Hopf algebras and quantum principal bundles

Thomas Weber (Charles University in Prague)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

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Comenius University in Bratislava, Slovakia

Differential geometry dictionary

	Classical differential geometry	**Quantum** differential geometry
observables	smooth manifold M	associative unital algebra A
differential forms	cotangent bundle $\Lambda^\bullet T^*M$	DGA $\Omega^\bullet = \bigoplus_{k\geq 0}\Omega^k$
symmetry	Lie group action $M\times G\to M$	Hopf algebra coaction $\Delta_A\colon A\to A\otimes H$
principal bundle	$\pi\colon P\xrightarrow{\circlearrowleft G} M$ $P\times G\xrightarrow{\cong} P\times_M P$	$B:=\{a\in A\mid \Delta_A(a)=a\otimes 1\}$ $\chi\colon A\otimes_B A\xrightarrow{\cong} A\otimes H$

We call $B:=A^{\mathrm{co}H}=\{a\in A\mid \Delta_A(a)=a\otimes 1\}\subseteq A$ a **Hopf–Galois extension** or **quantum principal bundle (QPB)** if

$$\chi\colon A\otimes_B A\xrightarrow{\cong} A\otimes H,\qquad \chi(a\otimes_B a'):=a\Delta_A(a')$$

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Quantum principal bundles

***G*-Principal Bundle**	**Quantum Principal Bundle**
$$\begin{tikzcd} P \arrow[d, two heads] & P \times G \arrow[l, "\triangleleft"'] \arrow[d, "\cong"] \\ M & P \times_M P \end{tikzcd}$$	$$\begin{tikzcd} A \arrow[r, "\Delta_A"] & A \otimes H \\ B \arrow[u, hook] & A \otimes_B A \arrow[u, "\cong"'] \end{tikzcd}$$

$M \cong P/G$ $\qquad$ $B = A^{\mathrm{co}H} := \{a \in A \mid \Delta_A(a) = a \otimes 1_H\} \subseteq A$

$$\begin{aligned} P \times G &\to P \times_M P \\ (p, g) &\mapsto (p, p \triangleleft g) \end{aligned} \qquad \begin{aligned} \chi \colon A \otimes_B A &\to A \otimes H \\ a \otimes_B a' &\mapsto a\Delta_A(a') \end{aligned}$$

Hopf–Galois extensions generalize G-principal bundles. They are **quantum principal bundles**:

- A total space algebra
- B base space algebra
- H structure Hopf algebra

Quantum principal bundles

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Construct differential calculi $\Omega^\bullet(A)$, $\Omega^\bullet(B)$, $\Omega^\bullet(H)$ with

$$\begin{tikzcd} \Omega^\bullet(A) \arrow[r, "\Delta_A^\bullet"] & \Omega^\bullet(A) \otimes \Omega^\bullet(H) \\ \Omega^\bullet(B) \arrow[u, hook] & \end{tikzcd}$$

such that

- $\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$ is a graded Hopf–Galois extension
- the Atiyah sequence is exact

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

- $\Omega^\bullet(A)$ is braided-commutative w.r.t. a canonical braiding

$$\sigma^\bullet \colon \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$$

...and discuss examples!

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- Đurđević, M.: *Geometry of Quantum Principal Bundles II - Extended Version*. Rev. Math. Phys. **9**, 5 (1997) 531-607.
- Đurđević, M.: *Quantum Principal Bundles as Hopf-Galois Extensions*. Preprint arXiv:q-alg/9507022.
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Differential calculi

Fix a field $\Bbbk$ and an (associative unital) algebra A.

Definition

A **differential calculus** on A is a differential graded algebra (DGA) $(\Omega^\bullet(A), \wedge, \mathrm{d})$ with $\Omega^\bullet(A) = \bigoplus_{n\geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$ and

$$\Omega^n(A) = \mathrm{span}_\Bbbk\{a^0 \mathrm{d}a^1 \wedge \ldots \wedge \mathrm{d}a^n \mid a^0, a^1, \ldots, a^n \in A\}$$

for $n > 0$. $\rightsquigarrow$ noncommutative de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}} := \frac{\ker \mathrm{d}|_{\Omega^n(A)}}{\mathrm{Im}\ \mathrm{d}|_{\Omega^{n-1}(A)}}$.

Truncating a differential calculus $\Omega^\bullet(A)$ in degree 1 gives the following

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We call $(\Omega^1(A), \mathrm{d})$ a **first order differential calculus** (FODC) on A, if

i.) $\Omega^1(A)$ is an A-bimodule.

ii.) $\mathrm{d}\colon A \to \Omega^1(A)$ is $\Bbbk$-linear s.t. for all $a, b \in A$ the **Leibniz rule** holds:

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b).$$

iii.) $\Omega^1(A) = A\mathrm{d}A := \left\{ \sum_i a^i \mathrm{d}b^i \mid a^i, b^i \in A \text{ finite sum} \right\}$.

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The universal FODC

Theorem (Woronowicz '89)

For every associative unital algebra A there exists a FODC $(\Omega^1_u(A), \mathrm{d}_u)$ on A, where

$$\Omega^1_u(A) := \ker m_A = \left\{ \sum_i a^i \otimes b^i \mid \sum_i a^i b^i = 0 \right\} \subseteq A \otimes A$$

and $\mathrm{d}_u(a) = 1 \otimes a - a \otimes 1$ for all $a \in A$.

Moreover, every FODC $(\Omega^1(A), \mathrm{d})$ on A is a quotient of $(\Omega^1_u(A), \mathrm{d}_u)$.

$\rightsquigarrow$ $\exists$ *an A-subbimodule $\mathcal{N} \subseteq \Omega^1_u(A)$ s.t. $\Omega^1_u(A)/\mathcal{N} \cong \Omega^1(A)$ and*

$$\begin{tikzcd} \Omega^1_u(A) \arrow[r, "\pi"] & \Omega^1_u(A)/\mathcal{N} \arrow[d, "\cong"] \\ A \arrow[u, "\mathrm{d}_u"] \arrow[r, "\mathrm{d}"] & \Omega^1(A) \end{tikzcd}$$

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The maximal prolongation

- A associative unital algebra over $\Bbbk$
- (Ω^1, d) FODC on A

$\Rightarrow (\Omega^1)^{\otimes_A} := A \oplus \Omega^1 \oplus (\Omega^1 \otimes_A \Omega^1) \oplus \ldots$ is graded associative unital algebra with product $\otimes_A$ and unit $1 \in A$.

Consider the graded ideal $S^\wedge \subseteq (\Omega^1)^{\otimes_A}$ generated by the elements

$$\sum_i \mathrm{d}a^i \otimes_A \mathrm{d}b^i, \qquad \text{where } a^i, b^i \in A, \text{ s.t. } \sum_i a^i \mathrm{d}b^i = 0.$$

Define the graded assoc. unital algebra $\Omega^\bullet := (\Omega^1)^{\otimes_A} / S^\wedge$ with induced product $\wedge$.

Theorem (The maximal prolongation)

i.) $\mathrm{d}\colon A \to \Omega^1$ *uniquely extends to a differential on* $\Omega^\bullet$ *s.t.* $\Omega^\bullet$ *is a DC on* A.

ii.) *Let* $\tilde{\Omega}^\bullet$ *be any DC on* A *s.t.* $\tilde{\Omega}^1 = \Omega^1$ *and* $\tilde{\mathrm{d}}|_A = \mathrm{d}$. *Then* $\exists$ *a surjective morphism* $\Omega^\bullet \to \tilde{\Omega}^\bullet$ *of DGAs, i.e.* $\tilde{\Omega}^\bullet$ *is a quotient of* $\Omega^\bullet$.

$\rightsquigarrow$ truncation and maximal prolongation determine adjoint functors

$$\begin{tikzcd} \underline{\mathrm{FODC}}_A \arrow[r, bend left] & \underline{\mathrm{DC}}_A \arrow[l, bend left] \end{tikzcd}$$

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ii.) *Let* $\tilde{\Omega}^\bullet$ *be any DC on* A *s.t.* $\tilde{\Omega}^1 = \Omega^1$ *and* $\tilde{\mathrm{d}}|_A = \mathrm{d}$. *Then* $\exists$ *a surjective morphism* $\Omega^\bullet \to \tilde{\Omega}^\bullet$ *of DGAs, i.e.* $\tilde{\Omega}^\bullet$ *is a quotient of* $\Omega^\bullet$.

$\rightsquigarrow$ truncation and maximal prolongation determine adjoint functors

$$\begin{tikzcd} \underline{\mathrm{FODC}}_A \arrow[r, bend left] & \underline{\mathrm{DC}}_A \arrow[l, bend left] \end{tikzcd}$$

The maximal prolongation

- A associative unital algebra over $\Bbbk$
- (Ω^1, d) FODC on A

$\Rightarrow (\Omega^1)^{\otimes_A} := A \oplus \Omega^1 \oplus (\Omega^1 \otimes_A \Omega^1) \oplus \dots$ is graded associative unital algebra with product $\otimes_A$ and unit $1 \in A$.

Consider the graded ideal $S^\wedge \subseteq (\Omega^1)^{\otimes_A}$ generated by the elements

$$\sum_i \mathrm{d}a^i \otimes_A \mathrm{d}b^i, \qquad \text{where } a^i, b^i \in A, \text{ s.t. } \sum_i a^i \mathrm{d}b^i = 0.$$

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Example

For a smooth manifold M and de Rham 1-forms Ω^1_{dR} the maximal prolongation $(\Omega^1_{\mathrm{dR}})^\bullet = \Omega^\bullet_{\mathrm{dR}}$ coincides with the de Rham complex.

Example

The maximal prolongation of the **universal FODC** $(\Omega^1_u, \mathrm{d}_u)$ on an associative unital algebra A is $(\Omega^\bullet_u, \wedge_u, \mathrm{d}_u)$, with

- $\Omega^n_u := \bigcap_{k=1}^n \ker m^k$, where $m^k := \mathrm{Id}_{A^{\otimes(k-1)}} \otimes m \otimes \mathrm{Id}_{A^{\otimes(n-k)}} : A^{\otimes(n+1)} \to A^{\otimes n}$.
- $$(a^0 \otimes \cdots \otimes a^n) \wedge_u (b^0 \otimes \cdots \otimes b^m) := a^0 \otimes \cdots \otimes a^n b^0 \otimes \cdots \otimes b^m,$$
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Example (The algebraic noncommutative torus $\mathcal{O}_\theta(\mathbb{T}^2)$)

is generated by u, v with relations $vu = e^{\mathrm{i}\theta} uv$ for $\theta \in \mathbb{R}$. Define Ω^1 as the free left $\mathcal{O}_\theta(\mathbb{T}^2)$-module with generators $\mathrm{d}u, \mathrm{d}v$ and right actions

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Hopf Algebras

$\Bbbk$ field (or even a commutative unital ring, e.g. $\mathbb{R}[[\hbar]], \mathbb{C}[[\hbar]], \ldots$)

$$H \text{ Hopf algebra} \begin{cases} \Bbbk\text{-algebra } (H, m, \eta) \\ \Bbbk\text{-coalgebra } \begin{cases} \text{coproduct } \Delta\colon H \to H \otimes H \\ \text{counit } \epsilon\colon H \to \Bbbk \end{cases} \begin{matrix} \text{algebra} \\ \text{morphisms} \end{matrix} \\ \text{antipode } S\colon H \to H \end{cases}$$

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\Delta\otimes\mathrm{id}"] \\ H \otimes H \arrow[r, "\mathrm{id}\otimes\Delta"] & H \otimes H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & H \otimes H \arrow[d, "\epsilon\otimes\mathrm{id}"] \\ & H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[d, "\mathrm{id}\otimes\epsilon"'] & H \arrow[l, "\Delta"'] \arrow[dl, "\mathrm{id}"] \\ H & \end{tikzcd}$$

$$\begin{tikzcd} & H \otimes H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[r, "\epsilon"] \arrow[dr, "\Delta"'] & \Bbbk \arrow[rr, "\eta"] & & H \\ & H \otimes H \arrow[r, "\mathrm{id}\otimes S"'] & H \otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

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Example (Hopf Algebras)

1. Group algebra $\mathbb{k}[G]$ of group G. $\Delta(g)=g\otimes g$, $\epsilon(g)=1$, $S(g)=g^{-1}$ for $g\in G$ and extended $\mathbb{k}$-linearly.
2. Universal enveloping algebra $\mathscr{U}\mathfrak{g}$ of Lie algebra $\mathfrak{g}$.
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3. Regular functions $\mathcal{F}(G)$ on algebraic group G. $\Delta(f)(g,h)=f(gh)$, $\epsilon(f)=f(e)$, $S(f)(g)=f(g^{-1})$ for $g,h\in G$.
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$$\begin{aligned} \beta\alpha &= q\alpha\beta, \quad \gamma\alpha = q\alpha\gamma, \quad \delta\beta = q\beta\delta, \quad \delta\gamma = q\gamma\delta, \\ \gamma\beta &= \beta\gamma, \quad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma, \quad \alpha\delta - q^{-1}\beta\gamma = 1 \end{aligned}$$

with $\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q\beta \\ -q^{-1}\gamma & \alpha \end{pmatrix}$.

Calculi on Hopf algebras

H Hopf algebra, $\Delta(h) = h_1 \otimes h_2$, $\varepsilon\colon H \to \mathbb{k}$, $S\colon H \to H$

$H^+ := \ker \varepsilon \subseteq H$
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Def: A FODC $\Omega^1(H)$ is **left covariant** if ${}_{\Omega^1(H)}\Delta\colon \Omega^1(H) \to H \otimes \Omega^1(H)$ coaction s.t. $\mathrm{d}\colon H \to \Omega^1(H)$ left colinear and ${}_{\Omega^1(H)}\Delta(h \cdot \omega \cdot g) = \Delta(h) \cdot {}_{\Omega^1(H)}\Delta(\omega) \cdot \Delta(g)$.

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Theorem (Woronowicz '89)

$$\left\{ \begin{matrix} \text{left covariant} \\ \text{FODC on } H \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{right } H\text{-ideal} \\ I \subseteq H^+ \end{matrix} \right\}$$

$\Omega^1 \cong H \otimes \Lambda^1 \cong H \otimes H^+/I$ *via* ***Maurer–Cartan form*** $\varpi\colon H^+ \twoheadrightarrow \Lambda^1$, $\varpi(h) := S(h_1)\mathrm{d}h_2$
Then $\Omega^1(H)$ *is even bicovariant iff* $\mathrm{Ad}(I) \subseteq I \otimes H$.

Theorem (Beggs–Majid '20)

The maximal prolongation of a left covariant FODC $\Omega^1(H)$ *is* $\Omega^\bullet(H) \cong H \otimes \Lambda^\bullet$, *where* $\Lambda^\bullet$ *is generated by* Λ^1 *modulo* $\wedge \circ (\varpi \circ \pi_\varepsilon \otimes \varpi \circ \pi_\varepsilon)\Delta(I) = 0$.

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Graded Hopf algebra

Let $\Omega^1(H)$ bicovariant with maximal prolongation $\Omega^\bullet(H)$.

Lemma (Beggs–Majid '20)

$\Delta\colon H \to H \otimes H$ extends to a morphism of DGAs $\Delta^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H)$.

$\rightsquigarrow$ write $\Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]} \in \bigoplus_{k+\ell=n} \Omega^k(H) \otimes \Omega^\ell(H)$ for $\omega \in \Omega^n(H)$.

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$\Omega^\bullet(H)$ is a graded Hopf algebra with

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Hopf algebra H

Comodule algebra A, i.e. assoc. unital algebra with right H-coaction $\Delta_A\colon A \to A \otimes H$ s.t. Δ_A is algebra morphism.

$\rightsquigarrow B = A^{\mathrm{co}H} := \{a \in A \mid \Delta_A(a) = a \otimes 1_H\} \subseteq A$ is subalgebra

Recall: **quantum principal bundle** (QPB) also called **Hopf–Galois extension**

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] & A \otimes H \\ B \arrow[u, hook] & A \otimes_B A \arrow[u, "\cong"'] \end{tikzcd} \qquad \chi\colon A \otimes_B A \to A \otimes H, \quad a \otimes_B a' \mapsto a\Delta_A(a')$$

- A total space algebra
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Definition ([Đurđević '97])

A DC $\Omega^\bullet(A)$ on A is called **complete** if the right H-coaction $\Delta_A \colon A \to A \otimes H$ extends to a morphism

$$\Delta_A^\bullet \colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$$

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We use the short notation

$$\Delta_A^\bullet(\omega) = \omega_{[0]} \otimes \omega_{[1]}$$

Proposition

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$$(a \otimes \vartheta) \wedge (a' \otimes \vartheta') := aa'_0 \otimes S(a'_1)\vartheta a'_2 \wedge \vartheta',$$
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For a complete calculus $\Omega^\bullet(A)$ on a QPB A we define the **horizontal forms** as the preimage

$$\mathrm{hor}^\bullet := (\Delta_A^\bullet)^{-1}(\Omega^\bullet(A) \otimes H)$$

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- $\mathrm{hor}^\bullet$ is a right H-comodule algebra
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Definition

Let $\Omega^\bullet(A)$ be a complete calculus on a QPB $B = A^{\mathrm{co}H} \subseteq A$. The corresponding **base forms** are defined as the graded subspace

$$\Omega^\bullet(B) := \{\omega \in \Omega^\bullet(A) \mid \Delta_A^\bullet(\omega) = \omega \otimes 1_H\}.$$

- $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ is a DG subalgebra
- but $\Omega^\bullet(B)$ **might not** be generated in degree 0: $B\mathrm{d}B \subseteq \Omega^1(B)$
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Let $\Omega^\bullet(A)$ be a complete calculus on a QPB $B = A^{\mathrm{co}H} \subseteq A$. The corresponding **base forms** are defined as the graded subspace

$$\Omega^\bullet(B) := \{\omega \in \Omega^\bullet(A) \mid \Delta_A^\bullet(\omega) = \omega \otimes 1_H\}.$$

- $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ is a DG subalgebra
- but $\Omega^\bullet(B)$ **might not** be generated in degree 0: $B\mathrm{d}B \subseteq \Omega^1(B)$
- however in all examples we encounter $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ equals the pullback calculus

Theorem

For any complete calculus $\Omega^\bullet(A)$ on a QPB $B = A^{\mathrm{co}H} \subseteq A$ the ***Atiyah sequence***

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

is exact in the category ${}_A\mathcal{M}^H_A$ of right H-covariant A-bimodules.

Given a complete calculus $\Omega^\bullet(A)$ we define

$$\begin{aligned}\chi^\bullet \colon \Omega^\bullet(A \otimes_B A) &\to \Omega^\bullet(A) \otimes \Omega^\bullet(H)\\ \omega \otimes_{\Omega^\bullet(B)} \eta &\mapsto \omega \Delta^\bullet_A(\eta) = \omega \wedge \eta_{[0]} \otimes \eta_{[1]}\end{aligned}$$

Theorem

$\Omega^\bullet(A)$ complete calculus on QPB $B = A^{\mathrm{co}H} \subseteq A$. Then

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$$

is a graded Hopf–Galois extension.

The inverse of $\chi^\bullet$ is $\omega \otimes \theta \mapsto \omega \wedge \tau^\bullet(\theta)$ with well-defined graded map

$$\tau^\bullet \colon \Omega^\bullet(H) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$$

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Example: Bicovariant calculi [DelDonno-Latini-TW '25]

Consider the Galois object $\Bbbk = H^{\mathrm{co}H} \subseteq H$.

For any bicovariant FODC $\Omega^1(H)$ with max. prolongation $\Omega^\bullet(H)$ we have that

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- horizontal and base forms are trivial, while $\mathrm{ver}^\bullet = \Omega^\bullet(H)$.
- $\chi^\bullet(\omega \otimes \eta) = \omega \wedge \eta_{[1]} \otimes \eta_{[2]}$ is invertible with inverse $\omega \otimes \eta \mapsto \omega \wedge S^\bullet(\eta_{[1]}) \otimes \eta_{[2]}$.

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Example: The noncommutative 2-torus

Let $A := \mathcal{O}_\theta(\mathbb{T}^2) := \mathbb{C}[u, u^{-1}, v, v^{-1}]/\langle vu - e^{i\theta} uv\rangle$ for $\theta \in \mathbb{R}$.

It is a right $H := \mathcal{O}(U(1))$-comodule algebra $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}$ and a faithfully flat Hopf–Galois extension, with coinvarint subalgebra $B := A^{\mathrm{co}H} = \mathrm{span}_{\mathbb{C}}\{(uv)^{\pm k}\}$ and cleaving map $j\colon H \to A$, $\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ for $k \geq 0$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{\mathrm{d}u,\mathrm{d}v\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{\mathrm{d}u\wedge\mathrm{d}v\}}$ and $\Omega^\bullet(H) = H \oplus \underbrace{\Omega^1(H)}_{=t^{\pm k}\mathrm{d}t}$.

Proposition (DelDonno-Latini-TW '25)

$\Omega^\bullet(A)$ is a complete calculus on the noncommutative 2-torus and $\Omega^\bullet(B)$ is the usual pullback calculus.

$$\Omega^1(A) \to \underbrace{\Omega^1(A) \otimes H}_{\Delta_{\Omega^1(A)}} \oplus \underbrace{A \otimes \Omega^1(H)}_{\mathrm{ver}}$$

$$\Omega^2(A) \to \underbrace{\Omega^2(A) \otimes H}_{\Delta_{\Omega^2(A)}} \oplus \underbrace{\Omega^1(A) \otimes \Omega^1(H)}_{\mathrm{ver}^{1,1}\colon\, \mathrm{d}u\wedge\mathrm{d}v \mapsto -\mathrm{d}(uv)\otimes t^{-1}\mathrm{d}t}$$

Already known in the literature.

Well-defined according to our calculations.

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It is a right $H := \mathcal{O}(U(1))$-comodule algebra $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}$ and a faithfully flat Hopf–Galois extension, with coinvarint subalgebra $B := A^{\mathrm{co}H} = \mathrm{span}_{\mathbb{C}}\{(uv)^{\pm k}\}$ and cleaving map $j\colon H \to A$, $\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ for $k \geq 0$.

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$\Omega^\bullet(A)$ is a complete calculus on the noncommutative 2-torus and $\Omega^\bullet(B)$ is the usual pullback calculus.

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Well-defined according to our calculations.

Example: The noncommutative 2-torus

The Đurđević braiding $\sigma\colon A\otimes_B A\to A\otimes_B A$ reads

$$\sigma(u\otimes_B f)=ufu^{-1}\otimes_B u,\qquad \sigma(v\otimes_B f)=vfv^{-1}\otimes_B v$$

for all $f\in A$. For generators of differential forms we obtain

$$\sigma^\bullet(\mathrm{d}u\otimes_{\Omega^\bullet(B)} u)=\mathrm{u}\otimes_{\Omega^\bullet(B)}\mathrm{d}u,\qquad \sigma^\bullet(\mathrm{d}v\otimes_{\Omega^\bullet(B)} v)=v\otimes_{\Omega^\bullet(B)}\mathrm{d}v,$$
$$\sigma^\bullet(\mathrm{d}u\otimes_{\Omega^\bullet(B)} v)=e^{-i\theta}v\otimes_{\Omega^\bullet(B)}\mathrm{d}u,\qquad \sigma^\bullet(\mathrm{d}v\otimes_{\Omega^\bullet(B)} u)=e^{i\theta}u\otimes_{\Omega^\bullet(B)}\mathrm{d}v,$$

$$\sigma^\bullet(\mathrm{d}u\otimes_{\Omega^\bullet(B)}\mathrm{d}u)=-\mathrm{d}u\otimes_{\Omega^\bullet(B)}\mathrm{d}u,\qquad \sigma^\bullet(\mathrm{d}v\otimes_{\Omega^\bullet(B)}\mathrm{d}v)=-\mathrm{d}v\otimes_{\Omega^\bullet(B)}\mathrm{d}v,$$
$$\sigma^\bullet(\mathrm{d}u\otimes_{\Omega^\bullet(B)}\mathrm{d}v)=-e^{-i\theta}\mathrm{d}v\otimes_{\Omega^\bullet(B)}\mathrm{d}u,\qquad \sigma^\bullet(\mathrm{d}v\otimes_{\Omega^\bullet(B)}\mathrm{d}u)=-e^{i\theta}\mathrm{d}u\otimes_{\Omega^\bullet(B)}\mathrm{d}v$$
$$\sigma^\bullet(\mathrm{d}u\wedge\mathrm{d}v\otimes_{\Omega^\bullet(B)} u)=e^{i\theta}u\otimes_{\Omega^\bullet(B)}\mathrm{d}u\wedge\mathrm{d}v,$$
$$\sigma^\bullet(\mathrm{d}u\wedge\mathrm{d}v\otimes_{\Omega^\bullet(B)} v)=e^{-i\theta}v\otimes_{\Omega^\bullet(B)}\mathrm{d}u\wedge\mathrm{d}v$$

etc...

In this case $\sigma^\bullet\colon\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)\to\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)$ is symmetric!

Let $A := \mathcal{O}_q(\mathrm{SU}_2)$ with $H := \mathcal{O}(U(1))$ coaction $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

This is known to be a faithfully flat Hopf–Galois extension with coinvariant subalgebra the Podleś sphere $B := A^{\mathrm{co}H}$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{e^\pm, e^0\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{e^\pm \wedge e^0, e^+ \wedge e^-\}} \oplus \underbrace{\Omega^3(A)}_{=\mathrm{span}_A\{e^+ \wedge e^- \wedge e^0\}}$ and

$\Omega^\bullet(H) = H \oplus \Omega^1(H)$ with $\mathrm{d}t \cdot t = q^2 t \mathrm{d}t$.

Proposition (DelDonno-Latini-TW '25)

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Let $A := \mathcal{O}_q(\mathrm{SU}_2)$ with $H := \mathcal{O}(U(1))$ coaction $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}$.

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Đurđević braiding

$$\begin{aligned}
\sigma(\alpha\otimes_B\alpha)&=\alpha\otimes_B\alpha, & \sigma(\alpha\otimes_B\beta)&=q^{-1}\beta\otimes_B\alpha,\\
\sigma(\alpha\otimes_B\gamma)&=q^{-1}\gamma\otimes_B\alpha, & \sigma(\alpha\otimes_B\delta)&=\delta\otimes_B\alpha+(q^{-1}-q)\beta\otimes_B\gamma,\\
\sigma(\beta\otimes_B\beta)&=\beta\otimes_B\beta, & \sigma(\beta\otimes_B\gamma)&=\gamma\otimes_B\beta,\\
\sigma(\beta\otimes_B\delta)&=q^{-1}\delta\otimes_B\beta, & \sigma(\gamma\otimes_B\gamma)&=\gamma\otimes_B\gamma,\\
\sigma(\gamma\otimes_B\delta)&=q^{-1}\delta\otimes_B\gamma, & \sigma(\delta\otimes_B\delta)&=\delta\otimes_B\delta.
\end{aligned}$$

$$\begin{aligned}
&\sigma^\bullet(e^+\otimes_{\Omega^\bullet(B)}e^+)=\sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}e^-)=0, && \sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}e^0)=-e^0\otimes_{\Omega^\bullet(B)}e^0,\\
&\sigma^\bullet(e^+\otimes_{\Omega^\bullet(B)}e^-)=-q^{-2}e^-\otimes_{\Omega^\bullet(B)}e^+, && \sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}e^+)=-q^2e^+\otimes_{\Omega^\bullet(B)}e^-,\\
&\sigma^\bullet(e^\pm\otimes_{\Omega^\bullet(B)}e^0)=-q^{\mp4}e^0\otimes_{\Omega^\bullet(B)}e^\pm,
\end{aligned}$$

$$\sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}e^\pm)=-e^\pm\otimes_{\Omega^\bullet(B)}e^0+(1-q^{\mp4})e^0\wedge e^\pm\otimes_{\Omega^\bullet(B)}1,$$

$$\begin{aligned}
\sigma^\bullet(e^\pm\otimes_{\Omega^\bullet(B)}(e^\pm\wedge e^0))&=0,\\
\sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}(e^+\wedge e^0))&=q^2(e^+\wedge e^0)\otimes_{\Omega^\bullet(B)}e^-,\\
\sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}(e^-\wedge e^0))&=(e^-\wedge e^0)\otimes_{\Omega^\bullet(B)}e^0,
\end{aligned}$$

etc... The braiding is symmetric on A but not symmetric on $\Omega^\bullet(A)$!

Crossed product algebras and cleft extensions

Definition

Let A be a right H-comodule algebra and $B := A^{\mathrm{co}H}$. We call $B \subseteq A$

- **trivial extension** if $\exists$ convolution invertible comodule algebra morphism $j\colon H \to A$.
- **cleft extension** if $\exists$ convolution invertible comodule morphism $j\colon H \to A$.

$$\big(\text{trivial extensions}\big) \subseteq \big(\text{cleft extensions}\big) \subseteq \big(\text{Hopf–Galois extensions}\big)$$

Theorem (Doi–Takeuchi '86)

$$\big(\textit{cleft extensions}\big) \xleftrightarrow{1:1} \big(\textit{crossed product algebras}\big)$$

where trivial extensions correspond to smashed product algebras.

B algebra with $\Bbbk$-linear map $\cdot\colon H \otimes B \to B$ such that $h \cdot (bb') = (h_1 \cdot b)(h_2 \cdot b')$ and $h \cdot 1 = \varepsilon(h)1$. $F\colon H \otimes H \to B$ convolution invertible 2-cocycle with values in B.
$\rightsquigarrow$ $B\#_F H$ is **crossed product algebra** with multiplication

$$(b \otimes h)(b' \otimes h') := b(h_1 \cdot b')F(h_2 \otimes h'_1) \otimes h_3 h'_2.$$

Crossed product algebras and cleft extensions

A FODC $\Omega^1(B)$ is called F-**compatible** if

$$h \cdot (b\mathrm{d}b') = (h_1 \cdot b)\mathrm{d}(h_2 \cdot b'), \qquad \mathrm{d} \circ F = 0.$$

Proposition (Sciandra-TW '25)

For every F-compatible FODC $\Omega^1(B)$ and bicov. $\Omega^1(H)$ there is a FODC on $B\#_F H$

$$\Omega^1_\# := (\Omega^1(B) \otimes H) \oplus (B \otimes \Omega^1(H))$$

with appropriate $B\#_F H$-bimodule structure and $\mathrm{d}_\# = \mathrm{d}_B + \mathrm{d}_H$.

Proposition (DelDonno-Latini-TW '25)

The above can be extended to a complete DC $\Omega^n_\# := \bigoplus_{i=0}^n \Omega^i(B) \otimes \Omega^{n-i}(H)$ on the crossed product algebra $B\#_F H$.

- *the base forms are $\Omega^\bullet(B)$, while $\mathrm{ver}^\bullet = B \otimes \Omega^\bullet(H)$ and $\mathrm{hor}^\bullet = \Omega^\bullet(B) \otimes H$.*
- *The Đurđević braiding reads*

$$\begin{aligned}&\sigma((b \otimes h) \otimes_B (b' \otimes h')) \\ &\quad = (b(h_1 \cdot b')F(h_2 \otimes h'_1)(h_3 h'_2 \cdot F^{-1}(S(h_8) \otimes h_9))F(h_4 h'_3 \otimes S(h_7)) \otimes h_5 h'_4 S(h_6)) \otimes_B (1 \otimes h_{10})\end{aligned}$$

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Thank you for your attention!