extending the action $S_{\text{not BV}}$ to the BV action S
the condition that $f(z)$ is holomorphic	the quantum master equation $\Delta e^{iS/\hbar}$
choosing the integration contour γ	choosing the Lagrangian $\mathcal{L}$
the integral $\int_\gamma f$ does not change if we deform γ	the integral $\int_{\mathcal{L}} e^{iS/\hbar}$ does not change if we deform $\mathcal{L}$

1.2 Idea 2: Relations between BV theories

following [Wei06], [GS79] and [Šev04]

Here, we try to answer the question "How to relate different QFTs in the Batalin-Vilkovisky formalism?". On the level of (-1)-shifted symplectic spaces of fields, the obvious guess is that a

[5]It is common (and sometimes confusing) to shorten "Lagrangian submanifold" to just "Lagrangian".

map between BV spaces of fields should be a function $\psi\colon \mathcal{M}_{\mathrm{BV}} \to \mathcal{N}_{\mathrm{BV}}$ such that the pullback of the symplectic form on the right hand side agrees with the symplectic form on the left. *This turns out to be too restrictive*, and a famous answer, due to Weinstein (extracted from the work of Hörmander on Fourier integral operators [Hör71]), is that *as relations, one should consider Lagrangian submanifolds of* $\mathcal{M}_{\mathrm{BV}} \times \mathcal{N}_{\mathrm{BV}}$, where the symplectic form is $-\omega_{\mathcal{M}} + \omega_{\mathcal{N}}$.

Examples of such relations are

- If $\psi\colon \mathcal{M}_{\mathrm{BV}} \to \mathcal{N}_{\mathrm{BV}}$ is a symplectic isomorphism, then $\{(m, \psi(m))) \mid m \in \mathcal{M}_{\mathrm{BV}}\} \subset \mathcal{M}_{\mathrm{BV}} \times \mathcal{N}_{\mathrm{BV}}$ is a Lagragian submanifold, i.e. a Lagrangian relation $\mathcal{M}_{\mathrm{BV}} \to \mathcal{N}_{\mathrm{BV}}$.

- If $\mu\colon \mathcal{M}_{\mathrm{BV}} \to \mathfrak{g}^*$ is a moment map, then the subset $\{(m, m \mod G) \mid m \in \mu^{-1}(0)\} \subset \mathcal{M}_{\mathrm{BV}} \times [\mu^{-1}(0)/G]$ is a Lagrangian relation $\mathcal{M}_{\mathrm{BV}} \to \mu^{-1}(0)/G$. Notice that in this case, there is no function $\mathcal{M}_{\mathrm{BV}} \to \mu^{-1}(0)/G$, as the relation is only defined for m such that $\mu(m) = 0$.

Now, let's add half-densities to the mix, following [Šev04]: a *generalized* Lagrangian relation $\mathcal{M}_{\mathrm{BV}} \to \mathcal{N}_{\mathrm{BV}}$ is a generalized Lagrangian submanifold of (distributional half-density on) $\mathcal{M}_{\mathrm{BV}} \times \mathcal{N}_{\mathrm{BV}}$. Thus, we additionally allow for smooth half-densities on the product, and more generally for half-densities which are distributional only in some directions.

There are some obvious problems. First, it is not clear what I mean by distributional half-densities. In addition, how to define a composition of such generalized Lagrangian relations?

In our work, we answer these questions with the following simplifying assumptions: we assume that $V := \mathcal{M}_{\mathrm{BV}}$ is a finite-dimensional vector space (which corresponds to discretizing the spacetime to a finite number of points), and that the (distributional) directions of the generalized Lagrangians are vector subspaces.

2 Background and Results

2.1 Symplectic linear algebra

Let me now built some of the background necessary to present our results.

Definition 2.1. A (-1)-shifted symplectic vector space is a $\mathbb{Z}$-graded vector space V with a non-degenerate, antisymmetric map $V \otimes V \to \mathbb{R}$ of degree -1, i.e. it vanishes if its inputs are of total degree different than 1.

For a (graded) vector subspace $X \subset V$, its symplectic complement is defined by

$$X^\omega = \{v \in V \text{ such that } \omega(v, x) = 0,\ \forall x \in X\}.$$

Unlike for a scalar product, the symplectic complement can intersect the original space. We will make use of two important cases:

- A vector subspace $C \subset V$ is called coisotropic if $C^\omega \subset C$.

- A vector subspace $L \subset V$ is called Lagrangian if $L^\omega = L$.

One can always find a basis $\{p_i, q^i\}_{i=1\ldots N}$ of V such that $\omega(p_i, q^j) = \delta_i^j$ and $\omega(p_i, p_j) = \omega(q_i, q_j) = 0$. A typical example of a coisotropic subspace is one spanned by all q's and some of the p's, say $p_1, \ldots, p_k$. The symplectic complement is spanned by $q_{k+1}, \ldots, q_N$, and we see that the quotient C/C^ω is given by $\{p_i, q^i\}_{i=1\ldots k}$.

Proposition 2.2. *For an arbitrary coisotropic subspace $C \subset V$, the quotient $R_C := C/C^\omega$ is again (-1)-shifted symplectic, with the symplectic form given by $\omega_{R_C}(c \mod C^\omega, c' \mod C^\omega) = \omega(c, c')$. This subquotient R_C is called the coisotropic reduction of C.*

There are two noteworthy special cases of coisotropic subspaces, $C = V$ and C Lagrangian. The coisotropic reductions are V and $\{0\}$, respectively.

2.2 Generalized Lagrangians

Our main definition, of a generalized Lagrangian, is as follows:

Definition 2.3 ([JPZ25])**.** A generalized Lagrangian subspace of V is a pair of

- a coisotropic subspace $C \subset V$
- a half-density[6] ρ on $R_C = C/C^\omega$

A generalized Lagrangian relation between V_1 and V_2 is a generalized Lagrangian subspace of $(V_1 \times V_2, -\omega_1 + \omega_2)$.

There are two extremal cases of generalized Lagrangians:

- if $C = L$ is Lagrangian, the coisotropic reduction R_L is the zero-dimensional vector space, and the half-density is a number. This represents the distributional half-density $\rho \cdot \delta_L$ from the beginning,
- if $C = V$, the coisotropic reduction is V again, and ρ is a smooth half-density on V.

In general, a generalized Lagrangian (C, ρ) should be understood as the tensor product of the smooth half-density ρ on R_C and distributional half-density δ_{C^ω}, where C^ω is a Lagrangian subspace in a (non-canonical) complement to R_C in V.

Finally, let me collect our results in the following omnibus theorem.

Theorem 2.4 (Jurčo–P.–Zika)**.** *Given two relations*

$$V_1 \xrightarrow{(C,\rho)} V_2 \xrightarrow{(C',\rho')} V_3$$

such that the total quadratic part of $\rho\rho'$ is nondegenerate on a suitable[7] $K_{C,C'} \subset R_C \times R_{C'}$, there is a generalized Lagrangian relation $V_1 \xrightarrow{(C',\rho')\circ(C,\rho)} V_3$, with the coisotropic subspace given by $C' \circ C$ and the half-density given by a fiber BV integral of $\rho\rho'$. This composition of generalized Lagrangians is unital, associative, compatible with the BV operator, and in the following case

$$\bullet \xrightarrow{(V,e^{iS/\hbar})} V \xrightarrow{(L,1)} \bullet$$

the composition is equal[8] to the BV integral (1) from the beginning.

[6] As mentioned above, I will not dwell much on half-densities. In coordinates x^i, a half-density can be uniquely written as $\rho(x) \otimes \sqrt{dx}$, and the second factor transforms with $(\det A_{\mathrm{even}}/\det A_{\mathrm{odd}})^{1/2}$, were the matrices A describe the change of the even and the odd basis vectors.

[7] See Section 4 of our paper [JPZ25] for details. In the paper, we separate the quadratic part of the action, and thus generalized Lagrangians in [JPZ25] are triples $(C, S_{\mathrm{free}}, \rho)$, with S_{free} a quadratic function on R_C. In this talk, we would understand this triple as the generalized Lagrangian $(C, e^{S_{\mathrm{free}}/\hbar}\rho)$. The fact that the composition is not always defined means that we get a "partial" category as a result.

[8] The composition is a half-density on the zero-dimensional vector space $\bullet$, i.e. a number (a formal power series in $\hbar$ in our formalism)

Other examples of compositions or generalized Lagrangians are

- If the original field theory on V has zero modes, then instead of computing the partition function, we can only compute an effective action functional, depending on the fields in the cohomology H of the differential $\{S_{\text{free}}, -\}$. In order to do this, one needs to choose a Lagrangian relation K between V and H (this is related to a choice of propagator, or a special deformation retract). Then the effective action W on H is given by the composition (the diagonal arrow below)

$$\begin{tikzcd} \bullet \arrow[r, "(V,e^{iS/\hbar})"] \arrow[dr, "(H,e^{iW/\hbar})"'] & V \arrow[d, "\delta_K=(K,1)"] \\ & H \end{tikzcd}$$

  If S satisfies the quantum master equation, it defines a so-called quantum L_∞ algebra on V; this procedure computes a "homotopy transfer" of this quantum L_∞ algebra to the homology H.

- In [CM09], Cattaneo and Mnëv notice that relaxing the notion of a "special deformation retract" from the previous point (specifically, not requiring that the homotopy squares to zero) changes K to a non-trivial generalized Lagrangian relation, with both smooth and distributional directions.

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