An algebraic formulation of nonassociative quantum mechanics

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based on 2311.03647 [quant-ph] with Richard Szabo

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Theoretical Physics Seminar, FMFI UK Bratislava, January 30, 2024

CONSTRUCTOR UNIVERSITY

Quick facts:

- small private research university in Bremen
- founded 2001
- > 1800 students from > 120 nations
- language of teaching + research: English

interaction via deformation

electromagnetic interaction, coupling to gauge field:

- start with free Hamiltonian H, then:
- *either* deform H ("minimal substitution"): $H' = H(p - A, q)$
- *or* deform ω and hence $\{\, ,\, \}$: $\alpha' = \sum pdq + A \to \omega' = \omega + dA$

e.g. relativistic particle in einbein formalism (no $\sqrt{\ldots}$!)

$$S = \int d\tau \left(\frac{1}{2e} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} e m^2 + A_\mu(x)\dot{x}^\mu \right) \rightsquigarrow p_\mu = \frac{1}{e} g_{\mu\nu}\dot{x}^\nu + A_\mu$$

$$S_H = \int p_\mu dx^\mu - \frac{1}{2} e \left((p_\mu - A_\mu)^2 + m^2 \right) d\tau \qquad \leftarrow p_\mu\text{: canonical momentum}$$

undeformed Hamiltonian

$$\boxed{S_H = \int \underbrace{(p_\mu + A_\mu)\, dx^\mu}_{\text{deformed symplectic potential}} - \frac{1}{2} e \left(p_\mu^2 + m^2 \right) d\tau} \qquad \leftarrow p_\mu\text{: physical momentum}$$

interaction via deformation

deformed symplectic 2-form

$$\omega' = d\big(p_\mu + A_\mu\big) \wedge dx^\mu \;\; \rightsquigarrow$$

$$\{p_\mu, p_\nu\}' = F_{\mu\nu},\;\; \{x^\mu, p_\nu\}' = \delta^\mu_\nu,\;\; \{x^\mu, x^\nu\}' = 0$$

$$\{p_\lambda, \{p_\mu, p_\nu\}'\}' + \text{cycl.} = (dF)_{\lambda\mu\nu} = (*j_m)_{\lambda\mu\nu} \qquad \leftarrow \text{magnetic 4-current}$$

free Hamiltonian $H = \frac{1}{2}p^2$, deformed Poisson brackets $\Rightarrow$ Hamilton-Lorentz equations:

$$\dot{x}^\nu = \{x^\nu, H\}' = p^\nu$$

$$\dot{p}_\mu = \{p_\mu, H\}' = F_{\mu\nu}\dot{x}^\nu$$

interaction via deformation

Quantization

- path integral ✓
- deformation quantization ✓(even non-associativity o.k.)
- canonical? (✓) :

Deformed CCR:

$$[p_\mu, p_\nu] = i\hbar F_{\mu\nu}, \quad [x^\mu, p_\nu] = i\hbar\delta^\mu_\nu, \quad [x^\mu, x^\nu] = 0, \quad [\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

Let $\mathbf{p} = \gamma^\mu p_\mu$ and $H = \frac{1}{2}\mathbf{p}^2 \rightsquigarrow$ correct coupling of fields to spin

$$H = \tfrac{1}{8}\big([\gamma^\mu, \gamma^\nu]_+[p_\mu, p_\nu]_+ + [\gamma^\mu, \gamma^\nu][p_\mu, p_\nu]\big) = \tfrac{1}{2}p^2 - \tfrac{i\hbar}{2}S^{\mu\nu}F_{\mu\nu}$$

Lorentz-Heisenberg equations of motion (ignoring spin)

$$\dot{p}_\mu = \tfrac{i}{\hbar}[H, p_\mu] = \tfrac{1}{2}(F_{\mu\nu}\dot{x}^\nu + \dot{x}^\nu F_{\mu\nu}) \quad \text{with} \quad \dot{x}^\nu = \tfrac{i}{\hbar}[H, x^\nu] = p^\nu$$

this formalism allows $dF \neq 0$: magnetic sources, non-associativity

interaction via deformation: monopoles

local non-associativity: $\frac{1}{3}[p_\lambda, [p_\mu, p_\nu]]\, dx^\lambda dx^\mu dx^\nu = \hbar^2 dF = \hbar^2 *j_m$

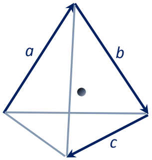
$j_m \neq 0 \Leftrightarrow$ no operator representation of the p_μ!

spacetime translations are still generated by p_μ (physical momentum), but magnetic flux Φ_m leads to path-dependence with phase $e^{i\phi}$; where $\phi = iq_e\Phi_m/\hbar$

globally:

$$\Phi_m = \int_S F = \int_{\partial S} A \qquad \leftrightarrow \quad \text{non-commutativity}$$

$$\Phi_m = \int_{\partial V} F = \int_V dF = \int_V *j_m = q_m \qquad \leftrightarrow \quad \text{non-associativity}$$



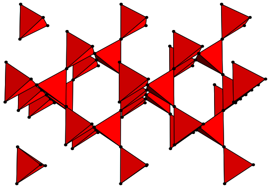
global associativity requires $\phi \in 2\pi\mathbb{Z} \;\Rightarrow\; \boxed{\frac{q_e q_m}{2\pi\hbar} \in \mathbb{Z}}$ Dirac quantization

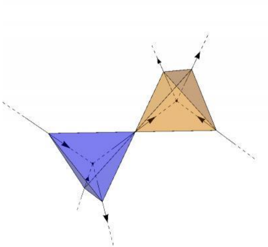
this is a fully relativistic version of previous work of Jackiw 1985, 2002

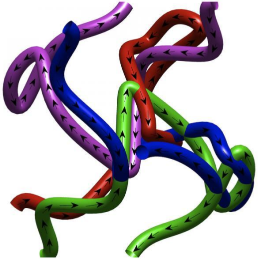
algebraic approach to non-associative QM: PS, Szabo: arXiv:2311.03647 [quant-ph]

magnetic monopoles in the lab

spin ice pyrochlore and Dirac monopoles







Castelnovo, Moessner, Sondhi (2008)
Fennell; Morris; Hall, . . . (2009)

frustrated spin system $\leftrightarrow$ huge degeneracy of classical ground state

Lieb, PS (1999), (2000); PS (2001)

Examples of nonassociative (quantum) algebras

magnetic sources (monopoles)

interaction via deformation (of CCR)

$$[p_i, p_j] = i\hbar F_{ij} \, , \quad [x^i, p_j] = i\hbar \delta^i_j \, , \quad [x^i, x^j] = 0$$

undeformed Hamiltonian $H = p^2/2m$

$\rightsquigarrow$ Lorentz-Heisenberg eom $\dot{p}_i = \frac{i}{\hbar}[H, p_i] = \dot{x}^j F_{ij}$

$\rightsquigarrow$ $\mathrm{Jac}(p_i, p_j, p_k)\, dx^i \wedge dx^j \wedge dx^k = dF = * j_{mag}$

$\neq$ for magnetic sources! $\rightsquigarrow$ NA!

Lie algebras $A = \mathbb{K} \oplus \mathfrak{g}$ with product $[ab] = a\cdot b - b\cdot a$

$$[[ab]c] = [a[bc]] + [[ac]b] \neq [a[bc]] \quad \text{NA!}$$

$\longrightarrow$ associative universal enveloping algebra

Jordan algebras $ab := \frac{1}{2}(a\cdot b + b\cdot a)$

$$\rightsquigarrow \quad a(bc) - (ab)c = \frac{1}{2}[[a;c];b] \quad \text{non-associative, flexible}$$

normed division algebras

$$\underbrace{\mathbb{R}, \mathbb{C}}_{\text{commutative}}, \quad \underbrace{\mathbb{H} \text{ quaternions}}_{\substack{\text{non-commutative} \\ \text{associative}}}, \quad \underbrace{\mathbb{O} \text{ octonions}}_{\substack{\text{non-associative,} \\ \text{alternative}}}$$

weakened non-associativity : (popular in literature)

alternative alg.: $\quad x(xy) = (xx)y$

e.g. octonions $\quad (yx)x = y(xx)$

$\Leftrightarrow$ associator $[x,y,z] = (xy)z - x(yz)$ alternating

flexible alg.: $\quad x(yx) = (xy)x$

e.g. Jordan alg.

power associative: $(xx)x = x(xx) = x^3$ etc.

$\longrightarrow$ we shall not assume any of these!

non-geometric string backgrounds

⇝ Poisson sigma model with 3-form R-flux :

$$S = \int_\Sigma \eta_I \wedge dX^I + \frac{1}{2}\Theta^{IJ}(X)\,\eta_I \wedge \eta_J \quad , \quad R = [\Theta,\Theta]_S$$

$$\Theta = \begin{pmatrix} R^{ijk}p_k & \delta^i{}_j \\ -\delta_j{}^i & 0 \end{pmatrix} \xrightarrow{\text{quant.}} \ast \text{ non-associative!}$$

$$f \ast g = \cdot \exp\left(\frac{i\hbar}{2}\left(R^{ijk}p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right)\right)$$

2-cyclic $\quad \int f \ast g = \int g \ast f$

3-cyclic $\quad \int f \ast (g \ast h) = \int (f \ast g) \ast h$

not Jordan,
not power-assoc

A. Held, PS
2014

Mylonas, PS, Szabo 1402.7306

nonassociative QM challenge

want: observables, states, probability interpretation
(↳ need notions of reality & positivity)

problem:

QM : operators on Hilbertspace
↳ automatically associative !

solution:

distinguish
→ universal enveloping algebra (associative)
→ quantum algebra (non-associative)

Universal enveloping algebra

A : unital alg. over $\mathbb{k}$ with unit 1

$ab \neq ba$ NC

$a(bc) \neq (ab)c$ NA

quantum algebra

A^{op} : opposite algebra , $\mathcal{U}(A) = \tilde{\mathcal{F}}(A \oplus A^{op})$

free assoc. alg. over $\mathbb{k}$

embedding $\hat{\ } : A \oplus A^{op} \to \mathcal{U}(A)$, $(a, b') \mapsto \hat{a} + \hat{b}'$

left action $\triangleright : \mathcal{U}(A) \times A \to A$:

$\hat{a} \triangleright x = ax$, $\hat{a}' \triangleright x = xa$ $(x \in A)$

$\to$ multiplication algebra of A

composition product $\circ : \mathcal{U}(A) \times \mathcal{U}(A) \to \mathcal{U}(A)$

$$(\hat{a}_1 \circ \hat{a}_2) \triangleright x = \hat{a}_1 \triangleright (\hat{a}_2 \triangleright x) = a_1(a_2 x)$$

$$(\hat{b}'_1 \circ \hat{b}'_2) \triangleright x = \hat{b}'_1 \triangleright (\hat{b}'_2 \triangleright x) = (x b_2) b_1$$

$$\underbrace{(\hat{a}_1 \circ \hat{b}'_2)' = (\hat{b}_2 \circ \hat{a}'_1)}_{' \text{ is } \circ\text{-antihom.}} \triangleright x = b_2 (x a_1) \quad \text{etc.}$$

relations in A

$$P : T_0(A \oplus A^{op}) \twoheadrightarrow \mathcal{U}(A), \quad \mathcal{U}(A) \simeq T_0(A \oplus A^{op}) / \ker(P)$$

we shall assume :

- A is a unital $*$-algebra
- field $\mathbb{k} = \mathbb{C}$

relates left & right action

as in traditional QM

States and GNS construction

state: $\mathbb{C}$-linear functional $\omega : \mathfrak{U}(A) \to \mathbb{C}$

positive: $\omega(A^* \circ A) \geq 0$, $A \in \mathfrak{U}(A)$

normalized: $\omega(1) = 1$

$\Updownarrow$ lemma (Cauchy-Schwartz)

$(A, B) \longmapsto \omega(A^* \circ B)$ is semi-def. sesquilinear

$$\omega(A^* \circ B) = \overline{\omega(B^* \circ A)}$$

$$|\omega(A^* \circ B)|^2 \leq \omega(A^* \circ A)\, \omega(B^* \circ B)$$

proof

$$0 \leq \omega\left((z_1 A + z_2 B)^* \circ (z_1 A + z_2 B)\right)$$

$$(\bar{z}_1, \bar{z}_2) \cdot \underbrace{\begin{pmatrix} \omega(A^* \circ A) & \omega(A^* \circ B) \\ \omega(B^* \circ A) & \omega(B^* \circ B) \end{pmatrix}}_{\text{positive-semidef matrix}} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$\Rightarrow$ hermitean, $\det(\cdots) \geq 0 \ldots$ $\square$

$\rightsquigarrow$ Gel'fand ideal of ω

$$\mathcal{J}_\omega := \{ A \in \mathfrak{U}(\mathcal{A}) \mid \omega(A^* \circ A) = 0 \}$$

is left ideal of $\mathfrak{U}(\mathcal{A})$ (proof via C.-S.)

quotient space $\rightsquigarrow$ pre Hilbert space

$$\mathcal{H}_\omega := \mathfrak{U}(\mathcal{A}) / \mathcal{J}_\omega$$

with equivalence classes $\psi_A = \{ \tilde{A} \in \mathfrak{U} \mid \tilde{A} - A \in \mathcal{J}_\omega \}$

inner product

$$(\psi_A, \psi_B)_\omega = \omega(A^* \circ B)$$

lin. representation $\Pi_\omega : \mathfrak{U}(\mathcal{A}) \to \mathrm{End}(\mathcal{H}_\omega)$

$$\Pi_\omega(A)\Psi_B := \Psi_{A\circ B}$$

$*$-rep.: $\Pi_\omega(A\circ B) = \Pi_\omega(A)\,\Pi_\omega(B)$

$$\Pi_\omega(A^*) = \Pi_\omega(A)^+$$

cyclic with vacuum vector Ψ_1

$$\Psi_A = \Psi_{A\circ 1} = \Pi_\omega(A)\Psi_1$$

$$\langle A\rangle_\omega := \omega(A) = (\Psi_1, \Pi_\omega(A)\Psi_1)_\omega$$

state $\omega_\Psi(A) := (\Psi, \Pi_\omega(A)\Psi)_\omega \Leftarrow$ ref. state

Observables

$$\mathcal{O} \in \mathfrak{U}(\mathcal{A}) \quad , \quad \mathcal{O}^* = \mathcal{O} \quad \text{(real)}$$

$\Rightarrow$ real expect. values : $\omega(\mathcal{O}) = \omega(\mathcal{O}^*) = \overline{\omega(\mathcal{O})} \in \mathbb{R}$

uncertainty $\quad \Delta_\omega \mathcal{O} := \sqrt{\omega\big((\mathcal{O} - \langle \mathcal{O} \rangle) \circ (\mathcal{O} - \langle \mathcal{O} \rangle)\big)}$

$$\Longrightarrow \quad \Delta_\omega \mathcal{O}_1 \cdot \Delta_\omega \mathcal{O}_2 \geq \frac{1}{2} \left| \langle [\mathcal{O}_1 , \mathcal{O}_2] \rangle_\omega \right|$$

proof via Cauchy-Schwartz

eigenstates

ω s.t. $\quad \omega(B \circ A) = \lambda\, \omega(B)\,, \quad \forall B \in \mathfrak{U}(A)$

λ: eigenvalue

note: observables have real eigenvalues

$\Longleftrightarrow \quad \omega((A-\lambda)^* \circ (A-\lambda)) = 0 \qquad$ (use C.-S.)

$\Longleftrightarrow \quad \pi_\omega(A)\,\Psi_1 = \lambda\, \Psi_1 \qquad (\Leftrightarrow \; \Psi_A = \lambda\, \Psi_1)$

eigenstates have zero uncertainty

(ω eigenstate of O $\Leftrightarrow$ $\Delta_\omega O = 0$.)

States from traces

zooming in to non associativity

note: not $\mathfrak{A}(A)$!

trace $\mathbb{C}$-linear functional $\tau : A \to \mathbb{C}$

- positive $\quad \tau(a^*a) \geq 0$
- normalized $\quad \tau(1) = 1$
- 2-cyclic $\quad \tau(ab) = \tau(ba)$
- 3-cyclic $\quad \tau(a(bc)) = \tau(c(ab))$

$\Bigg\}\ a, b, c \in A$

$$\Rightarrow \tau(a(bc)) = \tau((ab)c)$$

τ is a state directly on quantum algebra A!

$\rightsquigarrow$ <u>tracial GNS construction</u>

(gives smaller Hilbert space, "more nonassociative")

zero-norm vectors: $J_\tau = \{\psi \in A \mid \tau(\psi^*\psi) = 0\}$

$\psi \in J_\tau \Rightarrow A \triangleright \psi \in J_\tau$ (left ideal)

<u>proof</u>: $\tau((A \triangleright \psi)^* (A \triangleright \psi)) \overset{\text{3-cyclicity}}{=} \tau((A^{*} \triangleright \psi^*)(A \triangleright \psi))$

$= \tau(\psi^*(A^* \triangleright (A \triangleright \psi))) = \tau(\psi^*((A^* \circ A) \triangleright \psi))$

$\leq \underbrace{\tau(\psi^*\psi)}_{0}\, \tau(((A^* \circ A) \triangleright \psi)^* ((A^* \circ A) \triangleright \psi))$

further states: $\mu(a) := \tau(\psi^*(a\psi))$
$$= \tau((a\psi)\psi^*) = \tau(a\underbrace{(\psi\psi^*)}_{S_\psi})$$

more generally: $\mu(a) = \sum_i p_i \tau(\psi_i^*(a\psi_i))$

with $p_i \geq 0$, $\sum p_i = 1$, $\tau(\psi_i^*\psi_i) = 1$

positivity?

$\mu(\psi^*((a^*a)\psi))$ is not positive $\Downarrow$ no state on $\mathcal{A}$

$\wedge$ need to go to $\mathfrak{U}(A)$ ...

$$\tau\big((a\psi)^*(a\psi)\big) \geq 0$$

$$= \tau\big((\psi^* a^*)(a\psi)\big) \underset{\text{3-cycl.}}{=} \tau\big(\psi^*(a^*(a\psi))\big) = \tau\big(\psi^*((\hat{a}^* \circ \hat{a}) \triangleright \psi)\big)$$

$\wedge$ notions of positivity, probability need $\mathfrak{U}(A)$

tracial state : $\omega_\tau(A) = \tau(A \triangleright 1) =: \tau(A)$

$\rightarrow$ state on both $\mathcal{A}$ and $\mathcal{U}(\mathcal{A})$

proof uses 3-cyclicity

$\rightarrow$ but not a trace on $\mathcal{U}(\mathcal{A})$: $\tau(A \circ B) \neq \tau(B \circ A)$!

$\curvearrowright$ nonassociative effect

further states on $\mathcal{U}(\mathcal{A})$:

$$\omega_\Psi(A) = \sum p_i \, \tau(\Psi_i^*(A \triangleright \Psi_i)) \quad , \sum p_i = 1, \; p_i \geq 0$$

we will suppress $\sum p_i \ldots$ in the following

Dynamics

completely positive map:

$$\underbrace{\text{map } \phi \text{ from states to states}}_{\text{positive map}} \text{ that } \underbrace{\text{extends to } \phi \otimes \mathrm{id}}_{\text{CP}}$$

here: $\phi : \omega \mapsto \tilde{\omega}$ CPT map ~ "trace-preserving"

$$\tilde{\omega}(O) = \sum_{k=1}^{N} \omega(A_k^* \circ O \circ A_k), \quad \underbrace{\sum_{k=1}^{N} A_k^* \circ A_k = 1}$$

"Krauss operators"

Heisenberg pic.: $\tilde{O} = \sum A_k^* \circ O \circ A_k, \quad \tilde{1} = \sum A_k^* \circ A_k = 1$

unitary time evolution

$N = 1$, "Krauss op" $U_t = \exp_o\left(-\frac{i}{\hbar} t H\right)$, $H = H^*$

Schrödinger-Liouville eqn:

$$\frac{d\omega_t(O)}{dt} = \frac{i}{\hbar}\omega_t([H; O])$$

for S_ψ: $i\hbar \frac{d\psi}{dt} = H \triangleright \psi$

Heisenberg eqn:

$$\frac{dO}{dt} = \frac{i}{\hbar}[H; O]$$

examples:

Jordan algebras

$(A, \cdot)$ NC, associative, unital $*$-algebra

original multiplication

commutative Jordan product: $ab := \frac{1}{2}(a \cdot b + b \cdot a)$

non associative: $a(bc) - (ab)c = \frac{1}{4}[[a;c];b] \neq 0$ i.g.

consider $A = M_n(\mathbb{C})$ with $1 =$ unit matrix, $* = \dagger$

Prop.: $\mathcal{U}(M_n(\mathbb{C})) = M_{n^2}(\mathbb{C})$

2-cyclic & 3-cyclic trace = normalized matrix trace

$$\tau(a) := \frac{1}{n} \mathrm{Tr}(a)$$

unique square root

positive elements w.r.t. τ

$$a^* a = \frac{1}{2}(a^\dagger \cdot a + a \cdot a^\dagger) =: c = c^\dagger = \sqrt{c} \cdot \sqrt{c} = \sqrt{c}\sqrt{c}$$

$\rightsquigarrow$ density matrix

$$\rho_\psi = \psi\psi^* =: \varphi \cdot \varphi \quad , \quad \varphi = \sqrt{\rho_\psi}$$

... resolving ambiguities in previous treatment

$\rightsquigarrow$ pure states = projectors $\varphi^2 = \varphi = \varphi^*$

tracial GNS representation

Gelfan'd ideal trivial ($\mathrm{Tr}(a^\dagger a)=0 \Rightarrow a=0$)

$\rightsquigarrow \mathcal{H}_\tau = \mathbb{C}^{n^2}$, $\pi_\tau(A)\psi = A\cdot\psi$

eigenstates

proposition: $a\,\varphi = \lambda\,\varphi$, φ positive semi-def.

is solved by: $\varphi = \sum \sqrt{p_i}\, \phi_i \otimes \phi_i^\dagger$, $p_i \geq 0$

$a\cdot\phi_i = \lambda\,\phi_i$ (matrix - e.vec.)

Dynamics examples

Jordan: any polynomial of observables is an observable

for y, z observables, consider:

$$H = \underbrace{\hbar\omega}_{\text{units of energy}} \big(\underbrace{z(yy) - (zy)y}_{\text{would be zero for associative or even for alternative algebra!}} \big)$$

$$= \frac{1}{4}\hbar\omega [[z; y]; y] \neq 0$$

let $(x, y, z) = (\sigma_1, \sigma_2, \sigma_3)$ 2×2 Pauli matrices

$$\underbrace{[\sigma_i ; \sigma_j] = 2i\epsilon_{ijk}\sigma_k}_{su(2)\text{ Lie alg}} \quad , \quad \underbrace{\{\sigma_i ; \sigma_j\} = 2\delta_{ij}\mathbb{1}}_{Cl(3)\text{ Clifford alg}}$$

Jordan product: $\sigma_i \circ \sigma_j = \delta_{ij}\mathbb{1}$ $\left\{\begin{array}{l}\text{blind to } su(2) \\ \text{sees only } Cl(3)!\end{array}\right.$

$\in \mathbb{C}$

$\Lsh$ $H = \hbar\omega\hat{z}$ acts on $\psi = a\mathbb{1} + bx + cy + dz$

$H \circ \psi = \hbar\omega z\psi = \hbar\omega(az + d\mathbb{1}) = i\hbar\dot{\psi}$

$\Lsh$ $a(t) = \alpha\cos(\omega t + \phi)$, $b(t) = \beta$, $c(t) = \gamma$, $d(t) = -i\alpha\sin(\omega t + \phi)$

Octonions $\mathcal{A} = \mathbb{O}$

$e_0 = 1, \quad e_i e_j = -\delta_{ij} e_0 + \eta_{ijk} e_k \quad i,j,k \in \{1,\dots,7\}$

η_{ijk} completely skewsym

$\eta_{123} = \eta_{145} = \eta_{176} = \eta_{246} = \eta_{257} = \eta_{347} = \eta_{365} = 1$

$*$-involution: $e_0^* = e_0 \quad , e_i^* = -e_i$

2-cyclic & 3-cyclic trace: $\tau(e_0) = 1, \quad \tau(e_i) = 0$

enveloping algebra

left action of $\mathbb{O}$ on itself defines 8×8 matrix rep. Π

$$\mathbb{1} := \Pi(e_0), \quad E_i := \Pi(e_i)$$

The E_i act as signed permutations, e.g.:

$$E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(0\,1)_-\,(2\,3)_-\,(4\,5)_-\,(7\,6)_- \quad \text{and} \quad (0\,2)_-\,(3\,1)_-\,(4\,6)_-\,(5\,7)_-$$

let $\bar{E}_0 = E_0$, $\bar{E}_i = -E_i$ $\rightsquigarrow$ $\bar{E}_\mu \cdot E_\nu + \bar{E}_\nu \cdot E_\mu = 2\delta_{\mu\nu} \mathbb{1}$

$$\Gamma_\mu = \begin{pmatrix} 0 & E_\mu \\ \bar{E}_\mu & 0 \end{pmatrix} \text{ generate } Cl(8)$$

proposition: $\mathcal{U}(\mathbb{O}) = M_8(\mathbb{C})$

<u>tracial GNS</u> $\quad \tau(e_0) = 1$, $\tau(e_i) = 0$

$J_\tau = \{0\}$ trivial $\rightsquigarrow$ Hilbert space $\mathcal{H}_\tau = \mathbb{C}^8$

$\pi_\tau(e_0) = \mathbb{1}_{8\times 8}$, $\pi_\tau(e_i) = E_i$

$\tau(A) = \pi_\tau(A)_{00}$ ((0,0)-matrix entry)

tracial uncertainties

consider $A := i e_7 = A^*$, $B = e_1 \circ e_2 \circ e_4 = B^*$

$$A \circ A = B \circ B = \mathbb{1} \;,\; A \circ B = - B \circ A =: i E \;,\; E = E^*$$

$$E \circ E = \mathbb{1}$$

$$[A \overset{\circ}{,} B] = 2 i E$$

$$\tau(E) = 1 \implies \tau([A \overset{\circ}{,} B]) \neq 0$$

impossible for associative alg's in finite dim.

$$\underbrace{\Delta_\tau A}_{=1} \; \underbrace{\Delta_\tau B}_{=1} \geq \frac{1}{2} \left| \langle [A \overset{\circ}{,} B] \rangle_\tau \right| = 1$$

$\leadsto$ τ is minimum uncertainty state

Exotic quantum algebras, dual operator algebras

left operator alg

$$\hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

associative

commutative

unital

$*$-algebra

$$\hat{\partial} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

quantum algebra

$$\boxed{\begin{matrix} aa = a, & ab = b \\ ba = -b, & bb = a \end{matrix}}$$

$(bb)b = b \neq -b = b(bb)$

$ab = b \neq -b = ba$

non associative

non power-assoc.

non commutative

non unital

non $*$

right (dual) op. algebra

$$\check{a} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \quad \check{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

associative

non commutative

unital : $\check{a}^2 = \check{b}^2 = 1$

$*$-algebra

Summary

- natural generalization of QM:
  can handle nonassociative quantum algebras ✓
  equals traditional QM for associative q.-alg. ✓
- GNS-inspired representation of operator alg.
  on quantum alg. $\updownarrow$ univ. enveloping alg.
- preferred 3-cyclic state on quantum algebra
- completely positive dynamics ✓
- bonafide nonassociative effects ✓