Connected $(C_4, Diamond)$ -free Graphs Are Uniquely Reconstructible from Their Token Graphs

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Abstract

A diamond is the graph that is obtained from removing an edge from the complete graph on 4 vertices. A $(C_4, \text{diamond})$ -free graph is a graph that does not contain a diamond or a cycle on four vertices as induced subgraphs. Let G be a connected $(C_4, \text{diamond})$ -free graph on n vertices. Let $1 \le k \le n-1$ be an integer. The k-token graph, $F_k(G)$, of G is the graph whose vertices are all the sets of k vertices of G; two of which are adjacent if their symmetric difference is a pair of adjacent vertices in G. Let F be a graph isomorphic to $F_k(G)$. In this paper we show that given only F, we can construct in polynomial time a graph isomorphic to G. Let Aut(G) be the automorphism group of G. We also show that if $k \ne n/2$, then $Aut(G) \simeq Aut(F_k(G))$; and if k = n/2, then $Aut(G) \simeq Aut(F_k(G)) \times \mathbb{Z}_2$.

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1 Introduction

Let G be a graph on n vertices, and $1 \le k \le n-1$ a natural number. The k-token graph of G is the graph whose vertices are all the sets of k vertices of G; where two such sets A and B are adjacent if their symmetric difference $A \triangle B$ is a pair of adjacent vertices in G. We denote this graph by $F_k(G)$. The name "token graph" is motivated by the following interpretation. Take k indistinguishable tokens and place them on the vertices of G (at most one per vertex); form a new graph whose vertices are all possible token configurations; and make two configurations adjacent if one can be reached from the other by taking a token and sliding it along an edge to an unoccupied vertex. The resulting graph is isomorphic to $F_k(G)$. We often refer to the vertices of $F_k(G)$ as token configurations.

Token graphs have been defined independently at least four times:

- 1. In 1988, in his PhD thesis, Johns [13] called it the *k*-subgraph graph of *G*. He defined it as the graph whose vertices are all the subsets of *k* vertices of *G*; two of which are adjacent if their distance in *G* is equal to 1.
- 2. In 1991, Alavi, Behzad, Erdős and Lick [1] defined the 2-token graph; they called it the *double vertex* graph. In 1992, Zhu, Liu, Lick and Alavi [23] expanded the definition to k tokens and called it k-tuple graph. They defined it as the graph whose vertices are all the subsets of k vertices of G; two of which are adjacent if their symmetric difference is an edge of G.
- 3. In 2002, Rudolph [17] considered a cluster of n interacting q-bits. Each q-bit can be in a ground state $|0\rangle$ or in an excited state $|1\rangle$. At any given moment exactly k of the q-bits are in the excited state. He represented each q-bit by a vertex in a graph G where two are adjacent if they interact. The k-token graph of G represents the possible evolution of this cluster of q-bits. His aim was to translate physical quantities of a cluster of q-bits to graph invariants of the token graph. He called this construction the level k matrix of G.
- 4. In 2012, Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood [8] defined token graphs with the token configurations interpretation provided above.

Finally, the k-token graph of G has also been named the k-th symmetric power of G by Audenaert, Godsil, Royle and Rudolph [3]. This definition is excluded from the previous list because in [3] the definition is attributed to [17]. In this paper we follow the notation of [8].

In this paper we are interested in the existential and algorithmic problem of reconstructing a graph from its token graph. Specifically, let F be a graph isomorphic to $F_k(G)$.

- Given only F, can we find in polynomial time a graph G' such that $F_k(G') \simeq F$?
- is G' unique up to isomorphism?

A diamond is the graph that results from removing an edge from a complete graph on four vertices; C_4 is the cycle on four vertices. A graph is $(C_4, \text{diamond})$ -free if it does not contain a diamond or a C_4 as an induced subgraph. In this paper we consider the problem of reconstructing a graph G from its token graph, when G is connected and $(C_4, \text{diamond})$ -free.

The problem of reconstructing a graph from its token graph seems to be related to the Graph Isomorphism Problem. The Graph Isomorphism Problem is the algorithmic problem of determining whether two given graphs are isomorphic. The current best published algorithm for this problem was given by Babai and Luks [5]. This algorithm runs in $\exp(O(\sqrt{n \log n}))$ time for graphs on n vertices. In 2015, Babai [4] announced a $\exp((\log)^{O(1)})$ time algorithm for the Graph Isomorphism Problem. Helfgott discovered an error in the proof. In 2017, Babai announced a correction¹, which Helfgott verified².

There are many graph invariants, computable in polynomial time, that in many instances distinguish pairs of non isomorphic graphs. One of these is the spectra of a graph (the eigenvalues of its adjacency matrix). Two graphs are *cospectral* if they have the same spectra. As expected there are pairs of non-isomorphic cospectral graphs. In [17], Rudolph noted that the spectra of 2-token graphs may help in distinguishing two graphs. He gave an example of a pair of non-isomorphic cospectral graphs whose 2-token graphs are not cospectral. In [3], the authors showed that the 2-token graphs of two strongly regular graphs with the same parameters are cospectral. Thus, yielding a plethora of examples of pairs of non-isomorphic graphs whose 2-token graphs are cospectral. In the same paper it is noted that if for

¹http://people.cs.uchicago.edu/~laci/update.html

²https://valuevar.wordpress.com/2017/01/04/graph-isomorphism-in-subexponential-time/

some constant k it is the case that two graphs are isomorphic if and only if their k-token graphs are cospectral, then this would provide a polynomial time algorithm for the Graph Isomorphism Problem. This was shown not to be the case independently by Barghi and Ponomarenko [6], and Alzaga, Iglesias and Pignol [2]. Recently, Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete and Zaragoza-Martínez [7], considered the Laplacian spectra of token graphs. They showed that the Laplacian spectra of a graph is closely related to the Laplacian spectra of its token graphs. There is no known example of a pair of non-isomorphic graphs whose token graphs have the same Laplacian spectra.

Underlying the question of whether token graphs may help in distinguishing pairs of non-isomorphic graphs, is the question of how much information from G is carried out to the k-token graphs of G. In [8] the authors made the following conjecture.

Conjecture 1.1. Let G and H be two graphs such that for some k their k-token graphs are isomorphic. Then G and H are isomorphic.

Conjecture 1.1 was posed as a question for 2-token graphs by Jacob, Goddard and Laskar [12]. An equivalent formulation is that $F_k(G)$ determines G completely (up to isomorphism). If this is the case for some graph G we say that G can be reconstructed from its token graph. We believe this to be a hard problem, even in the case of only two tokens. There are very few results in this direction. We mention some of them. In [12], it is shown that if G is regular and does not contain a 4-cycle as a subgraph then G is reconstructible from it 2-token graph. They also show that cubic graphs can be reconstructed from their 2-token graphs. In [1] it is claimed (without proof) that trees can be reconstructed from their 2-token graphs. Trujillo-Negrete [20] in her Master's thesis gave an example of two non-isomorphic graphs G and H, and a pair of distinct integers k and l, such that $F_k(G)$ and $F_l(H)$ are isomorphic (and non-trivial). For completeness we provide this example in Section 7.

1.1 Notation

We now provide some of the notation used throughout the paper. Let G=(V,E) be a graph. We denote with |G| and |G| the number of vertices and edges of G, respectively. Let U,W be two sets of vertices of G or two subgraphs of G. We denote with E(U,W) the set of edges of G with one endpoint in G and the other endpoint in G. If G is an edge in G is always assume that G is an edge in G in G is an edge in G in

Two graphs G and H are isomorphic if there exists a bijection, φ , between the vertices of G and the vertices of H that satisfies the following. A vertex x is adjacent to a vertex y in G if and only if $\varphi(x)$ is adjacent to $\varphi(y)$ in H. We say that φ is an isomorphism between G and H. We write $G \simeq H$ to denote that G and H are isomorphic. We denote with $\operatorname{Iso}(G,H)$ the set of isomorphisms from G to G to G form a group under function composition; we denote this group by $\operatorname{Aut}(G)$.

Let G_1, \ldots, G_n be graphs. The Cartesian product of G_1, \ldots, G_n is the graph $G_1 \square \cdots \square G_n$ with vertex set $V(G_1) \times \cdots \times V(G_n)$; where (x_1, \ldots, x_n) is adjacent to (y_1, \ldots, y_n) if and only if there exists an index $1 \le i \le n$ such that x_i is adjacent to y_i and $x_j = y_j$ for all $j \ne i$. Let $v := (x_1, \ldots, x_n)$ be a vertex of $G_1 \square \cdots \square G_n$; we denote the *i*-th coordinate of (x_1, \ldots, x_n) with $v(i) := x_i$. Cartesian products of graphs play an important role throughout this paper. The *d*-dimensional hypercube is the Cartesian product of *d* copies of K_2 . We denote it with Q_d . A graph is composite if it is isomorphic to the Cartesian product of two or more nontrivial graphs. Otherwise, we say it is a prime graph.

The line graph of G is the graph, L(G), whose vertex set is the edge set of G. Two vertices of L(G) are adjacent if as edges of G they are incident to the same vertex. Whitney [22] showed that, except for the cases of a triangle and $K_{1,3}$, if G and G' are two graphs such that $L(G) \simeq L(G')$ then $G \simeq G'$. For |G| > 3, Roussopoulos [16] and Lehot [14] gave an O(|G| + |G|) time algorithm that given a graph isomorphic to L(G) constructs a graph isomorphic to G.

1.2 Main results

We mention the main results of this paper. Our first result is the following.

Theorem 1.2. Let G be a connected $(C_4, diamond)$ -free graph. Given only a graph isomorphic to $F_k(G)$, we can compute in polynomial time a graph isomorphic to G.

Let F be a graph. Let φ be an isomorphism from F to $F_k(G)$. We call the pair (G, φ) a k-token reconstruction of F. We say that a graph G is k-token reconstructible if for every (G', φ') , k-token reconstruction of $F_k(G)$, we have that $G \simeq G'$. Thus, Conjecture 1.1 states that all graphs are k-token reconstructible for every $1 \le k \le |G| - 1$. We prove the following result, which is stronger than Theorem 1.2.

Theorem 1.3. Let G be a connected $(C_4, diamond)$ -free graph. Given a graph, F, isomorphic to $F_k(G)$ $(k \le n/2)$, we can compute in polynomial time a k-token reconstruction of F.

In Section 2, we introduce the notion of a graph being uniquely k-token reconstructible as the k-token graph of G. Informally, a graph F is uniquely reconstructible as the k-token graph of G if all its k-token reconstruction as the k-token graph of G, are unique up to automorphisms of G. We show the following.

Theorem 1.4. Let G be a connected $(C_4, diamond)$ -free graph. Then $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

We prove the following consequence of being uniquely k-token reconstructible.

Proposition 1.5. Suppose that $F_k(G)$ is uniquely k-token reconstructible as the k-token graph of G.

$$\operatorname{Aut}(F_k(G)) \simeq \begin{cases} \operatorname{Aut}(G) \times \mathbb{Z}_2 & \text{for } k = n/2 \text{ and } n \geq 4, \\ \operatorname{Aut}(G) & \text{otherwise.} \end{cases}$$

Roadmap

In Section 2, we introduce the notion of unique k-token reconstructibility. In Theorem 2.4, we present three conditions equivalent to being uniquely k-token reconstructible. In Section 3, we consider the token graph of stars. Token graphs of stars play an instrumental role in our reconstruction algorithm. We show that token graphs of stars are uniquely reconstructible as the k-token graph of $K_{1,n}$. We also show that if F is isomorphic to $F_k(K_{1,n})$, then a reconstruction of F as the k-token graph of $F_k(K_{1,n})$ can be found in polynomial time. In Section 4, we characterize how 4-cycles are generated in $F_k(G)$; we derive some consequences of this characterization. In Section 5, we prove Theorem 1.2. In Section 6, we prove Theorems 1.3 and 1.4. Finally, in Section 7, we consider the case when G is a disconnected $(C_4, \text{diamond})$ -free graph.

2 Uniquely k-token Reconstructible Graphs

Let H be a graph isomorphic to G. We define a function $\iota : \operatorname{Iso}(H,G) \to \operatorname{Iso}(F_k(H),F_k(G))$ as follows. Let $\iota \in \operatorname{Iso}(H,G)$. Let $\iota(\iota \psi)$ be the function that maps every $A \in V(F_k(G))$ to

$$\iota(\psi)(A) := \{ \psi(v) : v \in A \}.$$

It is straightforward to show that $\iota(\psi) \in \operatorname{Iso}(F_k(H), F_k(G))$. We show that ι is injective. Let $\phi \in \operatorname{Iso}(H, G)$ such that $\psi \neq \phi$. Let $v \in V(H)$ such that $\phi(v) \neq \psi(v)$ and let $u \in V(G)$ be such that $\phi(u) = \psi(v)$. Thus, $u = \phi^{-1}\psi(v)$ and $u \neq v$. Let $A \in V(F_k(H))$ such that $v \in A$ and $v \notin A$. We have that $\psi(v) \notin \iota(\phi)(A)$ and $\psi(v) \in \iota(\psi)(A)$. Therefore, $\iota(\phi)(A) \neq \iota(\psi)(A)$. Let J be a graph isomorphic to G, and let ϕ now be an isomorphism from G to J. It is straightforward to show that

$$\iota(\phi \circ \psi) = \iota(\phi) \circ \iota(\psi).$$

Ibarra and Rivera [10] recently showed that when G = H, i is an injective group homomorphism from Aut(G) to $Aut(F_k(G))$. Thus,

$$Aut(G) < Aut(F_k(G)). \tag{1}$$

Let \mathfrak{c} be the map that sends every set of k vertices of G to its complement in V(G). This map is an isomorphism from $F_k(G)$ to $F_{n-k}(G)$. If k=n/2, then \mathfrak{c} is an automorphism of $F_k(G)$, which we call the complement automorphism of $F_k(G)$.

Proposition 2.1. For n > 2 even and k = n/2, $\mathfrak{c} \notin \iota(\operatorname{Aut}(G))$.

Proof. Suppose for a contradiction that there exists $\phi \in \operatorname{Aut}(G)$ such that $\iota(\phi) = \mathfrak{c}$. Note that ϕ is not the identity; thus, there exists a vertex v_1 of G such that $v_1 \neq \phi(v_1)$. Let $A := \{v_1, \phi(v_1), v_2, \dots, v_{k-1}\}$ be a vertex in $F_k(G)$. Then $V(G) \setminus A = \{\phi(v_1), \phi(\phi(v_1)), \dots, \phi(v_{k-1})\}$. This implies that $\phi(v_1) \in A$ and $\phi(v_1) \notin A$ —a contradiction.

Observation 2.2. Suppose that G is the edge uv. Let swap be the automorphism of G that interchanges these two vertices, so that swap(u) = v and swap(v) = u. Then $\mathfrak{c} = \iota(swap)$.

Note that for every $\psi \in \operatorname{Aut}(G)$ we have that $\mathfrak{c} \circ \iota(\psi) = \iota(\psi) \circ \mathfrak{c}$. Since \mathfrak{c}^2 is the identity, when $n \geq 3$, the group generated by $\operatorname{Aut}(G)$ and \mathfrak{c} is isomorphic to $\operatorname{Aut}(G) \times \mathbb{Z}_2$. Thus, when k = n/2 we have that

$$\operatorname{Aut}(G) \times \mathbb{Z}_2 \le \operatorname{Aut}(F_k(G)).$$
 (2)

The inclusions(1) and (2) can be proper. Using the SageMath [19] and GAP [9] softwares we determined that

$$Aut(K_{2,3}) = \mathbb{Z}_2 \times S_3 < \mathbb{Z}_2 \times S_4 = Aut(F_2(K_{2,3}))$$

and

$$\operatorname{Aut}(C_4) \times \mathbb{Z}_2 = D_4 \times \mathbb{Z}_2 < S_4 \times \mathbb{Z}_2 = \operatorname{Aut}(F_2(C_4)).$$

We define an equivalence relation between k-token reconstructions. Let (G, ψ) and (G, φ) be two k-token reconstructions of a graph F. We say that (G, φ) and (G, ψ) are equivalent k-reconstructions of F if there exists an automorphism $\mathfrak{s}(\varphi, \psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \text{ or } \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi.$$

We say that F is uniquely reconstructible as the k-token graph of G if any two k-reconstructions of F as the k-token graph of G are equivalent. For a given $\varphi \in \text{Iso}(F, F_k(G))$ let

$$I_{\varphi} := \{ \psi \in \operatorname{Iso}(F, F_k(G)) : \psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \}$$

and

$$C_{\varphi} := \{ \psi \in \operatorname{Iso}(F, F_k(G)) : \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \}.$$

By Proposition 2.1, if $|G| \ge 3$, then I_{φ} and C_{φ} are disjoint. Since ι is injective we have the following. Lemma 2.3.

$$I_{\varphi} = \iota(\operatorname{Aut}(G)) \circ \varphi \text{ and } C_{\varphi} = \mathfrak{c} \circ \iota(\operatorname{Aut}(G)) \circ \varphi.$$

For a given vertex $u \in G$ let

$$\kappa_G(u,k) := \{ A \in F_k(G) : u \in A \}$$

and

$$\overline{\kappa_G}(u,k) := \{ A \in F_k(G) : u \notin A \}.$$

In the following theorem we give three equivalent conditions for $F_k(G)$ to be uniquely reconstructible as the k-token graph of G.

Theorem 2.4. Let G and H be isomorphic graphs on at least 3 vertices; the following are equivalent:

1) $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

2)
$$\operatorname{Aut}(F_k(G)) \simeq \begin{cases} \operatorname{Aut}(G) \times \mathbb{Z}_2 & \text{for } k = n/2, \\ \operatorname{Aut}(G) & \text{otherwise.} \end{cases}$$

3) For every $\psi \in \text{Iso}(F_k(H), F_k(G))$ there exists a $f(\psi) \in \text{Iso}(H, G)$ such that

$$\psi = \iota(f(\psi)) \text{ or } \psi = \mathfrak{c} \circ \iota(f(\psi)).$$

4) There exists a function f that assigns to every $\psi \in \text{Iso}(F_k(H), F_k(G))$ a function $f(\psi) : V(H) \to V(G)$ such that the following holds. For every vertex $u \in H$ either

$$\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k)$$
 or $\psi(\kappa_H(u,k)) = \overline{\kappa_G}(f(\psi)(u),k)$.

Proof.

1) \Rightarrow 2) Let $\varphi \in \text{Iso}(F, F_k(G))$. Since $F_k(G)$ is uniquely reconstructible as the k-token graph of G, we have that $\text{Aut}(F_k(G)) = I_{\varphi} \cup C_{\varphi}$. By Lemma 2.3 we have that $|\text{Aut}(F_k(G))| \leq |\text{Aut}(G)|$ for $k \neq n/2$, and $|\text{Aut}(F_k(G))| \leq 2|\text{Aut}(G)|$ for k = n/2. By (1) and (2) we have 2).

2) \Rightarrow 3) Note that $|\operatorname{Iso}(H,G)| = |\operatorname{Aut}(G)|$ and $|\operatorname{Iso}(F_k(H), F_k(G))| = |\operatorname{Aut}(F_k(G))|$. Suppose that $k \neq n/2$. We have that $|\operatorname{Iso}(H,G)| = |\operatorname{Iso}(F_k(H), F_k(G))|$. Since ι is an injection from $\operatorname{Iso}(H,G)$ to $\operatorname{Iso}(F_k(H), F_k(G))$ it is also a bijection and we have 3). Suppose that k = n/2. Let

$$X := \{\iota(\phi) : \phi \in \operatorname{Iso}(H, G)\} \text{ and } Y := \{\mathfrak{c} \circ \iota(\phi) : \phi \in \operatorname{Iso}(H, G)\}.$$

Since ι is injective we have that $|X| = |\operatorname{Iso}(H,G)|$ and $|Y| = |\operatorname{Iso}(H,G)|$. By Proposition 2.1 we have that $X \cap Y = \emptyset$. Since $|\operatorname{Iso}(F_k(H), F_k(G))| = 2|\operatorname{Iso}(H,G)|$, we have that $\operatorname{Iso}(F_k(H), F_k(G)) = X \cup Y$; thus 3) holds.

3) \Rightarrow 4) Let u be a vertex of H. If $\psi = \iota(f(\psi))$, then

$$\psi(\kappa_H(u,k)) = \iota(f(\psi))(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k).$$

If $\psi = \mathfrak{c} \circ \iota(f(\psi))$, then

$$\psi(\kappa_H(u,k)) = \mathfrak{c} \circ \iota(f(\psi))(\kappa_H(u,k)) = \mathfrak{c} \circ \kappa_G(f(\psi)(u),k) = \overline{\kappa_G}(f(\psi)(u),k).$$

 $4) \Rightarrow 1)$ Note that

$$|\kappa_H(v,k)| = \binom{n-1}{k-1}$$
 and $|\overline{\kappa_H}(v,k)| = \binom{n-1}{k}$.

Therefore, if for some vertex v of H we have that $\psi(\kappa_H(v,k)) = \overline{\kappa_G}(f(\psi)(v),k)$, we would have that $\binom{n-1}{k-1} = \binom{n-1}{k}$. This would imply that n is even and k = n/2. Suppose that for some pair of vertices $u, v \in H$ we have that $\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k)$ and $\psi(\kappa_H(v,k)) = \overline{\kappa_G}(f(\psi)(v),k)$. Note that

$$|\kappa_H(u,k) \cap \kappa_H(v,k)| = \binom{n-2}{k-2};$$

we have that

$$|\varphi(\kappa_H(u,k)\cap\kappa_H(v,k))| = |\varphi(\kappa_H(u,k))\cap\varphi(\kappa_H(v,k))| = |\kappa_G(f(\psi)(u),k)\cap\overline{\kappa_G}(f(\psi)(v),k)| = \binom{n-2}{k-1}.$$

Thus, $\binom{n-2}{k-2} = \binom{n-2}{k-1}$, and n is odd—a contradiction. Therefore, for all vertices $u \in H$ either $\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u), k)$ or $\psi(\kappa_H(u,k)) = \overline{\kappa}_G(f(\psi)(u), k)$.

Suppose that there exist two vertices $u, v \in H$, such that $f(\psi)(u) = f(\psi)(v)$. We have that $\binom{n-1}{k-1} = \binom{n-2}{k-2}$. This implies that n = k, a contradiction. Thus, $f(\psi)$ is injective; thus, it is also bijective.

Note that u is not adjacent to v if and only if

$$E(\kappa_H(u,k) \setminus \kappa_H(v,k), \kappa_H(v,k) \setminus \kappa_H(u,k)) = \emptyset.$$

Similarly, u is not is adjacent to v if and only if

$$E\left(\overline{\kappa_H}(u,k)\setminus\overline{\kappa_H}(v,k),\overline{\kappa_H}(v,k)\setminus\overline{\kappa_H}(u,k)\right)=\emptyset.$$

Let $X := \kappa_H(u, k)$ and $Y := \kappa_H(v, k)$. Since

$$|E(X \setminus Y, Y \setminus X)| = |\psi(E(X \setminus Y, Y \setminus X))| = |E(\psi(X) \setminus \psi(Y), \psi(Y) \setminus \psi(X))|,$$

we have that u is adjacent to v if and only if $f(\psi)(u)$ is adjacent to $f(\psi)(v)$. Thus, $f(\psi) \in \text{Iso}(H, G)$. Moreover, if $A \in F_k(H)$ then

$$\psi(A) = \psi\left(\bigcap_{v \in A} \kappa_H(v, k)\right) = \bigcap_{v \in A} \psi(\kappa_H(v, k)) = \bigcap_{v \in A} \kappa_G(f(\psi)(v), k) = \iota(f(\psi))(A)$$

or

$$\psi(A) = \psi\left(\bigcap_{v \in A} \kappa_H(v, k)\right) = \bigcap_{v \in A} \psi(\kappa_H(v, k)) = \bigcap_{v \in A} \overline{\kappa_G}(f(\psi)(v), k) = \mathfrak{c} \circ \iota(f(\psi))(A).$$

Fix an isomorphism ψ from $F_k(H)$ to $F_k(G)$ and let (G, φ) and (G, ϕ) be two k-token reconstructions of $F_k(G)$. Let $\mathfrak{s}(\varphi, \phi) := f(\phi \psi) \circ f(\varphi \psi)^{-1}$. Note that $\phi = \iota(\mathfrak{s}(\varphi, \phi)) \circ \varphi$ or $\phi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \phi)) \circ \varphi$ and we have 1)

Observation 2.5.

- If 3) of Theorem 2.4 holds, then $f(\psi)$ is unique;
- if 4) of Theorem 2.4 holds, then $f(\psi)$ is unique and an isomorphism from H to G, and either

$$\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k) \text{ or } \psi(\kappa_H(u,k)) = \overline{\kappa_G}(f(\psi)(u),k),$$

for all vertices $u \in H$

We now present some consequences of Theorem 2.4.

2.1 k-reconstruction Families

Suppose that F is a graph on $\binom{n}{k}$ vertices. Inspired by property 4) of Theorem 2.4, we define the concept of a k-reconstruction family of F. Let \mathcal{R} be a family of subsets of vertices of F. For every vertex $A \in F$, let

$$S_{\mathcal{R}}(A) := \{ X \in \mathcal{R} : A \in X \}.$$

We say that \mathcal{R} is a k-reconstruction family of F if it satisfies the following properties.

- 1) $|X| = \binom{n-1}{k-1}$ for all $X \in \mathcal{R}$;
- 2) $|S_{\mathcal{R}}(A)| = k$ for all $A \in V(F)$; and
- 3) for every edge $AB \in F$ we have that

$$|S_{\mathcal{R}}(A) \cap S_{\mathcal{R}}(B)| = k - 1.$$

Note that 1) and 2) imply that $|\mathcal{R}| = n$. Let (G, φ) be a k-reconstruction of F. Note that

$$\mathcal{R}_{\varphi} := \{ \varphi^{-1}(\kappa_G(u, k)) : u \in V(G) \}$$

is a k-reconstruction family of F. Conversely, from a k-reconstruction family we can obtain a k-token reconstruction of F as follows. Let $G_{\mathcal{R}}$ be the graph whose vertex set is \mathcal{R} ; and such that X is adjacent to Y in $G_{\mathcal{R}}$ if and only if there exists an edge AB of F such that

$$S_{\mathcal{R}}(A)\triangle S_{\mathcal{R}}(B) = \{X, Y\}.$$

Proposition 2.6. If \mathcal{R} is a k-reconstruction family of F, then $(G_{\mathcal{R}}, S_{\mathcal{R}})$ is a k-token reconstruction of F.

Proof. By 2), for every $A \in V(F)$ we have that $S_{\mathcal{R}}(A) \in V(F_k(G_{\mathcal{R}}))$. 1) and 2) imply that $S_{\mathcal{R}}$ is a bijection from V(F) to $V(F_k(G_{\mathcal{R}}))$. The definition of $G_{\mathcal{R}}$ and 3) implies that $S_{\mathcal{R}}$ is an isomorphism from F to $F_k(G_{\mathcal{R}})$.

Suppose that \mathcal{R} is a k-reconstruction family of F; let

$$\overline{\mathcal{R}} := \{ V(F) \setminus X : X \in \mathcal{R} \}.$$

Proposition 2.7. Suppose that \mathcal{R} is k-reconstruction family of F and that k = n/2. Then $\overline{\mathcal{R}}$ is a k-reconstruction family of F.

Proof. For every $X \in \mathcal{R}$ we have that $|V(F) \setminus X| = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k-1}$; thus, $\overline{\mathcal{R}}$ satisfies 1). For every $A \in V(F)$ we have that $|S_{\overline{\mathcal{R}}}(A)| = |\mathcal{R} \setminus S_{\mathcal{R}}(A)| = n - k = k$; thus, $\overline{\mathcal{R}}$ satisfies 2). For every edge $AB \in F$ we have that

$$|S_{\overline{\mathcal{R}}}(A) \cap S_{\overline{\mathcal{R}}}(B)| = |(\mathcal{R} \setminus S_{\mathcal{R}}(A)) \cap (\mathcal{R} \setminus S_{\mathcal{R}}(B))|$$
$$= n - (k - 1) - 2$$
$$= k - 1.$$

thus, $\overline{\mathcal{R}}$ satisfies 3).

Proposition 2.8. Let (G, φ) and (G, ψ) be two k-token reconstructions of F. Then (G, φ) and (G, ψ) are equivalent k-token reconstructions of F if and only if $\mathcal{R}_{\varphi} = \mathcal{R}_{\psi}$ or $\mathcal{R}_{\varphi} = \overline{\mathcal{R}}_{\psi}$.

Proof. Suppose that (G, φ) and (G, ψ) are equivalent k-token reconstructions of F. Then there exists an automorphism $\mathfrak{s}(\varphi, \psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \text{ or } \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi.$$

In the first case we have that

$$R_{\varphi} = \{ \varphi^{-1}(\kappa_{G}(u, k) : u \in V(G) \}$$

$$= \{ \psi^{-1} \circ \iota(\mathfrak{s}(\varphi, \psi))^{-1}(\kappa_{G}(u, k)) : u \in V(G) \}$$

$$= \{ \psi^{-1}(\kappa_{G}(\mathfrak{s}(\varphi, \psi)^{-1}(u), k)) : u \in V(G) \}$$

$$= \{ \psi^{-1}(\kappa_{G}(u, k) : u \in V(G) \}$$

$$\mathcal{R}_{\psi}.$$

In the second case we have that

$$R_{\varphi} = \{ \varphi^{-1}(\kappa_{G}(u, k) : u \in V(G) \}$$

$$= \{ \psi^{-1} \circ \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi))^{-1}(\kappa_{G}(u, k)) : u \in V(G) \}$$

$$= \{ \mathfrak{c} \circ \psi^{-1}(\kappa_{G}(\mathfrak{s}(\varphi, \psi)^{-1}(u), k)) : u \in V(G) \}$$

$$= \{ V(F) \setminus \psi^{-1}(\kappa_{G}(u, k) : u \in V(G) \}$$

$$\overline{\mathcal{R}}_{\psi}.$$

Suppose that $\mathcal{R}_{\varphi} = \mathcal{R}_{\psi}$ or $\mathcal{R}_{\varphi} = \overline{\mathcal{R}}_{\psi}$. We define an automorphism f of G as follows. Let $u \in V(G)$ and let f(u) be the vertex of G such that

$$\varphi^{-1}(\kappa_G(u,k)) = \psi^{-1}(\kappa_G(f(u),k)) \text{ or } \varphi^{-1}(\kappa_G(u,k)) = V(F) \setminus \psi^{-1}(\kappa_G(f(u),k)).$$

Condition 3) in the definition of k-reconstruction family implies that f is an automorphism of G. We have that

$$\psi = \iota(f) \circ \varphi$$
 or $\psi = \mathfrak{c} \circ \iota(f) \circ \varphi$.

Thus, (G, φ) and (G, ψ) are equivalent k-reconstructions of F.

Proposition 2.9. Let \mathcal{R} be a k-reconstruction family of F. Then F is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$ if and only if for every automorphism φ of F we have that

$$\mathcal{R} = \{ \varphi(X) : X \in \mathcal{R} \} \text{ or } \overline{\mathcal{R}} = \{ \varphi(X) : X \in \mathcal{R} \}.$$

Proof. Let $X \in \mathcal{R}$. Note that

$$\begin{split} \kappa_{G_{\mathcal{R}}}(X,k) &= \{W \subset \mathcal{R} : |W| = k \text{ and } X \in W\} \\ &= \{S_{\mathcal{R}}(A) : A \in X\} \\ &= S_{\mathcal{R}}(X). \end{split}$$

If k = n/2, we also have that

$$\begin{split} \overline{\kappa_{G_{\mathcal{R}}}}(X,k) &= \{W \subset \mathcal{R} : |W| = k \text{ and } X \notin W\} \\ &= \{S_{\mathcal{R}}(A) : A \notin X\} \\ &= S_{\mathcal{R}}(V(F) \setminus X). \end{split}$$

Let

$$\varphi' := S_{\mathcal{R}} \circ \varphi \circ S_{\mathcal{R}}^{-1}.$$

Note that φ' is an automorphism of $F_k(G_{\mathcal{R}})$.

Suppose that F is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$. Thus, $F_k(G_{\mathcal{R}})$ is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$. Let $X \in \mathcal{R}$. By 4) of Theorem 2.4 we have that $\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \kappa_{G_{\mathcal{R}}}(Y,k)$ or $\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \overline{\kappa_{G_{\mathcal{R}}}}(Y,k)$, for some $Y \in \mathcal{R}$. In the first case we have that

$$\varphi(X) = Y;$$

In the second case case we have that

$$\varphi(X) = V(F) \setminus Y$$
.

As in the proof of 4) \Rightarrow 1), we have either the first case happens for all $X \in \mathcal{R}$ or the second case happens for all $X \in \mathcal{R}$. Thus,

$$\mathcal{R} = \{ \varphi(X) : X \in \mathcal{R} \} \text{ or } \overline{\mathcal{R}} = \{ \varphi(X) : X \in \mathcal{R} \}.$$
 (3)

Suppose that (3) holds. Then for all $X \in \mathcal{R}$ we have that

$$\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \kappa_{G_{\mathcal{R}}}(Y,k) \text{ or } \varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \overline{\kappa_{G_{\mathcal{R}}}}(Y,k), \text{ for some } Y \in \mathcal{R}.$$

Proposition 2.10. Let (G, φ) and (H, ϕ) be two k-token reconstructions of F. Then $G \simeq H$ if and only if there exists an automorphism ψ of F such that

$$\mathcal{R}_{\phi} = \{ \psi(X) : X \in \mathcal{R}_{\varphi} \}.$$

Proof. Suppose that $G \simeq H$. Let f be an isomorphism from G to H. Let

$$\psi := \phi^{-1} \circ \iota(f) \circ \varphi.$$

Let $X \in \mathcal{R}_{\varphi}$. Let $x \in V(G)$ be such that $\varphi(X) = \kappa_G(x, k)$. We have that $\iota(f) \circ \varphi(X) = \kappa_H(f(x), k)$. Let $Y \in \mathcal{R}_{\varphi}$ be such that $Y = \varphi^{-1}(\kappa_H(f(x), k))$. Thus, $Y = \psi(X)$, and $\mathcal{R}_{\varphi} = \{\psi(X) : X \in \mathcal{R}_{\varphi}\}$.

Suppose that there exists ψ Aut(F) such that $\mathcal{R}_{\phi} = \{\psi(X) : X \in \mathcal{R}_{\varphi}\}$. We define an isomorphism, f, from G to H. Let $x \in V(G)$. Let $Y = \psi(\varphi^{-1}(\kappa_G(x,k)))$. Let f(x) be the vertex of H such that $\phi^{-1}(\kappa_H(f(x),k)) = Y$. Condition 3) in the definition of k-reconstruction family implies that f is an isomorphism.

We can use Proposition 2.10, to rephrase Conjecture 1.1:

Conjecture 2.11. Let G be a graph. For every two k-token reconstruction families \mathcal{R} and \mathcal{R}' of $F_k(G)$, there exists an automorphism ψ of $F_k(G)$ such that

$$\mathcal{R}' = \{ \psi(X) : X \in \mathcal{R}_{\omega} \}.$$

Example 2.12. Consider $C_4 =: (1, 2, 3, 4)$. Let

$$X_1 := \{\{2,3\}, \{1,3\}, \{1,4\}\}, X_2 := \{\{2,3\}, \{1,2\}, \{2,4\}\},$$

 $X_3 := \{\{1,2\}, \{1,3\}, \{2,4\}\}, X_4 := \{\{1,4\}, \{2,4\}, \{3,4\}\}.$

 $\mathcal{R} := \{X_1, X_2, X_3, X_4\}$ is a 2-token reconstruction family of $F_2(C_4)$.

We now consider the token graphs of stars; these graphs play a crucial role in our reconstruction algorithm.

3 Token graphs of stars

For $n \geq 2$, we call the complete bipartite graph $K_{1,n}$ a star. Throughout this section let $n \geq 2$ and $k \leq (n+1)/2$. Let $\{x_0, x_1, \ldots, x_n\}$ be the vertices of $K_{1,n}$ so that x_0 is the vertex of degree greater than one. $F_k(K_{1,n})$ is a bipartite graph: one set in the partition corresponds to the token configurations without a token at x_0 and the other set corresponds to the token configurations with a token at x_0 . Let V_0 and V_1 be these sets, respectively. Every vertex in V_0 has degree equal to k and every vertex in k has degree equal to k and every vertex in k has degree equal to k and every vertex in k has degree equal to k and every vertex in k has degree equal to k has degree equal to

Lemma 3.1. Suppose that F is isomorphic to the token graph of a star. Then there exist unique positive integers n and $k \leq (n+1)/2$, such that $F \simeq F_k(K_{1,n})$; these integers can be found in polynomial time.

Proof. Note that every vertex in W_0 has the same degree d_0 , and every vertex in W_1 has the same degree d_1 . Without loss of generality assume that $d_0 \leq d_1$. If $d_0 < d_1$, an isomorphism from F to $F_k(K_{1,n})$ must map W_0 to V_0 and W_1 to V_1 . If $d_0 = d_1$, an isomorphism from F to $F_k(K_{1,n})$ can map W_0 to V_0 or to V_1 . In both cases $d_0 = k$ and $d_1 = n - k + 1$. Therefore, k and k are uniquely determined, and computable in polynomial time.

In view of Lemma 3.1, in what follows assume that n and k are such that every vertex in W_0 has degree k, and every vertex in W_1 has degree n - k + 1.

Lemma 3.2. Let

- v^* be a vertex in W_0 ;
- w_1, \ldots, w_k be the neighbors of v^* ;
- $v_{k+1}, v_{k+2}, \ldots, v_n$ be the neighbors of w_1 distinct from v^* ; and
- f be any injective function that maps $\{v^*\} \cup \{w_1, \ldots, w_k\} \cup \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ to the vertices of $F_k(K_{1,n})$ such that

```
- f(\{w_1, \dots, w_k\}) = N(f(v^*)) \text{ and } 
- f(\{v_{k+1}, v_{k+2}, \dots, v_n\}) = N(f(w_1)) \setminus \{f(v^*)\}.
```

If F and $F_k(K_{1,n})$ are isomorphic, then in polynomial time we can extend f to a unique isomorphism from F to $F_k(K_{1,n})$. Moreover, if F and $F_k(K_{1,n})$ are not isomorphic then we can determine in polynomial time that such an extension does not exist.

Proof. We provide an algorithm that attempts to extend f to an isomorphism from F to $F_k(K_{1,n})$. The algorithm succeeds if and only if F and $F_k(K_{1,n})$ are isomorphic. Our algorithm proceeds by labeling the vertices of F. Let v be a vertex of F. If v is in W_0 then v will be labeled with a string of integers $s_1s_2\cdots s_k$; this means that the isomorphism maps v to the token configuration $\{x_{s_1},\ldots,x_{s_k}\}$. If v is in W_1 then v will be labeled with a string of integers $s_1\cdot s_2\cdots s_{k-1}$; this means that our isomorphism maps v to the token configuration $\{x_0,x_{s_1},\ldots,x_{s_{k-1}}\}$. We denote with $\ell(v)$ the label assigned to vertex v. Let s be one of these labelings. For a given integer j, we denote with v0 the label that results from v1 by removing the appearance of v2. Similarly, we denote with v3 the label that results from adding v3 to v4.

If necessary we relabel the vertices of $K_{1,n}$ so that $\ell(v^*) = 1 \cdot 2 \cdots k$. Note that the neighbors of v^* receive a label of the form $\ell(v^*) \ominus j$ for some $1 \leq j \leq k$. We relabel the neighbors of v so that $\ell(w_j) := \ell(v^*) \ominus j$. Note that the neighbors of w_1 distinct from v^* receive a label of the form $\ell(v^*) \ominus 1 \oplus j$ for some $k+1 \leq j \leq n$. We relabel the neighbors of w_1 distinct from v^* so that $\ell(v_j) = \ell(v^*) \ominus 1 \oplus j$. This first labeling can be made if F and $F_k(K_{1,n})$ are indeed isomorphic. In what follows, we show that this first labeling determines the labels of the remaining vertices of F.

We now label the neighbors of each w_j with $j \neq 1$. Note that the neighborhoods of distinct w_j only intersect at v^* . Let $j \neq 1$. Let u be an unlabeled neighbor of w_j . Note that u should receive a label of the form $\ell(v^*) \ominus j \oplus t$ for some $k+1 \leq t \leq n$. Further note that u should receive the label $\ell(v^*) \ominus j \oplus t$ if and only if there is a path of length two from u to v_t . This corresponds to the following token moves: starting from the token configuration assigned to v_t move the token at x_j to x_0 ; then move this token from x_0 to x_1 to arrive to the token configuration assigned to u. We label each such u by checking the paths of length 2 from u to $v_{k+1}, v_{k+2}, \ldots, v_n$. In the process we check whether there are conflicting labelings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic.

So far we have labeled all the vertices in W_1 at distance one from v^* and all vertices in W_0 at distance two from v^* . Let $d \geq 3$ be an odd integer. Suppose we have labeled all the vertices in W_1 at distance at most d-2 from v^* and all the vertices in W_0 at distance at most d-1 from v^* . We now label the vertices in W_1 at distance d from v^* and the vertices in W_0 at distance d+1 from v^* .

Let u be a vertex in W_1 at distance d from v^* . Let y_1 and y_2 be two neighbors of u at distance d-1 from v^* . Note that there exists two integers t_1 and t_2 such that $\ell(y_2) = \ell(y_1) \ominus t_1 \oplus t_2$; thus, u should be labeled with $s := \ell(y_1) \ominus t_1 = \ell(y_2) \ominus t_2$. We label each such u by checking all its pairs of neighbors at distance d-1 from v^* . In the process we check whether there are conflicting labelings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic.

Let now u be a vertex in W_0 at distance d+1 from v^* . Let y_1 and y_2 be two neighbors of u at distance d from v^* . Note that there exists two integers t_1 and t_2 such that $\ell(y_2) = \ell(y_1) \ominus t_1 \oplus t_2$; thus, u should be labeled with $s := \ell(y_1) \oplus t_2 = \ell(y_2) \oplus t_1$. We label each such u by checking all its pairs of neighbors at distance d from v^* . In the process we check whether there are conflicting labelings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic. If the algorithm succeeds in labeling the vertices of F, then F and $F_k(K_{1,n})$ are isomorphic.

Lemma 3.3. We can determine in polynomial time whether F and $F_k(K_{1,n})$ are isomorphic. Moreover, if $F \simeq F_k(K_{1,n})$, then we have the following.

- 1. We can find an isomorphism between F and $F_k(K_{1,n})$ in polynomial time;
- 2. F is uniquely reconstructible as the k-token graph of $K_{1,n}$.

Proof. It only remains to show that if $F \simeq F_k(K_{1,n})$, then F is uniquely reconstructible as the k-token graph of $K_{1,n}$. Suppose that $F \simeq F_k(K_{1,n})$. Pick a vertex $v^* \in W_0$. Let $\{w_1, \ldots, w_k\}$ be the neighbors of v^* . Let $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ be the neighbors of w_1 distinct from v^* . Choose any injective function, f, that maps $\{v^*\} \cup \{w_1, \ldots, w_k\} \cup \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ to the vertices of $F_k(K_{1,n})$ such that

- $f(v^*) \in V_0$ if k < (n+1)/2;
- $f(\{w_1, \ldots, w_k\}) = N(f(v^*))$; and
- $f({v_{k+1}, v_{k+2}, \ldots, v_n}) = N(f(w_1)) \setminus {f(v^*)}.$

By Lemma 3.2, we can extend f to an isomorphism ψ from F to $F_k(K_{1,k})$. Iterating over all possible choices for f, we generate all isomorphisms, ψ , from F to $F_k(K_{1,n})$. This allows us to compute the size of $\operatorname{Iso}(F, F_k(K_{1,k}))$ by counting the number of possible choices for f. If k = (n+1)/2 we have that $f(v^*) \in V_0$ or $f(v^*) \in V_1$. Once this choice is made, there are $\binom{n}{k}$ possible choices for $f(v^*)$. Once the value of $f(v^*)$ is fixed there are k! possible choices for $\{f(w_1), \ldots, f(w_k)\}$. Once these values are fixed, there are (n-k)! possible choices for $\{f(v_{k+1}), f(v_{k+2}), \ldots, f(v_n)\}$. We have that

$$|\operatorname{Iso}(F, F_k(K_{1,n}))| = \begin{cases} n! & \text{if } k \neq (n+1)/2, \\ 2n! & \text{if } k = (n+1)/2. \end{cases}$$

Since $\operatorname{Aut}(K_{1,n}) = S_n$ and $|\operatorname{Aut}(F_k(K_{1,n}))| = |\operatorname{Iso}(F, F_k(K_{1,n}))|$, we have that by Theorem 2.4, F is uniquely reconstructible as the k-token graph of $K_{1,n}$.

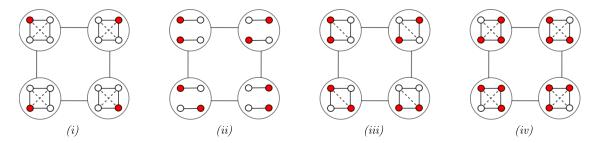


Figure 1: These are the four ways to generate an induced 4-cycle in $F_k(G)$; tokens that are not shown are assumed to remain fixed; and dashed lines are forbidden in G.

4 Induced 4-cycles of $F_k(G)$ and Ladders

In this section we study how induced 4-cycles in $F_k(G)$ can be generated. In particular we show that if G is a $(C_4, \text{diamond})$ -free graph then all induced 4-cycles of $F_k(G)$ are generated by moving two tokens along two independent edges of G. We use this characterization to define an equivalence relation on the edges of $F_k(G)$. This equivalence relationship is computable in polynomial time.

4.1 Induced 4-cycles of $F_k(G)$

Proposition 4.1. Every induced 4-cycle of a k-token graph is generated in one of the four ways depicted in Figure 1.

Proof. Let G be a graph. Let $\mathcal{C} := (A, B, C, D)$ be an induced 4-cycle of $F_k(G)$. Let

- $A \triangle B := \{a_1, b_1\} \text{ with } a_1 \in A, b_1 \in B;$
- $B\triangle C := \{b_2, c_1\}$ with $b_2 \in B, c_1 \in C$; and
- $C\triangle D := \{c_2, d_1\}$ with $c_2 \in C, d_1 \in D$.

We proceed by case analysis.

- Suppose that $\{a_1, b_1\} \cap \{b_2, c_1\} = \emptyset$. This implies that $A \triangle C = \{a_1, b_2, b_1, c_1\}$. There are three possible values for $C \triangle D$. $C \triangle D = \{a_1, b_1\}$; $C \triangle D = \{a_1, c_1\}$; or $C \triangle D = \{b_1, b_2\}$.
 - If $C\triangle D = \{a_1, b_1\}$ then $A\triangle D = \{b_2, c_1\}$ and C is generated as in (ii) of Figure 1.
 - If $C\triangle D = \{a_1, c_1\}$ then $A\triangle D = \{b_1, b_2\}$ and C is generated as in (iii) of Figure 1.
 - If $C\triangle D = \{b_1, b_2\}$ then $A\triangle D = \{c_1, a_1\}$ and C is generated as in (iii) of Figure 1.
- Suppose that $\{a_1, b_1\} \cap \{b_2, c_1\} \neq \emptyset$. Thus, $b_1 = b_2$ or $a_1 = c_1$.
 - Suppose that $b_1 = b_2$.
 - * Suppose that $c_1 = c_2$. Thus, $d_1 \neq a_1$ and $A \triangle D = \{a_1, d_1, \}$. This implies that C is generated as in (i) of Figure 1.
 - * Suppose that $c_1 \neq c_2$. If $d_1 \neq a_1$ then A and D are not adjacent since $A \triangle D = \{a_1, c_1, c_2, d_1\}$ in this case. Therefore, $d_1 = a_1$. This implies that $D \triangle A = \{c_2, d_1\}$ and C is generated as in (iii) of Figure 1.
 - Suppose that $a_1 = c_1$.
 - * Suppose $c_2 = a_1$. Then $d_1 \neq b_2$ as otherwise D = B. But then A is not adjacent to D.
 - * Suppose that $c_2 = b_1$. Then $A \triangle D = \{d_1, b_2\}$ and C is generated as in (iii) of Figure 1.
 - * Suppose that $c_2 \notin \{a_1, b_1\}$. Then $d_1 = b_2$ as otherwise D would not be adjacent to A. Therefore, $A \triangle D = \{b_1, c_2\}$ and C is generated as in *(iv)* of Figure 1.

It may be the case that F can be reconstructed (even uniquely) as the token graph of two non-isomorphic graphs. The following lemma shows that if one of them is a $(C_4, \text{diamond})$ -free graph, then the other graph is also a $(C_4, \text{diamond})$ -free graph.

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Lemma 4.2. Let G be a $(C_4, diamond)$ -free graph. If (G', φ) is any k'-reconstruction of $F_k(G)$ then G' is also a $(C_4, diamond)$ -free graph.

Proof. We start by showing the following property of $F_k(G)$:

(P1) for each induced 4-cycle ABCD of $F_k(G)$ and each vertex $X \in F_k(G) - \{A, B, C, D\}$, if X is adjacent to two non-consecutive vertices of the cycle ABCD, then it is adjacent to the vertices A, B, C and D.

Let A, B, C, D and X as in (P1). Suppose that X is adjacent to A and C. Since G is $(C_4, \text{diamond})$ -free, the 4-cycle ABCD must be generated as in (ii) of Figure 2: by moving two tokens on two disjoint edges (a_1, b_1) and (a_2, b_2) of G, while the other k-2 tokens remain fixed on a subset S of $G-\{a_1, a_2, b_1, b_2\}$. Without loss of generality assume that

$$A = S \cup \{a_1, a_2\}, \qquad B = S \cup \{b_1, a_2\}, \qquad C = S \cup \{b_1, b_2\}, \qquad D = S \cup \{a_1, b_2\}.$$

Consider now the vertex X. Let us note that X must be obtained from C by moving a token at one of $\{b_1,b_2\}$ to a vertex in $\{a_1,b_1\}$, as otherwise we would have $|X\triangle A| > 2$, and so X and A cannot be adjacent, a contradiction. Clearly, X cannot be obtained from C by moving the token at b_2 to a_2 , as otherwise X = B, a contradiction. Similarly, X cannot be obtained from C by moving the token at b_1 to a_1 . Thus, either X is obtained from C by moving the token at b_1 to a_2 , or by moving the token at b_2 to a_1 , but these two cases are analogous. Without loss of generality let us assume that X is obtained from C by moving the token at b_1 to a_2 , and so, b_1 is adjacent to a_2 . Since X is adjacent to A, it follows that a_1 is adjacent to b_2 , and since G is a $(C_4, \text{diamond})$ -free graph, the vertex set $\{a_1, a_2, b_1, b_2\}$ must induce a complete graph in G. This fact implies that X is also adjacent to B and B, and so B.

Suppose that G' is not a $(C_4, \text{diamond})$ -free graph. Let uvwz be a 4-cycle in G', with at most one chord, let us assume that v and z are not adjacent. Let $S' \subseteq G' - \{u, v, w, z\}$ with |S'| = k - 2, and consider the vertices

$$A' = S' \cup \{u, v\}, \qquad B' = S' \cup \{u, w\}, \qquad C' = S' \cup \{u, z\}, \qquad D' = S' \cup \{v, z\} \qquad \text{and} \qquad X' = S' \cup \{z, w\}.$$

Note that A'B'C'D' induces a 4-cycle in $F_{k'}(G')$ and the vertex X' is adjacent to B' and D', however, X' cannot be adjacent to A', and so (P1) does not hold for $F_{k'}(G')$ —a contradiction. Thus, G' is a $(C_4, \text{diamond})$ -free graph.

4.2 Ladders, Cartesian Products and Line Graphs

A ladder is a graph isomorphic to the Cartesian product of K_2 and a path P_m of length $m \ge 1$. Let x and y be the two vertices of K_2 ; let v_1, \ldots, v_{m+1} be the vertices of P_m . For $m \ge 2$, we call the edges $(x, v_i)(y, v_i)$ the rungs of the ladder. In the case of $K_2 \square P_1$ the rungs may be either one of the two pairs of disjoint edges. Two edges e and f in F are said to be connected by a ladder if there exists an induced subgraph of F isomorphic to a ladder, such that e and f are rungs of this ladder. Being connected by a ladder is an equivalence relation on the edges of F. We refer to its equivalence classes as ladders classes. We denote the ladder class of e with R[e]. The ladder classes of F are easily computed in polynomial time as follows. Construct a graph F' whose vertices are the edges of F; two of which are adjacent if they are disjoint edges of an induced 4-cycle of F. The ladder classes of F correspond to the components of F'. In the case when F is the k-token graph of a $(C_4, \text{diamond})$ -free graph, we have the following.

Proposition 4.3. Let G be a $(C_4, diamond)$ -free graph and let AB, A'B' be two edges of $F_k(G)$ in the same ladder class. Then

$$A\triangle B = A'\triangle B';$$

that is, AB and A'B' correspond to moving a token along the same edge of G.

Proof. Let $H \simeq K_2 \square P_m$ be a ladder of $F_k(G)$ such that AB is the first rung of H and A'B' is the last rung of H. Since G is a $(C_4, \text{diamond})$ -free graph, every induced 4-cycle of $F_k(G)$ is generated as in (ii) of Figure 1. If m=1 then the result follows from this observation. Suppose that m>1 and that the result holds for smaller values of m. Let A''B'' be the rung of H previous to A'B'. By induction $A\triangle B = A''\triangle B''$ and by the previous argument $A''\triangle B'' = A'\triangle B'$; the result follows. \square

Although Proposition 4.3 implies that every edge in a given ladder class of $F_k(G)$ corresponds to moving a token along the same edge of G, two edges in different ladder classes may correspond to moving a token along the same edge ab of G. The next lemma shows this does not happen when $G \setminus \{a, b\}$ is connected.

Lemma 4.4. Let G be a $(C_4, diamond)$ -free graph. Let e := ab be an edge of G such that $G \setminus \{a, b\}$ is connected. Then the set of edges of $F_k(G)$ that correspond to moving a token along e form a ladder class.

Proof. Let A_1B_1 and A_2B_2 edges of $F_k(G)$ such that $A_1\triangle B_1=A_2\triangle B_2=\{a,b\}$. Without loss of generality assume that $a\in A_1,\ a\in A_2,\ b\in B_1$ and $b\in B_2$. Let $A_1':=A_1\setminus\{a,b\},\ B_1':=B_1\setminus\{a,b\},\ A_2':=A_2\setminus\{a,b\}$ and $B_2':=B_2\setminus\{a,b\}$. Note that A_1',B_1',A_2',B_2' are vertices of $F_{k-1}(G\setminus\{a,b\})$. Since G' is connected then so is $F_{k-1}(G\setminus\{a,b\})$ [8]. Let $(A_1':=C_1,C_2,\ldots,C_m:=A_2')$ be a path from A_1' to A_2' in $F_{k-1}(G\setminus\{a,b\})$. The set of vertices

$$\{C_i \cup \{a\} : 1 \le i \le m\} \cup \{C_i \cup \{b\} : 1 \le i \le m\}$$

induces a ladder that connects A_1B_1 to A_2B_2 in $F_k(G)$.

Note that if G is a 3-connected (C_4 ,diamond)-free graph, then the edges of G and the ladder classes of $F_k(G)$ are in a one to one correspondence. By Proposition 4.1, the edges corresponding to two ladder classes R_1 and R_2 are incident to a same vertex if and only if no edge of R_1 is contained in an induced 4-cycle of $F_k(G)$ simultaneously with an edge of R_2 . In particular this implies that if G is 3 connected, then we can recover the line graph L(G) of G from the ladder classes of F in polynomial time. We have the following corollary.

Corollary 4.5. Let G be a 3-connected $(C_4, diamond)$ -free graph; let F be a graph isomorphic to $F_k(G)$. Given only F and the information that G is 3-connected, we can compute in polynomial time a graph H isomorphic to G.

In the Section 5 we give an algorithm that given $F \simeq F_k(G)$, in polynomial time finds a graph isomorphic to G. A key step in our algorithm is to find a large composite graph in F. The following lemma characterizes how certain large composite graphs are generated in $F_k(G)$.

Theorem 4.6. Let G be a connected $(C_4, diamond)$ -free graph. Let H be a subgraph of $F_k(G)$, such that H is maximal with the property of being isomorphic to a graph $H' = H'_1 \square \cdots \square H'_r$, where each H'_i is connected and with at least two vertices. Then there exists a partition V_1, \ldots, V_r of V(G), and positive integers k_1, \ldots, k_r with $k = k_1 + \cdots + k_r$, such that the following holds. H is generated by moving k_i tokens on $G_i := G[V_i]$ and each H'_i is isomorphic to $F_{k_i}(G_i)$

Proof. Let f be an isomorphism from H' to H. Fix an index $1 \le i \le r$. Let u_1u_2 and v_1v_2 be two edges of H' such that

$$u_1(i) = x = v_1(i)$$
 and $u_2(i) = y = v_2(i)$

for some pair of adjacent vertices x, y in H'_i . We first show that

 $f(u_1)f(u_2)$ and $f(v_1)f(v_2)$ are generated by moving a token along the same edge of G. (*)

Let $(u_1 = w_1, \dots, w_m = v_1)$ be a shortest path from u_1 to v_1 in H' such that for all $1 \leq j \leq m$, we have that

$$w_j(i) = x.$$

Let $(u_2 = w_1', \dots, w_m' = v_2)$ be the path in H' such that for all $1 \le j \le m$ and all $1 \le l \le r$, we have that

$$w'_j(l) := \begin{cases} y & \text{if } l = i, \\ w_j(l) & \text{if } l \neq i. \end{cases}$$

Note that the set of vertices

$$\{f(w_i): 1 \le j \le m\} \cup \{f(w_i'): 1 \le j \le m\}$$

induces a ladder in H. By Proposition 4.3, $f(u_1)f(u_2)$ and $f(v_1)f(v_2)$ are generated by moving a token along the same edge of G. This proves (*).

We now define the sets V_i 's. Fix a vertex $v^* \in H'$. Let H_i be subgraph of H induced by the set of vertices

$$\{f(u) : u \in V(H') \text{ and } u(j) = v^*(j) \text{ for all } j \neq i\}.$$

Clearly, $H_i \simeq H_i'$. Let

$$V_i := \{x \in V(G) : \text{ there exist } A, B \in V(H_i) \text{ such that } x \in A \text{ and } x \notin B\}.$$

By (*) and the fact that H_i is connected we have that V_i does not depend on the choice of v^* .

We show that the V_i are pairwise disjoint. Suppose that for some distinct V_i and V_j there exists a vertex $x \in V_i \cap V_j$. Since H_i is connected, there exist adjacent vertices A_1 and B_1 of H_i , such that $x \in A_1$ and $x \notin B_1$; let y_1 be the vertex of V_i such that B_1 is obtained from A_1 by moving the token from x to y_1 . Since H_j is connected there exists adjacent vertices A_2 and B_2 of H_j such that $x \in A_2$ and $x \notin B_2$; let y_2 be the vertex of V_j such that B_2 is obtained from A_2 by moving the token from x to y_2 . Note that $f^{-1}(A_1)f^{-1}(B_1)$ is an edge of H' and $f^{-1}(A_1)(i)f^{-1}(B_1)(i)$ is an edge of H'_i . Similarly, $f^{-1}(A_2)f^{-1}(B_2)$ is an edge of H' and $f^{-1}(A_2)(j)f^{-1}(B_2)(j)$ is an edge of H'_j . Let w_1, w_2, w_3, w_4 be vertices of H' defined as follows. For all $1 \le l \le r$ and $l \ne i, j$ we have

$$w_1(l) = w_2(l) = w_3(l) = w_4(l) = v^*(l).$$

For i, we have

$$w_1(i) = f^{-1}(A_1)(i), w_2(i) = f^{-1}(A_1)(i), w_3(i) = f^{-1}(B_1)(i) \text{ and } w_4(i) = f^{-1}(B_1)(i).$$

For j, we have

$$w_1(j) = f^{-1}(A_2)(j), w_2(j) = f^{-1}(B_2)(j), w_3(j) = f^{-1}(B_2)(j) \text{ and } w_4(j) = f^{-1}(A_2)(j).$$

Note that (w_1, w_2, w_3, w_4) is an induced 4-cycle of H'. By Proposition 4.1, $f(w_1)f(w_2)$ and $f(w_1)f(w_4)$ are generated each by moving a token along disjoint edges of G. However, by (*) these edges are xy_1 and xy_2 , respectively—a contradiction.

Let A be a vertex of H_i , we define $k_i := |A \cap V_i|$. Let B a vertex of H_i distinct from A. Let $(A =: A_1, A_2, \ldots, A_m := B)$ be a path from A to B in H_i . Note that for every $1 \le l < m$, $A_l \triangle A_{l+1} \subset V_i$. Therefore, $|A \cap V_i| = |B \cap V_i|$. Thus, k_i does not depend on our choice of A. For every $1 \le i \le r$, let G_i be the subgraph of G induced by V_i . Let H'' be the subgraph of F generated by moving k_i tokens on each G_i . Note that $H'' \simeq F_{k_1}(G_1) \square \cdots \square F_{k_r}(G_r)$. Since V_i does not depend on the choice of v^* , we have that H is a subgraph of H''. The maximality of H implies that H'' = H, $H_i \simeq F_{k_i}(G_i)$, $k = k_1 + \cdots + k_r$ and that $V(G) = V_1 \cup \cdots \cup V_r$.

We have the following corollary to Theorem 4.6.

Corollary 4.7. If G is a connected $(C_4, diamond)$ -free graph, then $F_k(G)$ is a prime graph.

5 Reconstructing G

Throughout this section let:

- G be a connected (C_4 ,diamond)-free graph;
- F be a graph isomorphic to $F_k(G)$, with 1 < k < |G| 1; and
- φ be a fixed isomorphism from F to $F_k(G)$.

In this section we present a polynomial time algorithm that given only F constructs a graph isomorphic to G. Note that we are not given $n,k,\,\varphi,\,F_k(G)$ nor G. In particular, we use φ only as a tool to help us reason about F.

Our general strategy is as follows. We run an algorithm, called PRODUCTSUBGRAPH, on F. The first step of PRODUCTSUBGRAPH is to find a vertex A of F with the following property. The number of independent edges of G incident to exactly one vertex of $\varphi(A)$ is maximum. Let r be this number of independent edges of G. Afterwards, PRODUCTSUBGRAPH finds a subgraph H of F, that is maximal with the property of being isomorphic to a Cartesian product $H_1 \square \cdots \square H_r$ of connected graphs H_i , each

with at least two vertices. PRODUCTSUBGRAPH also finds these H_i . By Theorem 4.6, we know that there exist induced disjoint subgraphs G_1, \ldots, G_r of G, and integers k_1, \ldots, k_r that sum up to k, such that $V(G) = \bigcup_{i=1}^r V(G_i)$ and $H_i \simeq F_{k_i}(G_i)$. The structure of the H_i is such that we can construct in polynomial time a graph isomorphic to each G_i . Finally, we reconstruct the adjacencies between the G_i 's.

The information stored in the ladder equivalence relations of the edges of F allows us to locally reconstruct small parts of G. Let A be a vertex of $F_k(G)$; let

$$E_A := \{A \triangle B : B \in N(A)\}.$$

Thus, E_A is the set of edges of G with exactly one vertex of A as one of their endpoints. Let G_A be the subgraph of G whose vertices are the endpoints of the edges in E_A , and its edge set is E_A .

Let AB and AC be two edges of $F_k(G)$; let e_1 and e_2 be the edges of G such that AB and AC correspond to moving a token along e_1 and e_2 , respectively. Since G is $(C_4, \text{diamond})$ -free and by Proposition 4.1, we have that AB and AC are in a common induced 4-cycle of $F_k(G)$ if and only if e_1 and e_2 are disjoint. By checking whether each pair of edges incident to A are contained in a 4-cycle (in $F_k(G)$) we can reconstruct the incidence relations in E_A . Thus, given a vertex B of F we can construct, in polynomial time, a graph isomorphic to the line graph of $G_{\varphi(B)}$. As mentioned above, for graphs with more than three vertices there is a polynomial time algorithm that can reconstruct a graph from its line graph. [16, 14]. Since triangles in $F_k(G)$ are generated by moving one or two tokens in a triangle of G [8], we have the following result.

Lemma 5.1. Given only F we can construct in polynomial time a set of graphs

$${J_A: A \in V(F)},$$

where each J_A is isomorphic to $G_{\varphi(A)}$.

PRODUCTSUBGRAPH has two subroutines: Initialize and Extend. Initialize does the following. In line 1 it constructs the set of graphs J_A described in Lemma 5.1. In lines 2-5 for every vertex A of F it computes a maximum cardinality matching M_A of J_A ; this can be done in polynomial time [15]. In line 5, a vertex $A \in F$ is chosen so that $|M_A|$ is maximum. Assuming $k \leq n/2$, this matching corresponds to a matching of G of maximum cardinality with the property of having at most k edges. The 1-token graphs of these edges are the starting H_i . Afterwards, Productsubgraph iteratively calls extend for each i in turn. Extend attempts to extend H_i into a larger graph isomorphic to the token graph of some subgraph G_i of G. The initial choice of A is what enable us to reconstruct the G_i from their H_i . At the end of its execution Productsubgraph outputs a subgraph H of F, graphs H_1, \ldots, H_r and an isomorphism π from H to $H_1 \square \cdots \square H_r$.

${\bf Procedure}\,\,{\rm Initialize}$

```
1 Construct a set of graphs \{J_A : A \in V(F)\} where each J_A is isomorphic to G_{\varphi(A)};
 2 for A \in V(F) do
      Compute a maximum cardinality matching M_A of J_A;
 4 end
 5 Find A \in V(F) maximizing |M_A|;
 6 Let e_1, \ldots, e_r be the edges incident to A in F corresponding to the edges of M_A;
 7 Find the r-cube, Q_r \subset F containing A as a vertex and e_1, \ldots, e_r as edges;
8 H=Q_r;
9 for i \leftarrow 1 to r do
       Initialize two new vertices x_i and y_i and a new graph H_i;
10
       V(H_i) \leftarrow \{x_i, y_i\};
11
       E(H_i) \leftarrow \{x_i y_i\};
12
13 end
14 for B \in Q_r do
       Compute a shortest path P in Q_r from A to B;
15
       for i \leftarrow 1 to r do
16
           if P contains an edge in R[e_i] then
17
               \pi(B)(i) \leftarrow y_i;
18
19
           else
               \pi(B)(i) \leftarrow x_i;
\mathbf{20}
21
           end
22
       end
23 end
```

Procedure Extend(i)

```
1 Let A_1 be any vertex of H;
 2 Let A_2 be the neighbor of A_1 in H such that \pi(A_1)(i) \neq \pi(A_2)(i);
 \mathbf{3} \ Q = \text{Queue}();
 4 Q. Insert(A_1);
 5 Q. Insert(A_2);
 6 while Q not empty do
       A = Q. Dequeue();
       for every edge AB of F that is not an edge of H do
          if every C \in V(H), such that \pi(C)(i) == \pi(A)(i), is incident to an edge in R[AB] then
 9
              if B \notin H then
10
                 Initialize a new vertex y;
11
                 Add the vertex y to H_i;
12
                 for every X \in V(H), such that \pi(X)(i) == \pi(A)(i) do
13
                     Let Y be the neighbor of X in F such that XY is in R[AB];
14
                     Add the vertex Y to H;
15
                     \pi(Y) = \pi(X);
16
                     \pi(Y)(i) = y;
17
18
                 end
                 Q. Insert(B);
19
              end
20
              x = \pi(A)(i);
21
              y = \pi(B)(i);
22
              Add the edge xy to H_i;
23
              for every X \in V(H), such that \pi(X)(i) == \pi(A)(i) do
24
                 Let Y be the neighbor of X in H such that XY is in R[AB];
25
                 Add the edge XY to H;
26
              end
27
          end
28
       end
29
30 end
```

Algorithm 1: ProductSubgraph

```
Input: A graph F \simeq F_k(G) where G is a graph without induced 4-cycles as subgraphs.

Output: A subgraph H of F, graphs H_1, \ldots, H_r, and an isomorphism \pi from H to H_1 \square \ldots \square H_r.

1 Compute the set R of ladder classes of E(F);

2 Initialize (); // Initializes H and H_1, \ldots, H_r

3 for i \leftarrow 1 to r do

4 | Extend (i);

5 end
```

The following lemma provides structural properties of the output of PRODUCTSUBGRAPH; along the way, its proof also analyses PRODUCTSUBGRAPH, INITIALIZE and EXTEND in detail.

Lemma 5.2. There exist disjoint induced subgraphs G_1, \ldots, G_r of G, and positive integers k_1, \ldots, k_r such that the following holds.

- (1) $k = k_1 + \cdots + k_r$ and $V(G) = V(G_1) \cup \cdots \cup V(G_r)$.
- (2) For every pair of vertices $A_1, A_2 \in H$ and index $1 \le i \le r$ we have that $\pi(A_1)(i) = \pi(A_2)(i)$ if and only if $\varphi(A_1) \cap V(G_i) = \varphi(A_2) \cap V(G_i)$.
- (3) For every index $1 \le i \le r$, and vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to $F_{k_i}(G_i)$.
- (4) For every $A \in V(H)$,

$$\varphi(A) = \bigcup_{i=1}^{r} \varphi_i(\pi(A)(i)).$$

That is, the following diagram commutes.

$$H \xrightarrow{\varphi} F_k(G)$$

$$\downarrow^{\pi} \bigcup_{i=1}^r \varphi_i(\cdot)$$

$$H_1 \square \cdots \square H_r$$

Proof. H, H_1, \ldots, H_r and π are initialized when Initialize is called in line 2 of ProductSubgraph. Afterwards, these graphs and π are updated throughout the execution of ProductSubgraph. In what follows we show that throughout the execution of ProductSubgraph there exist disjoint subgraphs G_1, \ldots, G_r of G, and integers k_1, \ldots, k_r whose sum is at most k, such that at key steps of the execution of ProductSubgraph, (2) and the following properties hold.

- (3') For every index $1 \le i \le r$, and vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to a subgraph of $F_{k_i}(G_i)$.
- (4') For every $A \in V(H)$,

$$\varphi(A) = \left(\bigcup_{i=1}^r \varphi_i(\pi(A)(i))\right) \cup \left(\varphi(A) \setminus \bigcup_{i=1}^r V(G_i)\right).$$

Afterwards, we show that (1),(3) and (4) hold at the end of the execution of PRODUCTSUBGRAPH. We also show that at the end of the execution of PRODUCTSUBGRAPH the k_i sum up to k and that G_i are induced subgraphs of G; this proves the lemma.

Let A be as in line 5 of Initialize. Since M_A is a matching of J_A , its edges are in correspondence with $r:=|M_A|$ independent edges in G, such that there is exactly one token of $\varphi(A)$ in each edge. Moving these tokens on their respective edges produces an r-cube in $F_k(G)$. Therefore, the r-cube, Q_r , of line 7 exists. Q_r can be computed as follows. Let e_1, \ldots, e_r be the edges of F incident to A that correspond to the edges of M_A (line 6 of initialize). The vertices of Q_r are all the vertices of F that are reachable from F by a path with all its edges contained in $F[e_1] \cup \cdots \cup F[e_r]$. Thus, $F[e_r] \cap F[e_r]$ is found by computing the subgraph of F with edge set $F[e_1] \cup \cdots \cup F[e_r]$ and then finding the component containing $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ is each $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ is each $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ in the subgraph of $F[e_r] \cap F[e_r]$ be the edges of $F[e_r] \cap F[e_r]$ corresponds to moving the token along $F[e_r] \cap F[e_r]$ be the subgraph of $F[e_r] \cap F[e_r]$ and let $F[e_r] \cap F[e_r]$ in lines $F[e_r] \cap F[e_r]$ is constructed so that $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ in lines $F[e_r] \cap F[e_r]$ is constructed so that $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ in lines $F[e_r] \cap F[e_r]$ in lines $F[e_r] \cap F[e_r]$ is constructed so that $F[e_r] \cap F[e_r]$ and $F[e_r] \cap F[e_r]$ in lines $F[e_r] \cap F[e_r$

We now consider the i-th call to EXTEND in line 4 of PRODUCTSUBGRAPH. Assume that (2), (3') and (4') hold before the i-th call to EXTEND. Throughout the execution of EXTEND we have the following invariant.

Every vertex X in Q satisfies that
$$\pi(X)(j) = \pi(A_1)(j)$$
 for all $j \neq i$. (*)

This is certainly the case before the first execution of the while in line 6, since Q contains only the

vertices A_1 and A_2 . We show that (2), (3'), (4') and (*) hold at the end of each execution of the **for** of line 8.

Let AB be the edge in line 8 and let e := uv be the edge of G such that $\varphi(B)$ is obtained from $\varphi(A)$ by moving a token along e. We show that

the condition of line 9 is satisfied if and only if one of u and v is in G_i while the other is not in any G_j with $j \neq i$. (\dagger)

Suppose that one of u and v is in G_i while the other is not in any G_j with $j \neq i$. Let $C \in V(H)$ with $\pi(C)(i) = \pi(A)(i)$. Let $(A = C_1, \ldots, C_m = C)$ be a shortest path in H from A to C. Note that $\pi(C_l)(i) = \pi(A)(i)$ for all $1 \leq l \leq m$. Since (2) holds we have that for every $1 \leq l \leq m$ there exists a vertex D_l such that $\varphi(D_l)$ is obtained from $\varphi(C_l)$ by sliding a token along e. Thus, the set of vertices

$${C_l: 1 \le l \le m} \cup {D_l: 1 \le l \le m}$$

induce a ladder from AB to CD_m . Therefore, the condition of line 9 is satisfied.

Suppose that u and v are not in $\bigcup_{j=1}^r V(G_i)$. Then $\{e, e_1, \ldots, e_r\}$ is a matching of size r+1 of J_A , where A is as in line 5 of Initialize; this is a contradiction to the fact that M_A is maximum. Therefore, at least one of u and v is in $\bigcup_{j=1}^r V(G_i)$. Suppose that one of u and v is in G_j for some $j \neq i$. Without loss of generality suppose it is u. Then there exist vertices C_1 and C_2 of H with $\pi(C_1)(i) = \pi(C_2)(i) = \pi(A)(i)$, such that in $\varphi(C_1)$ there is a token at u, and in $\varphi(C_2)$ there is no token at u. Depending on whether there is a token at v in $\varphi(A)$, for one of $\varphi(C_1)$ and $\varphi(C_2)$ either e contains two tokens at its endpoints or no endpoint of e contains a token. In either case, there is no token move possible along e. Therefore, there exists a vertex $C \in H$ with $\pi(C)(i) = \pi(A)(i)$ that is not incident to an edge in R[AB]. Thus, the condition of line 9 does not hold. Therefore, (\dagger) holds.

Suppose that B is not a vertex of H. If $v \notin V(G_i)$, update $V(G_i)$ to $V(G_i) \cup \{v\}$, and $E(G_i)$ to $E(G_i) \cup \{uv\}$. If $\varphi(B)$ is obtained from moving a token from v to u, then this token has not been moved before. In this case update k_i to $k_i + 1$. Otherwise, if $\varphi(B)$ is obtained from moving a token from u to v, then k_i remains unchanged. In line 12 a new vertex y is added to H_i . Consider lines 13 – 15. For every vertex $X \in H$ with $\pi(X)(i) = \pi(A)(i)$, let Y be its neighbor such that $XY \in R[AB]$; we add Y to V(H). In lines 16 and 17, $\pi(Y)$ is defined so that $\pi(Y)(i) := y$ and $\pi(Y)(j) := \pi(X)(j)$ for all $j \neq i$. Thus (2) is satisfied after the execution of line 18. Since $\varphi(Y)$ is obtained from $\varphi(X)$ by sliding a token along e we have that e0 holds after the execution of line 18. In line 19, e1 is inserted to e2, and e3 sliding a token along e3 is obtained from e4. Since e5 since e6 we have that e7 by sliding a token along e8 we have that e8 may or may not be a vertex of e9. Let e9 as in lines 24 and 25. Since e9 is obtained from e1 by sliding a token along e1 by sliding a token along e3 by sliding a token along e4 by sliding after the execution of line 27.

Suppose that the *i*-th execution of EXTEND has ended. Let uv be an edge of G_i . Let X be any vertex of H such that $\varphi(X)$ contains a token at u and no token at v. Let $Y \in F$ be such that $\varphi(Y)$ is obtained from $\varphi(X)$ by sliding a token along uv. Note that Y is also in H. At some point during the execution of EXTEND, in line 23 the edge $\pi(A)(i)\pi(B)(i)$ was added to H_i . Therefore, we have that

(3") For every vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to $F_{k_i}(G_i)$.

Assume that the execution of Productsubgraph has ended. Since (3") holds for every $1 \leq i \leq r$ we have that (3) holds. Let $G' = \bigcup_{i=1}^r G_i$. Suppose that $G \setminus G' \neq \emptyset$. Let uv be a $G' - G \setminus G'$ edge. Let G_i be such that $u \in G_i$. Let A_1 and A_2 be vertices of H such that in $\varphi(A_1)$ there is a token at u and in $\varphi(A_2)$ there is no a token at u. Note that either there is a token at v in both $\varphi(A_1)$ and $\varphi(A_2)$ or there is no token at v in neither of $\varphi(A_1)$ and $\varphi(A_2)$. For exactly one of $\varphi(A_1)$ and $\varphi(A_2)$ we have that there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is a token at v in holds at the endpoints of v. Let v is a such that v is obtained from v is a such that v is a such that v is obtained from v is a such that v is a such that v is obtained from v is a such that v is a such that v is obtained from v is a such that v is obtained from v is a such that v is a such that

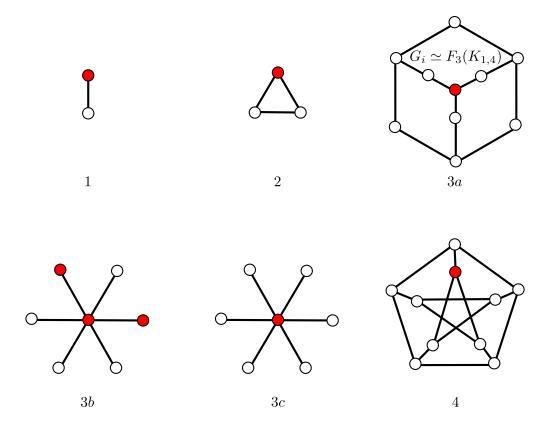


Figure 2: A sample token configuration of $F_{k_i}(G_i)$ for an H_i of each possible class.

5.1 Reconstructing the G_i

Suppose that ProductSubgraph has been executed; let G_1, \ldots, G_r and k_1, \ldots, k_r be as in Lemma 5.2. In this section we show how to construct graphs isomorphic to the G_i . We classify each H_i into the following four classes.

- 1. H_i is a an edge.
- 2. H_i is a triangle.
- 3. H_i is isomorphic to the token graph of a star of at least three vertices. By Lemma 3.3, there are unique integers l and m, with $l \leq (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$. There are three more possibilities in this case:
 - 3a. $G_i \simeq F_l(K_{1,m}), k_i = 1 \text{ or } k_i = |F_l(K_{1,m})| 1, \text{ and } 1 < l < m; \text{ or } k_i = |F_l(K_{1,m})| 1$
 - 3b. $G_i \simeq K_{1,m}$, $1 < k_i < |G_i| 1$, and $k_i = l$ or $k_i = m + 1 l$.
 - 3c. $G_i \simeq K_{1,m}$ and $k_i = 1$ or $k_i = m$.
- 4. H_i is not a triangle, an edge, nor isomorphic to the token graph of a star.

See Figure 2.

We now show how to determine the class of each H_i in polynomial time. The following lemma is useful for restricting the possible values of the k_i .

Lemma 5.3. If some G_i contains two disjoint edges, then all k_j are equal to 1 or all k_j are equal to $|G_j| - 1$.

Proof. Consider the vertex A and the edges e_1, \ldots, e_r in lines 5 and 6 of INITIALIZE. The edges $\varphi(e_1), \ldots, \varphi(e_r)$ of $F_k(G)$ correspond to e'_1, \ldots, e'_r disjoint edges in G, each with exactly one token of $\varphi(A)$ at one of their endpoints. For every $1 \leq i \leq r$, we have that e_i is in G_i . Let e_1^* and e_2^* be two disjoint edges of G_i . By the maximality of M_A , in $\varphi(A)$ at least one of e_1^* and e_2^* contains either:

no token, or two tokens at its endpoints. Without loss of generality assume it is e_1^* . This implies that $e_1^* \neq e_i'$.

For a contradiction suppose that some k_j is different from 1 and $|G_j|-1$. This implies that in $\varphi(A)$, G_j contains both a vertex $u \notin e'_j$ without a token, and vertex $v \notin e'_j$ with a token. If e_1^* contains no token of $\varphi(A)$, then let $\varphi(A')$ be the token configuration that is produced from $\varphi(A)$ by removing the token at v and placing it at e_1^* . If e_1^* contains two tokens of $\varphi(A)$, then let $\varphi(A')$ be the token configuration that is produced from $\varphi(A)$ by removing one token from e_1^* and placing it at u. We have that $e_1^*, e_1', \ldots, e_r'$ is a set of disjoint edges each with exactly one token of $\varphi(A')$. This implies that $|M'_A| = |M_A| + 1$, which contradicts our choice of A.

Lemma 5.4. We can determine in polynomial time the class of every H_i .

Proof. By Lemma 3.3, we can determine in polynomial time whether each H_i is of class 1, 2, 3c or 4. We show how to distinguish between the classes 3a and 3b. By Lemma 5.3 there cannot simultaneously exists an H_i of class 3a and an H_j of class 3b. Assume that at least one H_i is of class 3a or 3b as otherwise we are done. Suppose that r = 1; since we are assuming that 1 < k < |G| - 1, we have that H_1 is of class 3b and we are done in this case. Assume that r > 1.

We claim that

all the H_i of class 3a or 3b, are of class 3a if and only if F contains three edge disjoint graphs F_1, F_2 and M with the following properties. (*)

- (1) F_1 is an induced subgraph of H;
- (2) there exists an H_i of class 3a or 3b, and vertices $u \in H_i$ and $v \in H_j (j \neq i)$, such that the set of vertices of F_1 is of the form

$${A \in V(H) : \pi(A)(i) \neq u \text{ and } \pi(A)(j) = v}.$$

- (3) F_2 is disjoint from H;
- (4) M is a matching from the vertices of F_1 to the vertices of F_2 ;
- (5) all the edges in M are in the same ladder class;
- (6) the map that sends every vertex in F_1 to its matched vertex in M is an isomorphism from F_1 to F_2 .

Let F_1 , F_2 and M be as above. Since all the edges in M are in the same ladder class, the set of edges of $\varphi(M)$ corresponds to moving a token along the same edge xy of G. This implies that every token configuration in $\varphi(F_1)$ either: contains a token at x and no token at y, or contains a token at y and no token at x. By (2) of Lemma 5.2 there exist token configurations $B_1 \in F_{k_i}(G_i)$ and $B_2 \in F_{k_j}(G_j)$ such that

$$\varphi(V(F_1)) = \{ C \in \varphi(V(H)) : C \cap V(G_i) \neq B_1 \text{ and } C \cap V(G_j) = B_2 \}$$

Thus, either $x \in G_i$ and $y \in G_j$, or $x \in G_j$ and $y \in G_i$. Without loss of generality assume it is the former. If H_i is of type 3b then there exists token configurations C_1 and C_2 of $F_{k_i}(G_i)$ distinct from B_1 such that $x \in C_1$ and $x \notin C_2$. This is a contradiction to the fact that in every token configuration of $\varphi(F_1)$ either there is a token at x or there is no token at x. Therefore, if H contains subgraphs F_1, F_2 and M as above then every H_i of class 3a or 3b, is of class 3a.

Conversely, suppose that every H_i of class 3a or 3b, is of class 3a. Since r > 1 and G is connected there exists a pair of indices i and j, such that H_i is of class 3a and there exists an edge $xy \in G$ with $x \in G_i$ and $y \in G_j$. By Lemma 5.3 either all k_i are equal to 1, or all k_i are equal to $|G_i| - 1$. If all the k_i are equal to 1, then let F'_1 be the subgraph of $F_k(G)$ induced by the set of token configurations

$$\{B \in \varphi(H) : x \notin B \text{ and } y \in B\}.$$

If all k_i are equal to $|G_i| - 1$, then let F'_1 be the subgraph of $F_k(G)$ induced by the set of token configurations

$$\{B \in \varphi(H) : x \in B \text{ and } y \notin B\}.$$

Let $F_1 := \varphi^{-1}(F_1')$. By (2) of Lemma 5.2 and the fact that every k_i is equal to 1 or to $|G_i| - 1$, the vertex set of F_1 is of the form

$${A \in H : \pi(A)(i) \neq u \text{ and } \pi(A)(j) = v},$$

for some pair of vertices $u \in H_i$ and $v \in H_j$. Thus F_1 satisfies (1) and (2). Let F'_2 be the subgraph of $F_k(G)$ induced by the set of vertices

 $\{C \in F_k(G) : C \text{ is obtained from a vertex } B \in F_1' \text{ by sliding the token along } xy\}.$

Let $F_2 := \varphi^{-1}(F_2)$. Since xy is not an edge of $\bigcup_{i=1}^r G_i$, F_2 is disjoint from H. Thus, F_2 satisfies (3). Let

$$M' := \{ C_1 C_2 \in E(F_1', F_2') : \varphi(C_1) \triangle \varphi(C_2) = \{x, y\} \}.$$

Let $M := \varphi^{-1}(M')$. M' is a matching from F'_1 to F'_2 ; thus, M satisfies (4). By construction of F'_2 , the map that sends every vertex in F'_1 to its matched vertex in M' is an isomorphism from F'_1 to F'_2 . Therefore, M satisfies (6). It is not hard to show that H_i is 2-connected; this, in turn implies that F'_1 is connected. Thus, all the edges in M' are in the same ladder class, and M satisfies (5).

The existence of F_1 , F_2 and M can be determined in polynomial time as follows. First we iterate over all possible candidates for F by considering all subgraphs induced by a set of vertices satisfying (2); there are a polynomial number of these sets, and each can be constructed in polynomial time. Afterwards, we iterate over each ladder class of F and compute the subset of edges, M, in this ladder class such that exactly one of its endpoints is a vertex of F_1 . We compute the graph F_2 induced by the endpoints of these edges that are not in F_1 . Finally, we check whether M and F_2 satisfy (3) – (6). If the desired F_1 , F_2 and M exist they are found by this algorithm.

For every $1 \le i \le r$ we construct a graph J_i isomorphic to G_i as follows. If H_i is not of class 3b we set J_i to be a copy of H_i . If H_i is of class 3b, we use Lemma 3.1 to compute m and $l \le (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$; and set J_i to be a copy of $K_{1,m}$. Let $J := \bigcup_{i=1}^r J_i$; note that J is isomorphic to $\bigcup_{i=1}^r G_i$.

5.2 Reconstructing the adjacencies between the G_i 's

To reconstruct G all that remains to be done is to reconstruct the adjacencies between the G_i 's. This information is encoded in the adjacencies between H and $F \setminus H$. We start by labeling each H_i as a token graph of J_i . First note that each H_i is uniquely reconstructible as the k_i -token graph of J_i : when H_i is not of class 3b this is straightforward; and when H_i is of class 3b it follows from Lemmas 3.1 and 3.3. There are at most two possible values, l_i and \bar{l}_i , for each k_i :

- if H_i is of class 1, then $k_i = 1$; in this case we set $l_i := 1$ and $\bar{l}_i := 1$;
- If H_i is not of class 3b nor 1, then by Lemma 5.3, we have that $k_i = 1$ or $k_i = |J_i| 1$; in this case we set $l_i := 1$, and $\bar{l}_i := |J_i| 1$;
- If H_i is of class 3b, then by Lemma 3.1, there exist unique integers m and $l \leq (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$; we set $l_i := l$, and $\bar{l}_i := m+1-l$ in this case.

We construct in polynomial time an isomorphism $\psi_i: H_i \to F_{l_i}(J_i)$. This is straightforward when H_i is not of class 3b; when H_i is of class 3b it can be done in polynomial time by Lemma 3.3. For each H_i we also construct the following isomorphism from H_i to $F_{\bar{l}_i}(J_i)$.

$$\overline{\psi_i} := \left\{ egin{array}{ll} \mathfrak{c} \circ \psi_i & ext{if } H_i ext{ is not of class 1;} \\ & \psi_i & ext{if } H_i ext{ is of class 1;} \end{array}
ight.$$

Let φ_i be the isomorphism from H_i to $F_{k_i}(G_i)$ given by (3) of Lemma 5.2. Using φ_i and one of ψ_i and $\overline{\psi_i}$, we define an isomorphism ϕ'_i from $F_{k_i}(J_i)$ to $F_{k_i}(G_i)$, and an isomorphism ϕ_i from J_i to G_i such that

$$\phi_i' = \iota(\phi_i).$$

• Suppose that H_i is not of class 1. By 3) of Theorem 2.4, there exists a unique $f(\varphi_i \circ \psi_i^{-1}) \in \text{Iso}(J_i, G_i)$ such that

$$\varphi_i \circ {\psi_i}^{-1} = \iota(f(\varphi_i \circ {\psi_i}^{-1})) \text{ or } \varphi_i \circ {\psi_i}^{-1} = \mathfrak{c} \circ \iota(f(\varphi_i \circ {\psi_i}^{-1})).$$

Let

$$\phi_i' := \begin{cases} & \varphi_i \circ {\psi_i}^{-1} \text{ if } \varphi_i \circ {\psi_i}^{-1} = \iota(f(\varphi_i \circ {\psi_i}^{-1})); \\ & \varphi_i \circ \overline{\psi_i}^{-1} \text{ if } \varphi_i \circ {\psi_i}^{-1} = \mathfrak{c} \circ \iota(f(\varphi_i \circ {\psi_i}^{-1})). \end{cases}$$

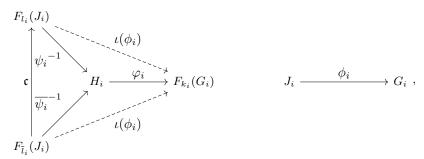
Let $\phi_i := f(\varphi_i \circ \psi_i^{-1})$. If $\phi_i' = \varphi_i \circ \psi_i^{-1}$, then $\phi_i' = \iota(\phi_i)$. If $\phi_i' = \varphi_i \circ \overline{\psi_i}^{-1}$, then

$$\begin{aligned} \phi_i' &= \varphi_i \circ \overline{\psi_i}^{-1} \\ &= \varphi_i \circ (\mathfrak{c} \circ \psi_i)^{-1} \\ &= \varphi_i \circ {\psi_i}^{-1} \circ \mathfrak{c} \\ &= \mathfrak{c} \circ \iota(f(\varphi_i \circ {\psi_i}^{-1})) \circ \mathfrak{c} \\ &= \mathfrak{c} \circ \iota \circ \iota(f(\varphi_i \circ {\psi_i}^{-1})) \\ &= \iota(f(\varphi_i \circ {\psi_i}^{-1})) \\ &= \iota(\phi_i). \end{aligned}$$

• Suppose that H_i is of class 1. Let $\phi'_i := \varphi_i \circ \psi_i^{-1}$, and let $\phi_i \in \text{Iso}(J_i, G_i)$ such that

$$\phi_i' = \iota(\phi_i).$$

We have the following diagram.



where we have exactly one of the dashed lines.

In what follows we always use the same letter to denote a vertex of J_i and its image under ϕ_i in G_i . We use a prime to distinguish the vertex in J_i . So that if $u' \in J_i$ then $u := \phi_i(u') \in G_i$. Let $e \in E(H, F \setminus H)$. Note that there exist indices $1 \le i < j \le r$ such that $\varphi(e)$ corresponds to moving a token along a $G_i - G_j$ edge. Let $\mathrm{idx}_H(e) := \{i, j\}$; and let $E(H, F \setminus H)_{ij}$ be the set of edges $e' \in E(H, F \setminus H)$ such that $\mathrm{idx}_H(e') = \{i, j\}$.

Lemma 5.5. For every $H - F \setminus H$ edge, e, we can compute $idx_H(e)$ in polynomial time.

Proof. Let AB be an $H-F \setminus H$ edge. For every pair $1 \le i < j \le r$ we check whether every vertex in the set

$$\{C \in H : \pi(C)(i) = \pi(A)(i) \text{ and } \pi(C)(j) = \pi(A)(j)\}\$$

is incident to an edge in the same ladder class as AB. The pair where this is the case is the pair of indices we are looking for.

5.2.1 Labeling the $H - F \setminus H$ edges with vertices in J

Consider an edge $e \in E(H, F \setminus H)_{ij}$. Note that there exist vertices $x' =: \text{endpoint}_J(e)(i) \in J_i$ and $y' =: \text{endpoint}_J(e)(j) \in J_j$ such that $\varphi(e)$ corresponds to moving a token along the edge xy. In this section we show how to compute endpoint J(e)(i) in polynomial time when H_i is not of class 1. We define some subgraphs of F, that are useful for this and other purposes.

Let A be a vertex of F.

• Let $\mathbf{Move}(A, i)$ be the subgraph of F induced by all the vertices $B \in F$ such that

$$\varphi(B) \cap G_i = \varphi(A) \cap G_i$$
 for all $j \neq i$.

Thus, $\varphi(\mathbf{Move}(A, i))$ is the subgraph of $F_k(G)$ induced by all the token configurations that can be reached from $\varphi(A)$ by moving the tokens within G_i while leaving the tokens at the other G_j fixed.

• Let $\mathbf{Move}(A)$ be the subgraph of F induced by all the vertices $B \in F$ such that

$$|\varphi(B) \cap G_i| = |\varphi(A) \cap G_i|$$
 for all $1 < i < r$.

Thus, $\varphi(\mathbf{Move}(A))$ is the subgraph of $F_k(G)$ induced by all the token configurations that can be reached from $\varphi(A)$ by token moves that do not involve moving tokens between different G_i .

Note that if $A \in V(H)$, then

$$\mathbf{Move}(A, i) \simeq H_i \text{ and } \mathbf{Move}(A) \simeq H.$$

In particular, in this case $\mathbf{Move}(A, i)$ is the subgraph of H induced by the set of vertices

$$\{B \in H : \pi(B)(j) = \pi(A)(j) \text{ for all } j \neq i\}.$$

Thus, when A is a vertex of H we can compute $\mathbf{Move}(A, i)$ in polynomial time.

Let $e = AB \in E(H, F \setminus H)_{ij}$.

• Let $\mathbf{FixEdge}(e,i)$ be the component, that contains A, of the subgraph of F induced by the set of vertices

$$\{C \in \mathbf{Move}(A, i) : C \text{ is incident to an edge in the ladder class of } e\}.$$

Thus, $\varphi(\mathbf{FixEdge}(e, i))$ is the subgraph of $F_k(G)$ induced by token configurations in $\varphi(\mathbf{Move}(A, i))$ that are reachable from $\varphi(A)$ by a path in $\varphi(\mathbf{Move}(A, i))$, such that at every token move of the path no token has been moved from or placed at the endpoints of $\varphi(e)$.

• Let $\mathbf{NFixEdge}(e, i)$ be the subgraph of $\mathbf{Move}(A, i) \setminus \mathbf{FixEdge}(e, i)$ induced by neighbors of $\mathbf{FixEdge}(e, i)$ in $\mathbf{Move}(A, i) \setminus \mathbf{FixEdge}(e, i)$.

We now prove some lemmas that use the structure of the previously defined subgraphs of F, to compute endpoint $_{J}(e)(i)$ in polynomial time.

Lemma 5.6. Let $e \in E(H, F \setminus H)_{ij}$. Suppose that $|\mathbf{FixEdge}(e, i)| > 1$ or $|\mathbf{NFixEdge}(e, i)| > 1$. Then we can compute $\mathbf{endpoint}_{J}(e)(i)$ in polynomial time.

Proof. Let AB := e. If H_i is of class 1, then $|\mathbf{FixEdge}(e, i)| = 1$ and $|\mathbf{NFixEdge}(e, i)| = 1$. Thus, H_i is not of class 1. Note that

a) if A_1A_2 is an edge of $\mathbf{FixEdge}(e,i)$, then

$$x' \notin \psi_i(A_1) \triangle \psi_i(A_2) = \overline{\psi_i}(A_1) \triangle \overline{\psi_i}(A_2); \text{ and}$$
(*)

b) if A_1A_2 is a **FixEdge**(e,i) - **NFixEdge**(e,i) edge then

$$x' \in \psi_i(A_1) \triangle \psi_i(A_2) = \overline{\psi_i}(A_1) \triangle \overline{\psi_i}(A_2).$$

Suppose that H_i is not of class 3b. We have that $k_i = 1$ or $k_i = |G_i| - 1$. Let v' the only vertex in

$$\psi_i(A) = V(J_i) \setminus \overline{\psi_i}(A).$$

Let C be a vertex in $\mathbf{NFixEdge}(e,i)$. Let w' be the only vertex in

$$\psi_i(C) = V(J_i) \setminus \overline{\psi_i}(C).$$

Suppose that $|\mathbf{FixEdge}(e, i)| > 1$; by (*) we have that endpoint J(e)(i) = w'. Suppose that $|\mathbf{FixEdge}(e, i)| = 1$; thus, $|\mathbf{NFixEdge}(e, i)| > 1$; by (*) we have that endpoint J(e)(i) = v'.

Suppose that H_i is of class 3b. Thus, J_i is a star. Let v' be the center of J_i . If $|\mathbf{FixEdge}(e,i)| = 1$, then $|\mathbf{NFixEdge}(e,i)| > 1$ and $\mathrm{endpoint}_J(e)(i) = v'$. Suppose that $|\mathbf{FixEdge}(e,i)| > 1$, then $\mathrm{endpoint}_J(e)(i) \neq v'$. Let CD be a $\mathbf{FixEdge}(e,i) - \mathbf{NFixEdge}(e,i)$ edge. We have that $\mathrm{endpoint}_J(e)(i)$ is the vertex in

$$\psi_i(C)\triangle\psi_i(D) = \overline{\psi_i}(C)\triangle\overline{\psi_i}(D)$$

distinct from v'.

Lemma 5.7. Suppose that H_i is not of class 1. For every vertex $u \in G_i$ such that u is adjacent to a vertex $v \in G_j$, there exists $e \in E(H, F \setminus H)_{ij}$ such that endpoint J(e)(i) = u' and for which we can compute endpoint J(e)(i) in polynomial time.

Proof. Suppose that u is of degree greater than one in G_i . Let $A \in V(H)$ be such that: if $k_i = 1$, then in $\varphi(A)$ there is a token at u and no token at v; and if $k_i > 1$, then in $\varphi(A)$ there is no token at u, a token in at least two neighbors of u and a token at v. Let e := AB, such that $\varphi(B)$ is obtained from A by sliding the token along uv. We have that $|\mathbf{NFixEdge}(e,i)| > 1$ and by Lemma 5.6 we can compute endpoint I(e)(i).

Suppose that u is of degree equal to one in G_i , and let w be its neighbor in G_i . Note that since H_i is not of class 1, w is of degree greater than one in G_i . Let $A \in H$ be such that: if $k_i = 1$, then in $\varphi(A)$ there is a token at w, no token at u, and a token at v; if $k_i > 1$, then in $\varphi(A)$ there is no token at w, a token at u, a token at u, a token at u, and no token at v. Let e := AB, such that $\varphi(B)$ is obtained from A by sliding the token along uv. We have that $|\mathbf{FixEdge}(e,i)| > 1$ and by Lemma 5.6 we can compute endpoint I(e)(i).

Lemma 5.8. Let $e := AB \in E(H, F \setminus H)_{ij}$ such that H_i is not of class 1. If we know that it must be the case that either $k_i = k_j = 1$, or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$, then we can compute endpoint J(e)(i) in polynomial time.

Proof. Let x' be the vertex of J_i such that

$$\{x'\} = \psi_i(\pi(A)(i)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(i)).$$

Suppose that x' is of degree greater than one in J_i . If $x' = \text{endpoint}_J(e)(i)$ then $|\mathbf{NFixEdge}(e,i)| > 1$, and we are done by Lemma 5.6. If $x' \neq \text{endpoint}_J(e)(i)$ then $|\mathbf{FixEdge}(e,i)| > 1$, and we are done by Lemma 5.6. Assume that x' is of degree equal to one in J_i . Let x' be the neighbor of x' in x'. Assume that endpoint x' or endpoint x' or

Let y' be the vertex of J_j such that

$$\{y'\} = \psi_i(\pi(A)(j)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(j)).$$

Suppose that $|\mathbf{FixEdge}(e,j)| > 1$ or $|\mathbf{NFixEdge}(e,j)| > 1$. Note that H_j is not of class 1. By Lemma 5.6 we can compute $\operatorname{endpoint}_J(e)(j)$. If $\operatorname{endpoint}_J(e)(j) = y'$, then $\operatorname{endpoint}_J(e)(i) = v'$; if $\operatorname{endpoint}_J(e)(j) \neq y'$, then $\operatorname{endpoint}_J(e)(i) = x'$. Thus, we may assume that $|\mathbf{FixEdge}(e,j)| = 1$ and $|\mathbf{NFixEdge}(e,j)| = 1$. This implies that y' is of degree equal to one in J_j . Let w' be the neighbor of y' in J_j . We have that $\operatorname{endpoint}_J(e)(j) = y'$ or $\operatorname{endpoint}_J(e)(j) = w'$; otherwise, $|\mathbf{FixEdge}(e,j)| > 1$.

Let A' be the vertex of H such that $\psi_i(\pi(A'))(i) = V(J_i) \setminus \overline{\psi}_i(\pi(A)(i))$ and $\psi_j(\pi(A'))(j) = V(J_j) \setminus \overline{\psi}_i(\pi(A)(j))$. Let

$$S := \{ B' \in V(F \setminus H) : idx_H(A'B') = \{i, j\} \}.$$

Let $B' \in S$. Since v' is of degree greater than one in J_i we have that $|\mathbf{FixEdge}(A'B',i)| > 1$ or $|\mathbf{FixEdge}(A'B',i)| > 1$. By Lemma 5.6 we can determine endpoint J(A'B')(i). By a similar argument, if H_j is not of class 1 we can determine endpoint J(A'B')(j).

Suppose that H_j is not of class 1. We determine whether x is adjacent to w, and whether y is adjacent to v as follows. If x is adjacent to w but v is not adjacent to y, then endpoint $_J(e)(i) = x'$. If y is adjacent to v but x is not adjacent to w, then endpoint $_J(e)(i) = v'$. Assume that x is adjacent to w and that v is adjacent to y. We determine the vertex $B' \in S$ such that $\varphi(B')$ is obtained from $\varphi(A')$ by sliding the token along the edge xw, and the vertex $B'' \in S$ such that $\varphi(B'')$ is obtained from $\varphi(A')$ by sliding the token along the edge yv. Note that B = B' or B = B''. If B = B' then endpoint $_J(e)(i) = v'$; and if B = B'' then endpoint $_J(e)(i) = x'$.

Suppose that H_j is of class 1. Suppose that there exists a vertex $B' \in S$ such that endpoint J(A'B')(i) = v'. If B = B', then endpoint J(e)(i) = x'; otherwise, endpoint J(e)(i) = v'. Suppose that no such vertex B' exists. If $k_i = 1$, then v is not adjacent to y, and endpoint J(e)(i) = x'. If $k_i = |G_i| - 1$, then v is not adjacent to y, and endpoint J(e)(i) = x'. In either case we have that endpoint J(e)(i) = x'.

Suppose that we have computed endpoint $_J(e)(i)$ and endpoint $_J(e)(j)$ for some $e:=AB\in E(H,F\setminus H)_{ij}$. Note that ψ_i and $\overline{\psi_i}$ both interpret H_i as a token graph of J_i . With the difference being that there is a token at endpoint $_J(e)(i)$ in $\psi_i(\underline{\pi(A)})$ if and only if there is no token at endpoint $_J(e)(i)$ in $\overline{\psi_i(\underline{\pi(A)})}$. The same relationship holds for ψ_j , $\overline{\psi_j}$ and H_j . So ψ_i is compatible with exactly one of ψ_j and $\overline{\psi_j}$. We formalize this idea in Lemma 5.9. For every $1 \le i \le r$, and every $\psi_i' \in \{\psi_i, \overline{\psi_i}\}$, let

$$\overline{\psi_i'} := \begin{cases} \psi_i & \text{if } \psi_i' = \overline{\psi_i}, \\ \overline{\psi_i} & \text{if } \psi_i' = \psi_i. \end{cases}$$

Lemma 5.9. Suppose that H_j is not of class 1 and that we have computed both $\operatorname{endpoint}_J(e)(i)$ and $\operatorname{endpoint}_J(e)(j)$ for some $e \in E(H, F \setminus H)_{ij}$. Then we can determine in polynomial time $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ with the following property. For every edge $AB \in E(H, F \setminus H)_{ij}$, there is exactly one token at $\{\operatorname{endpoint}_J(AB)(i), \operatorname{endpoint}_J(AB)(j)\}$ in each of

$$\psi_i(\pi(A)(i)) \cup \psi'_j(\pi(A)(j))$$
 and $\overline{\psi_i}(\pi(A)(i)) \cup \overline{\psi'_j}(\pi(A)(j))$.

Proof. Let $AB := e, \ x' := \operatorname{endpoint}_J(e)(i)$ and $y' := \operatorname{endpoint}_J(e)(j)$. Since $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along the edge xy, we have that in $\varphi(A)$ there is exactly one token at each of x and y. By definition of $\overline{\psi_j}$ there is a token at y' in $\psi_j(\pi(A)(j))$ if and only if there is no token at y' in $\overline{\psi_j}(\pi(A)(j))$. Choose $\psi_j' \in \{\psi_j, \overline{\psi_j}\}$ so that

there is a token at y' in $\psi'_i(\pi(A)(j))$ if and only there is no token at x' in $\psi_i(\pi(A)(i))$. (*)

Let $CD \in E(H, F \setminus H)_{ij}$, $v' := \text{endpoint}_J(CD)(i)$ and $w' := \text{endpoint}_J(CD)(j)$. Recall that

$$\phi_i' = \varphi_i \circ {\psi_i}^{-1} \text{ or } \phi_i' = \varphi_i \circ \overline{\psi_i}^{-1}.$$

Suppose $\phi_i' = \varphi_i \circ \psi_i^{-1}$. Thus, the isomorphism ϕ_i from J_i to G_i is given by $\phi_i = \iota^{-1}(\varphi_i \circ \psi_i^{-1})$. By (*) we have that $\phi_j = \iota^{-1}(\varphi_j \circ {\psi_j'}^{-1})$. This implies that:

- there is a token at v in $\varphi(C)$ if and only if there is a token at v' in $\psi_i(\pi(C)(i))$; and
- there is a token at w in $\varphi(C)$ if and only if there is a token at w' in $\psi'_i(\pi(C)(j))$.

Therefore, there is exactly one token at $\{v', w'\}$ in each of

$$\psi_i(\pi(C)(i)) \cup \psi'_i(\pi(C)(j))$$
 and $\overline{\psi_i}(\pi(C)(i)) \cup \overline{\psi'_i}(\pi(C)(j))$.

Suppose $\phi_i' = \varphi_i \circ \overline{\psi_i}^{-1}$. Thus, the isomorphism ϕ_i from J_i to G_i is given by $\phi_i = \iota^{-1}(\varphi_i \circ \overline{\psi_i}^{-1})$. By (*) we have that $\phi_j = \iota^{-1}(\varphi_j \circ \overline{\psi_j'}^{-1})$ This implies that:

- there is a token at v in $\varphi(C)$ if and only if there is no token at v' in $\psi_i(\pi(C)(i))$; and
- there is a token at w in $\varphi(C)$ if and only if there is no token at w' in $\psi'_i(\pi(C)(j))$.

Therefore, there is exactly one token at $\{v', w'\}$ in each of

$$\psi_i(\pi(C)(i)) \cup \psi_i'(\pi(C)(j))$$
 and $\overline{\psi_i}(\pi(C)(i)) \cup \overline{\psi_i'}(\pi(C)(j))$.

When ψ_i and ψ'_j are as in Lemma 5.9, we say that ψ_i is *compatible* with ψ'_j , and that $\overline{\psi_i}$ is *compatible* with $\overline{\psi'_j}$. For convenience if H_i is of class 1, then for every $1 \leq j \leq r$, we say that ψ_j and $\overline{\psi_j}$ are both *compatible* with $\psi_i = \overline{\psi_i}$. We are now ready to prove the main result of this section.

Theorem 5.10. Let $e \in E(H, F \backslash H)_{ij}$ such that H_i is not of class 1. Then we can compute endpoint J(e)(i) in polynomial time.

Proof. We first show that

If there exist three disjoint edges
$$e'_1, e'_2$$
 and e'_3 in $G_i \cup G_j$, then $k_i = k_j = 1$, or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$.

Suppose for a contradiction that $k_i = 1$ and $k_j > 1$, or $k_i = |G_i| - 1$ and $k_j < |G_j| - 1$. Let $e_1, \ldots e_r$ be as in line 6 of Initialize. We can rearrange the tokens at $\varphi(A)$ and place exactly one token at the endpoints of each of $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}, e_{j+1}, \ldots e_r, e'_1, e'_2$ and e'_3 ; which contradicts our choice of A in line 5 of Initialize.

Let x' be the vertex of J_i such that

$$\{x'\} = \psi_i(\pi(A)(i)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(i)).$$

We assume that $|\mathbf{FixEdge}(e, i)| = 1$ and $|\mathbf{NFixEdge}(e, i)| = 1$; otherwise we are done by Lemma 5.6. Thus, x' is of degree one in J_i . Let v' be the neighbor of x' in J_i . We have that

endpoint_I
$$(e)(i) = x'$$
 or endpoint_I $(e)(i) = v'$;

otherwise, $|\mathbf{FixEdge}(e, i)| > 1$ or $|\mathbf{NFixEdge}(e, i)| > 1$.

Since both H_i and H_j are of class 3c, we have that J_i and J_j are stars. Thus, x' is a leaf of J_i and v' is the center of J_i . Let w' be the center of J_j . By Lemma 5.7, we can determine for every pair of vertices $v'_1 \in G_i$ and $v'_2 \in G_j$, whether v_1 is adjacent to v_2 in G. We assume that x is adjacent to a vertex of G_j as otherwise endpoint J(e)(i) = v'. We also assume that v is adjacent to a vertex of G_j as otherwise endpoint J(e)(i) = x'. Suppose that a leaf of J_i is adjacent to a leaf of J_i note that there exists three disjoint edges in $J_i \cup J_j$. We have that $J_i = J_j$ in the $J_i = J_j$ or $J_i = J_j$ and $J_i = J_j$ is adjacent to a leaf of $J_i = J_j$ thus, by Lemma 5.8, we can compute endpoint $J_i(e)(i)$. Assume that no leaf of $J_i = J_j$ is adjacent to a leaf of $J_i = J_j$ this implies that $J_i = J_j$ is adjacent to $J_i = J_j$ this implies that $J_i = J_j$ is adjacent to $J_i = J_j$ this implies that $J_i = J_j$ and $J_i = J_j$ and $J_i = J_j$ thus $J_i = J_j$ and $J_i = J_j$ thus $J_i = J_j$ and $J_i = J_j$ and $J_i = J_j$ thus $J_i = J_j$ and $J_i = J_j$

Suppose that y is a neighbor of v in G_j distinct from w. Then, (v, y, w, x) is a 4-cycle in G and x and y are adjacent—contradicting the assumption that no leaf of G_i is adjacent to a leaf of G_j . Thus, w is the only neighbor of v in G_j . Suppose that a vertex y of G_i distinct from x and v, is adjacent to w.

Then, (y, w, x, v) is a 4-cycle in G and x and y are adjacent—contradicting the assumption that G_i is a star. Summarizing, we have that

$$E(G_i, G_j) = \{xw, vw\}.$$

Let z be a leaf of G_i distinct from x. Let A_1 such that $\varphi(A_1)$ is obtained from $\varphi(A)$ by sliding a token along xv. Let A_2 such that $\varphi(A_2)$ is obtained from $\varphi(A_1)$ by sliding a token along vz. If there exists a vertex $B' \in F \setminus H$ such that A_2 is adjacent to B' and $\mathrm{idx}_H(A_2B') = \{i, j\}$, then $\mathrm{endpoint}_J(e)(i) = v'$; if no such vertex exists, then $\mathrm{endpoint}_J(e)(i) = x'$.

5.2.2 Labeling the $H - F \setminus H$ edges with respect to token movement direction

Consider an edge $e \in E(H, F \setminus H)_{ij}$. The edge $\varphi(e)$ corresponds to moving a token either from G_i to G_j or from G_j to G_i . We denote these two possibilities with the tuples $(e, i \to j)$ and $(e, j \to i)$, respectively. If $\varphi(e)$ corresponds to moving a token from G_i to G_j , then we say that $(e, i \to j)$ agrees with φ . For every H_i of class 1, let $V(G_i) = \{x_i, \bar{x}_i\}$. For every vertex $u' \in J_i$, let

$$\overline{u}' := \left\{ egin{array}{ll} x_i' & ext{if } u' = \overline{x}_i', ext{ and} \ & & & & & & \\ \overline{x}_i' & ext{if } u' = x_i; \end{array}
ight.$$

and for every vertex $u \in G_i$, let

$$\overline{u} := \left\{ \begin{array}{ll} x_i & \text{if } u = \overline{x}_i, \text{ and} \\ \\ \overline{x}_i & \text{if } u = x_i. \end{array} \right.$$

For convenience, for every vertex v' in some J_j , such that H_j is not of class 1, we define

$$\bar{v'} := v'$$
 and $\bar{v} := v$.

Lemma 5.11. For all $1 \le i < j \le r$, let

$$D'_{ij} := \{(e, i \to j) : e \in E(H, F \setminus H)_{ij}\}\} \cup \{(e, j \to i) : e \in E(H, F \setminus H)_{ij}\}\}.$$

In polynomial time we can find a partition of the set

$$\mathcal{D} := \bigcup_{1 \le i < j \le r} D'_{ij}$$

into two sets $\overrightarrow{\mathcal{D}}$ and $\overleftarrow{\mathcal{D}}$, such that the following holds. Either for all edges $e \in E(H, F \setminus H)$ there is a tuple containing e in $\overrightarrow{\mathcal{D}}$ that agrees with the direction of φ , or for all edges $e \in E(H, F \setminus H)$ there is a tuple containing e in $\overleftarrow{\mathcal{D}}$ that agrees with the direction of φ .

Proof. Let $1 \leq i < j \leq r$ be such that $E(G_i, G_j) \neq \emptyset$. We first show that in polynomial time we can find a partition of D'_{ij} into two sets D_{ij} and $\overline{D_{ij}}$ such that the following holds. Either for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ , or for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in $\overline{D_{ij}}$ that agrees with φ . For convenience we define $D_{ji} := D_{ij}$ and $\overline{D_{ji}} := \overline{D_{ij}}$.

Let $e := AB \in E(H, F \setminus H)_{ij}$. We decide which of D_{ij} and $\overline{D_{ij}}$ contains $(e, i \to j)$ as follows.

- Suppose that at least one of H_i and H_j is not of class 1.
 - Without loss of generality assume that H_i is not of class 1. We use Theorem 5.10 to compute endpoint J(e)(i). If endpoint $J(e)(i) \in \psi_i(\pi(A)(i))$, then $(e, i \to j) \in D_{ij}$ and $(e, j \to i) \in \overline{D_{ij}}$; if endpoint $J(e)(i) \notin \psi_i(\pi(A)(i))$, then $(e, i \to j) \in \overline{D_{ij}}$ and $(e, j \to i) \in D_{ij}$. Note that, either for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ , or for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in $\overline{D_{ij}}$ that agrees with φ .
- Suppose that both H_i and H_j are of class 1.

Fix a vertex $A^* \in H$ and let H' be the subgraph of H induced by the vertices $C \in H$ such that $\pi(C)(l) = \pi(A^*)(l)$ for all $l \neq i, j$. Note that $\varphi(H')$ is the graph generated by moving a token at each of G_i and G_j , while fixing the tokens at the other G_l . Since G_i and G_j are edges, H' is an

induced 4-cycle of F. The endpoint in $F \setminus H$ of every edge in $E(H', F \setminus H)_{ij}$ must be one of two vertices B_1^* and B_2^* , where $\varphi(B_1^*)$ and $\varphi(B_2^*)$ correspond to having two tokens in either G_i or in G_j . Let $P:=(A=A_1,\ldots,A_m=A')$ be a path from A to a vertex $A' \in H'$ such that for all $1 \leq s \leq m$, we have that $\pi(A_s)(i)=\pi(A)(i)$ and $\pi(A_s)(j)=\pi(A)(j)$. Thus, $\varphi(P)$ corresponds to a sequence of tokens moves that leaves the tokens at G_i and G_j fixed and arrives at a vertex of $\varphi(H')$. Note that there exists exactly one edge $A'B' \in E(H, F \setminus H)_{ij}$, such that A'B' and e are in the same ladder class. If $B'=B_1^*$ then $(e,i\to j)\in D_{ij}$ and $(e,j\to i)\in \overline{D_{ij}}$; if $B'=B_2^*$ then $(e,i\to j)\in \overline{D_{ij}}$ and $(e,j\to i)\in D_{ij}$. Note that, either for all edges $e\in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ , or for all edges $e\in E(H, F \setminus H)_{ij}$ there is a tuple in $\overline{D_{ij}}$ that agrees with φ .

Suppose that we have defined all such D_{ij} and $\overline{D_{ij}}$. For $D \in \{D_{ij}, \overline{D_{ij}}\}$, we define

$$\overline{D} := \begin{cases} \overline{D_{ij}} & \text{if } D = D_{ij}, \text{ and} \\ D_{ij} & \text{if } D = \overline{D_{ij}}. \end{cases}$$

Let $1 \leq i, j, l \leq r$ be indices such that $E(G_i, G_j) \neq \emptyset$ and $E(G_j, G_l) \neq \emptyset$. Let $D_1 \in \{D_{ij}, \overline{D_{ij}}\}$ and $D_2 \in \{D_{jl}, \overline{D_{jl}}\}$. We say that D_1 and D_2 are an adjacent pair; we say that D_1 and D_2 are a compatible pair, if in addition the following holds. Either for all edges $e \in E(H, F \setminus H)_{ij} \cup E(H, F \setminus H)_{jl}$ there is a tuple in $\overline{D_1} \cup \overline{D_2}$ that agrees with φ , or for all edges $e \in E(H, F \setminus H)_{ij} \cup E(H, F \setminus H)_{jl}$ there is a tuple in $\overline{D_1} \cup \overline{D_2}$ that agrees with φ . Note that if D_1 and D_2 are compatible, then $\overline{D_1}$ and $\overline{D_2}$ are compatible. Moreover, there is exactly one of D_{jl} and $\overline{D_{jl}}$ that is compatible to D_{ij} .

We determine in polynomial time whether D_1 and D_2 are compatible as follows.

• Suppose that H_j is not of class 1.

• Suppose that H_j is of class 1.

Let $e_1 := AB_1 \in E(H, F \setminus H)_{ij}$ and $e_2 := AB_2 \in E(H, F \setminus H)_{jl}$ such that $(e_1, i \to j) \in D_1$ and $(e_2, j \to l) \in D_2$. Let u' be the only vertex in $\psi_j(\pi(A)(j))$. If $(e_1, i \to j)$ and $(e_2, j \to l)$ agree with φ , then $\varphi(e_1)$ corresponds to moving a token from a vertex in G_i to \bar{u} ; and $\varphi(e_2)$ corresponds to moving a token from u to a vertex in G_l . D_1 and D_2 are compatible if and only if e_1 and e_2 are contained in an induced 4-cycle of F.

We now extend the definition of compatible pairs to not necessarily adjacent pairs. Let $1 \le i, j, l, s \le r$ be indices such that $E(G_i, G_j) \ne \emptyset$ and $E(G_l, G_s) \ne \emptyset$. Let $D_1 \in \{D_{ij}, \overline{D_{ij}}\}$ and $D_2 \in \{D_{ls}, \overline{D_{ls}}\}$. We say that D_1 and D_2 are compatible if there exists a sequence $D_1 = C_1, C_2, \ldots, C_m = D_2$, such that for all $1 < t \le m$, C_t and C_{t+1} is a compatible adjacent pair. Having computed all compatible adjacent pairs we compute all compatible pairs.

We finish the proof by computing \mathcal{D} and $\overline{\mathcal{D}}$. Without loss of generality assume that $E(G_1, G_2) \neq \emptyset$. Let

$$\overrightarrow{\mathcal{D}} := \bigcup_{\substack{1 \leq i < j \leq r \\ E(G_i, G_j) \neq \emptyset}} \{D : D \in \{D_{ij}, \overline{D_{ij}}\} \text{ and } D \text{ is compatible with } D_{12}\}.$$

Similarly, let

$$\overleftarrow{\mathcal{D}} := \bigcup_{\substack{1 \leq i < j \leq r \\ E(\overrightarrow{G}_i, G_j) \neq \emptyset}} \{D : D \in \{D_{ij}, \overline{D_{ij}}\} \text{ and } D \text{ is compatible with } \overline{D_{12}}\}.$$

Renaming ψ and $\overline{\psi}$

Let $\overrightarrow{\mathcal{D}}$ and $\overleftarrow{\mathcal{D}}$ be as in Lemma 5.11. If for every edge $e \in E(H, F \setminus H)$ there is a tuple $(e, i \to j) \in \overrightarrow{\mathcal{D}}$ that agrees with φ , then we say that $\overrightarrow{\mathcal{D}}$ agrees with φ ; if for every edge $e \in E(H, F \setminus H)$ there is a tuple $(e, i \to j) \in \overleftarrow{\mathcal{D}}$ that agrees with φ , then we say that $\overleftarrow{\mathcal{D}}$ agrees with φ . Note that φ agrees with exactly one of $\overrightarrow{\mathcal{D}}$ and $\overleftarrow{\mathcal{D}}$.

Suppose that H_i is not of class 1 and let $\psi_i' \in \{\psi_i, \overline{\psi_i}\}$. If for every tuple $(AB, i \to j) \in \overrightarrow{\mathcal{D}}$, we have that there is a token at endpoint J(AB)(i) in $\psi'_i(A)$, then we say that ψ'_i agrees with $\overrightarrow{\mathcal{D}}$; if for every tuple $(AB, i \to j) \in \overrightarrow{\mathcal{D}}$, we have that there is no token at endpoint J(AB)(i) in $\psi'_i(A)$, then we say that $\psi'_i(A)$ agrees with $\overleftarrow{\mathcal{D}}$. Note that ψ_i' agrees with exactly one of $\overrightarrow{\mathcal{D}}$ and $\overleftarrow{\mathcal{D}}$. Moreover, ψ_i agrees with $\overrightarrow{\mathcal{D}}$ if and only if $\overline{\psi_i}$ agrees with $\overline{\mathcal{D}}$. For all $1 \leq i \leq r$, such that H_i is not of class 1, we rename ψ_i and $\overline{\psi_i}$ so that ψ_i agrees with $\overrightarrow{\mathcal{D}}$ and $\overline{\psi_i}$ agrees with $\overleftarrow{\mathcal{D}}$. This implies also renaming l_i and $\overline{l_i}$. Note that if H_i and H_j are not of class 1 and $E(H, F \setminus H)_{ij} \neq \emptyset$, then ψ_i is compatible with ψ_j , and $\overline{\psi_i}$ is compatible with $\overline{\psi_j}$. By the definition of ϕ'_i , we now have that

$$\phi_i' := \begin{cases} \varphi_i \circ {\psi_i}^{-1} \text{ if } \varphi \text{ agrees with } \overrightarrow{\mathcal{D}}; \\ \varphi_i \circ \overline{\psi_i}^{-1} \text{ if } \varphi \text{ agrees with } \overleftarrow{\mathcal{D}}. \end{cases}$$

5.3 Constructing a graph isomorphic to G

We construct, in polynomial time two isomorphic graphs \overrightarrow{J} and \overleftarrow{J} . Let

$$V(\overrightarrow{J}) := V(\overleftarrow{J}) := \bigcup_{1 \le i \le r} V(J_i);$$

let $E(\overrightarrow{J})$ and $E(\overleftarrow{J})$ both contain

$$\bigcup_{1 \le i \le r} E(J_i)$$

For every $(e, i \to j) \in \overrightarrow{D}$ we add an additional edge to \overrightarrow{J} and \overleftarrow{J} as follows. Let $AB \in E(H, F \setminus H)_{ij}$ such that AB = e. Let

$$x' := \begin{cases} \text{endpoint}_J(e)(i) & \text{if } H_i \text{ is not of class 1;} \\ \text{the only vertex in } \psi_i(\pi(A)(i)) & \text{if } H_i \text{ is of class 1;} \end{cases}$$

Let

$$y' := \begin{cases} \text{endpoint}_J(e)(j) & \text{if } H_j \text{ is not of class 1;} \\ \text{the only vertex in } \psi_j(\pi(A)(j)) & \text{if } H_j \text{ is of class 1.} \end{cases}$$

We add the edge $x'\overline{y'}$ to \overrightarrow{J} and the edge $\overline{x'}y'$ to \overleftarrow{J} . Note that, if H_i and H_j are not of class 1, then $x'\overline{y'} = \overline{x'}y' = x'y'.$

Let $\phi: V(J) \to V(G)$ be the map defined by

$$\phi(u') = \phi_i(u') = u.$$

where $u \in J_i$. By Lemma 5.11 and the constructions of \overrightarrow{J} and \overleftarrow{J} we have that: if φ agrees with $\overrightarrow{\mathcal{D}}$, then ϕ is an isomorphism from \overrightarrow{J} to G; and if φ agrees with $\overleftarrow{\mathcal{D}}$, then ϕ is an isomorphism from \overleftarrow{J} to G. Let swap: $V(J) \to V(J)$ be the map defined by

$$\operatorname{swap}(x') := \overline{x'},$$

for all $x' \in V(J)$. Note that swap is an isomorphism from \overrightarrow{J} to \overleftarrow{J} ; we have proved Theorem 1.2:

Theorem 1.2. Let G be a connected $(C_4, diamond)$ -free graph. Given only a graph isomorphic to $F_k(G)$, we can compute in polynomial time a graph isomorphic to G.

Add all the edges of \overrightarrow{J} to J, so that throughout the remainder of the paper we assume that $J = \overrightarrow{J}$.

6 F is Uniquely Reconstructible

In this section we show that F is uniquely reconstructible as the k-token graph of G. In the process we often consider unions, intersections, differences and complements of token configurations in G and J.

Boolean Formulas and Boolean Combinations on $V(F_k(G))$

Let \mathcal{F} be a family of subsets of a set S. A Boolean formula on \mathcal{F} is recursively defined as follows.

- 1. For every $X \in \mathcal{F}$, X is a Boolean formula on \mathcal{F} , which we call a *term*;
- 2. if Γ_1 and Γ_2 are Boolean formulas on \mathcal{F} , then so are $(\neg \Gamma_1)$, $(\Gamma_1 \vee \Gamma_2)$ and $(\Gamma_1 \wedge \Gamma_2)$.

For every Boolean formula Γ on \mathcal{F} there is a corresponding Boolean combination of elements in \mathcal{F} ; let $\operatorname{eval}(\Gamma)$ be the subset of S that is obtained from Γ by interpreting: every appearance of \neg as complementation with respect to S; every appearance of \vee as set union; and every appearance of \wedge as set intersection. Let G and H be isomorphic graphs and let $\psi \in \operatorname{Iso}(G, H)$. Let Γ be a Boolean formula on $V(F_k(G))$. Let $\Gamma(\psi)$ be the Boolean formula on $V(F_k(H))$ that is obtained by replacing every term A of Γ with $\psi(A)$. We use the following result extensively throughout the proofs of Theorems 1.3 and 1.4.

Proposition 6.1. Let G and H be isomorphic graphs on at least three vertices. Suppose that $F_k(G)$ is uniquely reconstructible as the k-token graph of G. Let Γ be a Boolean formula on $V(F_k(G))$; let $\psi \in \text{Iso}(F_k(G), F_k(H))$; and let $f(\psi) \in \text{Iso}(G, H)$ as in 3) of Theorem 2.4. Then

$$f(\psi)(\operatorname{eval}(\Gamma)) = \begin{cases} \operatorname{eval}(\Gamma(\psi)) & \text{if } \psi = \iota(f(\psi)), \\ \operatorname{eval}(\Gamma(\mathfrak{c} \circ \psi)) & \text{if } \psi = \mathfrak{c} \circ \iota(f(\psi)). \end{cases}$$

Proof. Since $f(\psi)$ is a bijection, we have that

$$f(\psi)(\operatorname{eval}(\Gamma)) = \operatorname{eval}(\Gamma(\iota(f(\psi))).$$

By 3) of Theorem 2.4, we have that

$$\iota(f(\psi)) = \psi \text{ or } \iota(f(\psi)) = \mathfrak{c} \circ \psi.$$

6.1 Extending our Framework

In what follows we define isomorphisms from subgraphs of J to subgraphs of G. To avoid confusion, let

$$\phi(J,\varphi) := \phi,\tag{4}$$

where ϕ is defined as above. Note that φ agrees with $\overrightarrow{\mathcal{D}}$ if and only if $\mathfrak{c} \circ \varphi$ agrees with $\overleftarrow{\mathcal{D}}$. In what follows, we assume without loss of generality that φ agrees with $\overrightarrow{\mathcal{D}}$. Thus,

$$k = \sum_{i=1}^{r} l_i.$$

Since, if necessary, we can replace φ with $\mathfrak{c} \circ \varphi$, we also assume that $k \leq n/2$.

In the remainder of this section we compute, in polynomial time, an isomorphism ψ from F to $F_k(J)$ so that (J, ψ) is a k-token reconstruction of F. For this purpose we extend the theoretical framework developed in the previous sections. For every $1 \le i \le r$ let

$$\widehat{H}_i := \bigcup_{s=0}^{\min\{k,|J_i|\}} F_s(J_i).$$

For convenience we set $F_0(J_i)$ to be the empty graph on one vertex; it represents that no tokens are placed at the vertices of J_i . Thus, \widehat{H}_i is the disjoint union of all possible token graphs of J_i with at most k tokens. Let \widehat{F} be the subset of vertices $\widehat{A} \in V(\widehat{H}_1 \square \cdots \square \widehat{H}_r)$ that satisfy

$$\sum_{i=1}^{r} \left| \widehat{A}(i) \right| = k;$$

if $\widehat{A}(i)$ is the only vertex in $F_0(J_i)$, then we set $\widehat{A}(i) := \emptyset$. Let $\widehat{u} : \widehat{F} \to V(F_k(J))$ be the map defined by

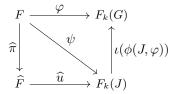
$$\widehat{u}(\widehat{A}) := \bigcup_{i=1}^r \widehat{A}(i),$$

for all $\widehat{A} \in V(\widehat{F})$. In the remainder of this section we prove the following theorem.

Theorem 6.2. Suppose that φ agrees with $\overrightarrow{\mathcal{D}}$ and that $k \leq n/2$. Then we can compute in polynomial time a map $\widehat{\pi}: V(F) \to \widehat{F}$ such that

- a) $\psi := \widehat{u} \circ \widehat{\pi}$ is an isomorphism from F to $F_k(J)$; and
- b) $\varphi = \iota(\phi(J,\varphi)) \circ \psi$.

That is the following diagram commutes.



Theorem 6.2 readily implies Theorem 1.4.

Theorem 1.4. Let G be a connected $(C_4, diamond)$ -free graph. Then $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

Proof. Let $g \in \text{Iso}(F_k(J), F_k(G))$. Then $g \circ \psi \in \text{Iso}(F, F_k(G))$. By b) of Theorem 6.2, with either $\varphi = g \circ \psi$ or $\varphi = \mathfrak{c} \circ g \circ \psi$ we have that

$$g \circ \psi = \iota(\phi(J, \varphi)) \circ \psi \text{ or } \mathfrak{c} \circ g \circ \psi = \iota(\phi(J, \varphi)) \circ \psi.$$

Therefore,

$$g = \iota(\phi(J,\varphi))$$
 or $g = \mathfrak{c} \circ \iota(\phi(J,\varphi))$.

By 3) of Theorem 2.4 we have that $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

Let A be a vertex of F. We say that we can define ψ on A, if we can find in polynomial time a vertex $\widehat{\pi}(A)$ of \widehat{F} such that the following holds. If we set $\psi(A) := \widehat{u} \circ \widehat{\pi}(A)$, then

$$\varphi(A) = \iota(\phi(J,\varphi)) \circ \psi(A)$$

We begin by defining $\widehat{\pi}$ on the vertices of H. For every vertex $A \in H$, let

$$\widehat{\pi}(A) := (\psi_1(\pi(A)(1), \dots, \psi_r(\pi(A)(r))).$$

We have that

$$\iota(\phi(J,\varphi)) \circ \psi(A) = (\iota(\phi(J,\varphi)) \circ \psi) (A)$$

$$= (\iota(\phi(J,\varphi)) \circ \widehat{u} \circ \widehat{\pi}) (A)$$

$$= (\iota(\phi(J,\varphi)) \circ \widehat{u}) (\psi_1(\pi(A)(1), \dots, \psi_r(\pi(A)(r)))$$

$$= \iota(\phi(J,\varphi)) \left(\bigcup_{i=1}^r \psi_i(\pi(A)(i)) \right)$$

$$= \bigcup_{i=1}^r \left\{ \bigcup_{x' \in \psi_i(\pi(A)(i))} \left\{ \phi(x') \right\} \right\}$$

$$= \bigcup_{i=1}^r \left\{ \bigcup_{x' \in \psi_i(\pi(A)(i))} \left\{ \phi_i(x') \right\} \right\}$$

$$= \bigcup_{i=1}^r \iota(\phi_i) (\psi_i(\pi(A)(i)))$$

$$= \bigcup_{i=1}^r (\phi_i' \circ \psi_i) (\pi(A)(i))$$

$$= \bigcup_{i=1}^r (\varphi_i \circ \psi_i^{-1} \circ \psi_i) (\pi(A)(i))$$

$$= \bigcup_{i=1}^r \varphi_i(\pi(A)(i))$$

$$= \varphi(A).$$

Before proceeding we define two subgraphs of F that are useful for defining $\widehat{\pi}$. Let A be a vertex of F.

• Let $s := |\varphi(A) \cap G_i|$. Let $\mathbf{Split}(s, i)$ be the subgraph of F induced by all the vertices $B \in F$ such that

$$|\varphi(B) \cap G_i| = |\varphi(A) \cap G_i|.$$

Thus, $\varphi(\mathbf{Split}(s,i))$ is the subgraph of $F_k(G)$ induced by all the token configurations in which there are s tokens at G_i and k-s tokens at $G \setminus G_i$.

• Let $\mathbf{Fix}(A,i)$ be the subgraph of F induced by all the vertices $B \in \mathbf{Split}(s,i)$ such that

$$\varphi(B) \cap G_i = \varphi(A) \cap G_i$$
.

Thus, $\varphi(\mathbf{Fix}(A,i))$ is the subgraph of $\varphi(\mathbf{Split}(s,i))$ in which the tokens at $\varphi(A) \cap G_i$ remain fixed.

We proved the following proposition with the aid of the SAGE software [19]. For the proof, we iterated over all $(C_4, \text{diamond})$ -free connected graphs and computed the automorphisms groups of them and their respective token graphs. We then used 2) of Theorem 2.4.

Proposition 6.3. If $n \leq 6$, then F is uniquely k-reconstructible as the k-token graph of G.

For the proof of Theorem 6.2, we extend the definition of ψ on certain subgraphs of F. Let F^* be an induced subgraph of F such that the following conditions hold.

- (a) We can determine in polynomial time which vertices of F belong to F^* .
- (b) There exists a connected induced subgraph G^* of G, on at least three and less than |G| vertices, such that $\varphi(F^*)$ is generated by moving $k^* \leq k$ tokens on the vertices of G^* while leaving $k k^*$ tokens fixed at the vertices of a subset T of $V(G \setminus G^*)$.
- (c) We can determine in polynomial time the subgraph J^* of J such that $\phi(J,\varphi)(J^*)=G^*$; and the set $T^*\subset V(J)$ such that

$$T^* = \phi(J, \varphi)^{-1}(T).$$

(d) Let W be the set of vertices of F^* for which we have defined ψ . We can compute in polynomial time a family $\{\Gamma_{u'}\}_{u'\in V(J^*)}$ of Boolean formulas on the set

$$\{B \in V(F_{k^*}(J^*)) : B = \psi(A) \cap V(J^*) \text{ for some } A \in W\},$$

such that

$$\{u'\} = \operatorname{eval}(\Gamma_{u'}),$$

for all $u' \in V(J^*)$.

We call F^* a definable subgraph of F, and $(F^*, \{\Gamma_{u'}\}_{u' \in V(J^*)})$ a definable pair. We now show that when certain conditions are met, we can extend the definition of ψ to every vertex of F^* .

Lemma 6.4. Let $(F^*, \{\Gamma_{u'}\}_{u' \in V(J^*)})$ be a definable pair. Suppose that one of the following conditions holds.

- (1) $k^* \neq |J^*|/2$.
- (2) There exists a vertex u' such that for every $q \in Aut(J^*)$, we have that q(u') = u'.
- (3) There exist an induced subgraph F_1^* of F^* , and an induced connected subgraph G_1^* of G^* , with the following properties.
 - $-V(F_1^*)\subset W$:
 - $-|G_1^*| \ge 3$; and
 - $-\varphi(F_1^*)$ is generated by moving $k_1^* \leq k^*$ tokens on the vertices of G_1^* , while leaving the remaining $k - k_1^*$ tokens fixed.

Then we can define ψ on every vertex of F^* .

Proof. Let φ^* be the map that sends every vertex $X \in F^*$ to

$$\varphi^*(X) := \varphi(X) \cap V(G^*).$$

Note that φ^* is an isomorphism from F^* to $F_{k^*}(G^*)$. We use Theorem 1.2 and induction on Theorem 6.2, with F^* as input, to obtain a graph J', and an isomorphism $\psi': F^* \to F_{k^*}(J')$ such that

$$\varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi' \text{ or } \varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ (\mathfrak{c} \circ \psi').$$

Note that exactly one of $\iota(\phi(J',\varphi^*)) \circ \psi'$ and $\iota(\phi(J',\varphi^*)) \circ (\mathfrak{c} \circ \psi')$ is equal to φ^* . Let q:= $\phi(J', \varphi^*)^{-1} \circ \phi(J, \varphi)(u')$ considered as a map from $V(J^*)$ to V(J'). Note that $q \in \text{Iso}(J^*, J')$. Thus, $\iota(q)$ is an isomorphism from $F_{k^*}(J^*)$ to $F_{k^*}(J')$.

For $A \in W$, let

$$\psi^*(A) = \psi(A) \cap V(J^*).$$

We have that

$$\varphi^*(A) = \iota \left(\phi \left(J, \varphi \right) \right) \circ \psi^*(A).$$

Thus,

$$\psi' \circ \psi^{*-1}(A) = \iota(g)(A) \text{ or } (\mathfrak{c} \circ \psi') \circ \psi^{*-1}(A) = \iota(g)(A).$$

By induction on Theorem 1.4, $F_{k^*}(J^*)$ is uniquely reconstructible as the k^* -token graph of J^* . By Proposition 6.1, for every vertex $u' \in J^*$ we have that

$$g(\lbrace u' \rbrace) = g(\text{eval}(\Gamma_{u'})) = \text{eval}(\Gamma_{u'}(\iota(g))).$$

For every vertex $u' \in J^*$, let

$$g_1(u) := \begin{cases} \text{the only vertex in eval}(\Gamma_{u'}(\psi' \circ \psi^{*-1})) & \text{if } |\Gamma_{u'}(\psi' \circ \psi^{*-1})| = 1. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$$g_1(u) := \begin{cases} \text{the only vertex in } \operatorname{eval}(\Gamma_{u'}(\psi' \circ \psi^{*-1})) & \text{if } |\Gamma_{u'}(\psi' \circ \psi^{*-1})| = 1. \\ \operatorname{undefined} & \text{otherwise.} \end{cases}$$

$$g_2(u) := \begin{cases} \text{the only vertex in } \operatorname{eval}(\Gamma_{u'}((\mathfrak{c} \circ \psi') \circ \psi^{*-1})) & \text{if } |\Gamma_{u'}((\mathfrak{c} \circ \psi') \circ \psi^{*-1})| = 1. \\ \operatorname{undefined} & \text{otherwise.} \end{cases}$$

Note that exactly one of g_1 and g_2 is equal to g. If only one of g_1 and g_2 defines an isomorphism from J^* to J', then we have computed g. Suppose that both g_1 and g_2 are isomorphisms from J^* to J'. We use the conditions of the lemma to determine which of g_1 and g_2 is equal to g.

• Suppose that (1) holds.

Since $k^* \neq |G^*|/2$, for every vertex $A \in F^*$, we have that $|\mathfrak{c} \circ \varphi^*(A)| = |G^*|/2 - k^* \neq k^*$. Therefore,

$$\varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi'.$$

Thus, $g = g_1$ in this case.

• Suppose that (2) holds.

Let A be any vertex in W. Note that $g(u') = g_1(u') = g_2(u')$. Suppose that $u' \in \psi(A)$. We have that $g(u') \in \iota(g)(A)$. If $g(u') \in \psi' \circ \psi^{*-1}(A)$, then $g = g_1$; and if $g(u') \in (\mathfrak{c} \circ \psi') \circ \psi^{*-1}(A)$, then $g = g_2$. Suppose that $u' \notin \psi(A)$. We have that $g(u') \notin \iota(g)(A)$. If $g(u') \in \psi' \circ \psi^{*-1}(A)$, then $g = g_2$; and if $g(u') \in (\mathfrak{c} \circ \psi') \circ \psi^{*-1}(A)$, then $g = g_1$.

• Suppose that (3) holds.

Let J_1^* be the subgraph of J^* such that $\phi(J,\varphi)(J_1^*)=G_1^*$. Since $V(F_1^*)\subset W$, $V(J_1^*)$ is equal to

$$\{u': u' \in \psi(A) \setminus \psi(B), \text{ for some } A, B \in V(F_1^*)\}.$$

Thus, we can compute J_1^* in polynomial time. We compute the subgraph J_1' of J' such that $\psi'(F_1^*)$ is generated by moving some k_1' tokens on the vertices of J_1' while leaving the remaining tokens fixed. We have that $k_1' = k_1^*$ or $k_1' = |J_1^*| - k_1^*$. Suppose that $|J_1^*|$ is odd. Note that $k_1^* \neq |J_1^*| - k_1^*$. If $k_1' = k_1^*$, then

$$\varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi'$$
 and $g = g_1$.

If $k_1' \neq k_1^*$, then

$$\mathfrak{c} \circ \varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi' \text{ and } g = g_2.$$

Suppose that $|J_1^*|$ is even. Let u' be a vertex of J_1^* such that $J_2^* = J_1^* \setminus u'$ is connected. Let F_2^* be the subgraph of F_1^* such that in all vertices of $\psi'(F_2^*)$ there is a token at u'. Let $G_2^* = \varphi(J_2^*)$. Note that F_2^* and G_2^* satisfy condition (d). Thus, we may proceed as in the case of when $|J_1^*|$ is odd.

In what follows suppose that we have computed g. If $g = g_2$, then replace ψ' with $\mathfrak{c} \circ \psi$, so that we have

$$\varphi^* = \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi'.$$

We now define ψ on every vertex of F^* . Let A be a vertex of F^* . Let

$$\psi(A) := (\iota(g^{-1}) \circ \psi'(A)) \cup T^*.$$

We define $\widehat{\pi}(A)$, accordingly, by setting

$$\widehat{\pi}(A)(i) = \psi(A) \cap V(J_i),$$

for $1 \le i \le r$. We have that

$$\begin{split} \varphi(A) &= \varphi^*(A) \cup T \\ &= \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi'(A) \cup T \\ &= \iota(\phi(J, \varphi)) \circ \iota(\phi(J, \varphi))^{-1} \circ \iota \left(\phi \left(J', \varphi^* \right) \right) \circ \psi'(A) \cup T \\ &= \iota(\phi(J, \varphi)) \circ \iota(g^{-1}) \circ \psi'(A) \cup T \\ &= \iota(\phi(J, \varphi)) (\iota(g^{-1}) \circ \psi'(A) \cup T^*) \\ &= \iota(\phi(J, \varphi)) \circ \psi(A). \end{split}$$

The result follows.

Lemma 6.5. Let $1 \le i \le r$ such that $J \setminus J_i$ is connected, and with at least three vertices. Suppose that there exists an integer $0 \le s \le |J_i|$ that satisfies the following.

- 1. There exists a vertex $A \in F$, with $|\widehat{\pi}(A)(i)| = s$, for which we have defined ψ on all the vertices of $\mathbf{Move}(A, i)$.
- 2. Let W be the set of vertices of Fix(A, i) for which we have defined ψ .

- a) There exist an induced subgraph F' of $\mathbf{Fix}(A,i)$, and an induced connected subgraph G' of $G \setminus G_i$, with the following properties.
 - $-V(F')\subset W;$
 - $|G'| \ge 3$; and
 - $-\varphi(F')$ is generated by moving $1 \le k' \le |G'| 1$ tokens on the vertices of G', while leaving the remaining tokens fixed.
- b) For every vertex $u' \in J \setminus J_i$, we can compute in polynomial time a Boolean combination $\Gamma_{u'}$ of elements in $\{\psi(B) : B \in W\}$, such that

$$\{u'\} = V(J \setminus J_i) \cap \operatorname{eval}(\Gamma_{u'}).$$

3. If s = 0, then we have defined ψ on all vertices of $\mathbf{Split}(1,i)$; and if $s = |J_i|$, then we have defined ψ on all vertices of $\mathbf{Split}(|J_i| - 1, i)$.

Then we can define ψ on every vertex of $\mathbf{Split}(s, i)$.

Proof. Suppose that s = 0 or $s = |J_i|$. Let

$$s^* := \begin{cases} 1 & \text{if } s = 0. \\ |J_i| - 1 & \text{if } s = |J_i|. \end{cases}$$

We use condition (3) of Lemma 6.4, with $F^* := \mathbf{Split}(s,i)$, $G^* := G \setminus G_i$, $k^* := k - s$, and

$$T^* := \begin{cases} \emptyset & \text{if } s = 0. \\ V(J_1) & \text{if } s = |J_i|. \end{cases}$$

Note that we have conditions (b) and (c) and (d) for F^* being definable graph of F. Note that every vertex in $\mathbf{Split}(s,i)$ is adjacent only to vertices in $\mathbf{Split}(s,i) \cup \mathbf{Split}(s^*,i)$. Since $\mathbf{Split}(s,i)$ is connected and we have defined ψ on every vertex of $\mathbf{Split}(s^*,i)$, we can determine which vertices of F are in $\mathbf{Split}(s,i)$; and we have condition (a).

Assume that $0 < s < |J_i|$. Since

$$V(\mathbf{Split}(s,i)) = \bigcup_{B \in \mathbf{Move}(A,i)} V(\mathbf{Fix}(B,i)),$$

it is sufficient to show that for every $B \in \mathbf{Move}(A, i)$, we can define ψ on the vertices of $\mathbf{Fix}(B, i)$. Let B be a vertex of $\mathbf{Move}(A, i)$. We use condition (3) Lemma 6.4, with $F^* := \mathbf{Fix}(B, i)$, $G^* := G \setminus G_i$, $k^* := k - s$, $T := \widehat{\pi}(B)(i)$ and $J^* := J \setminus J_i$. Note that we have conditions (b) and (c) for F^* being a definable graph of F. We show the remaining conditions for F being a definable graph, and condition (3) of Lemma 6.4. Let

$$Q := (A =: B_1, B_2, \dots, B_l := B)$$

be a closed walk in $\mathbf{Move}(A, i)$, such that for every $A' \in V(F_s(J_i))$ there exists an B_j such that $\widehat{\pi}(B_j)(i) = A'$.

• condition (a)

Let E_B be the set of edges in $\mathbf{Move}(B, i) = \mathbf{Move}(A, i)$ incident to B. Let Y be the set of vertices C of F with the following two properties.

- 1) No edge of P is in the same ladder as an edge in E_B .
- 2) For every vertex $C' \in P$ there is a closed walk $Q' := (C' = D_1, D_2, \dots, D_l = C')$ such that for every $1 \le j \le l$, the edge $D_j D_{j+1}$ is in the same ladder as $B_j B_{j+1}$.
- 1) and 2) imply that in the path $\varphi(P)$ no token was placed at or moved from G_i . Therefore, Y is precisely $V(\mathbf{Fix}(B,i))$.
- condition (d) and condition (3) of Lemma 6.4

For every $C \in W$, we proceed as follows. Let $(C =: C_1, \ldots, C_l =: C')$ be the walk in F starting at C such that for every $1 \leq j < l$, C_jC_{j+1} is in the same ladder as B_jB_{j+1} . Note that $\varphi(C') \cap G_i = \varphi(B) \cap G_i$, and $\varphi(C') \cap G_j = \varphi(C) \cap G_j$ for all $j \neq i$. Thus, $C' \in \mathbf{Fix}(B, i)$. Let

 $u' \in J \setminus J_i$. Let $\Gamma'_{u'}$ be the Boolean combination that results from replacing every term $\psi(C)$ in $\Gamma_{u'}$ with $\psi(C')$. We have that

$$\{u'\} = V(J \setminus J_i) \cap \operatorname{eval}(\Gamma'_{u'}).$$

Thus, we have condition (d). Let F'_B be the graph induced by the set of vertices $\{C': C \in F'\}$. Since F'_B is a subgraph of $\mathbf{Fix}(B,i)$, we have condition (3) of Lemma 6.4.

Lemma 6.6. Suppose that $J \setminus J_i$ is connected and with at least three vertices. Let A be a vertex of H, and let $s := |\widehat{\pi}(A)(i)|$. Then we can define ψ on every vertex of $\mathbf{Split}(s,i)$.

Proof. We use Lemma 6.5 to define ψ on every vertex of $\mathbf{Split}(s,i)$. Note that we have condition 1 of Lemma 6.5. For every vertex $x' \in J \setminus J_i$, let

$$S_{x'} := \{ B \in V(H) : \widehat{\pi}(B)(i) = \widehat{\pi}(A)(i) \text{ and } x' \in \psi(B) \}.$$

Note that $S_{x'} \subset V(\mathbf{Fix}(A, i))$ and

$$\{x'\} = V(J \setminus J_i) \cap \left(\bigcap_{B \in S_{x'}} \psi(B)\right).$$

Thus, we have condition 2b. If some J_j distinct from J_i , has at least three vertices then we have condition 2a. Suppose that all J_j different from J_i are of class 1. Let J_j and J_l be adjacent vertices of P. Let $A \in V(H)$. Let F^* be the subgraph of F induced by the set of vertices $B \in F$ such that

$$|\varphi(B) \cap (G_j \cup G_l)| = 2$$
 and $\varphi(B) \cap G_t = \varphi(A) \cap G_t$ for all $t \neq i, j$.

Let $AB \in E(H, F \setminus H)_{ij}$ such that $(AB, i \to j) \in \overrightarrow{D}$. Note that $B \in F^*$ and $\varphi(B)$ is obtained from $\varphi(A)$ by moving a token from G_i to G_j . We define $\widehat{\pi}(B)(i) := \emptyset$, $\widehat{\pi}(B)(j) := V(J_j)$ and $\widehat{\pi}(B)(t) := \widehat{\pi}(A)(t)$ for all $t \neq i, j$. Let $AC \in E(H, F \setminus H)_{ij}$ such that $(AC, j \to i) \in \overrightarrow{D}$. Note that $C \in F^*$ and $\varphi(C)$ is obtained from $\varphi(A)$ by moving a token from G_j to G_i . We define $\widehat{\pi}(C)(i) := V(J_i)$, $\widehat{\pi}(C)(j) := \emptyset$ and $\widehat{\pi}(C)(t) := \widehat{\pi}(A)(t)$ for all $t \neq i, j$. We have defined ψ on all the vertices of F^* . Since F^* is a subgraph of $\mathbf{Fix}(A,i)$, we have condition 2a. Thus, we can define ψ on all the vertices of $\mathbf{Split}(s,i)$.

6.2 Proof of Theorem 6.2

We proceed by induction on n. Suppose that Theorem 6.2 holds for smaller values of n. By Proposition 6.3, we may assume that $n \ge 7$. If r = 1, then there is nothing to show since F = H in this case. Assume that $r \ge 2$. We consider two cases: r = 2 and r > 2.

Suppose that r=2.

Without loss of generality assume that $|J_1| \ge |J_2|$. Since we are assuming that $n \ge 7$, J_1 is not a triangle nor an edge. For every $0 \le i \le \min\{|J_2|, k\}$, let F_i be the subgraph of F induced by the set of vertices $A \in V(F)$, such that

$$|\varphi(A) \cap V(G_2)| = i.$$

Let h be such that $H = F_h$. Let $z_1'z_2' \in E(J_1, J_2)$. Let

$$S_1 := \{ X \in V(H) : z_1' \notin \widehat{\pi}(X)(1) \text{ and } z_2' \in \widehat{\pi}(X)(2) \},$$

and

$$S_2 := \{ X \in V(H) : z_1' \in \widehat{\pi}(X)(1) \text{ and } z_2' \notin \widehat{\pi}(X)(2) \}.$$

For every $X \in S_1 \cup S_2$, let X' be the vertex in F such that $\varphi(X')$ is obtained from $\varphi(X)$ by sliding a token along z_1z_2 . By Theorem 5.10 we can find X' in polynomial time. Let $S'_1 := \{X' : X \in S_1\}$ and $S'_2 := \{X' : X \in S_2\}$. For every X' in S'_1 , we define

$$\widehat{\pi}(X')(1) := \widehat{\pi}(X)(1) \cup \{z_1'\} \text{ and } \widehat{\pi}(X')(2) := \widehat{\pi}(X)(1) \setminus \{z_2'\}.$$

For every X' in S'_2 , we define

$$\widehat{\pi}(X')(1) := \widehat{\pi}(X)(1) \setminus \{z_1'\} \text{ and } \widehat{\pi}(X')(2) := \widehat{\pi}(X)(1) \cup \{z_2'\}.$$

Note that $S_1' \subset V(F_{h-1})$ and $S_2' \subset V(F_{h+1})$.

Before proceeding we make the following observations.

If $k - (h - 1) < |J_1|$, we have that

$$\{z_1'\} = \bigcap_{X' \in S_1'} \psi(X'); \tag{5}$$

and for every vertex $w' \in J_1 \setminus \{z_1'\},$

$$\{w'\} = \left(\bigcap_{X' \in S'_1, w' \in \widehat{\pi}(X')(1)} \psi(X')\right) \setminus \{z'_1\}. \tag{6}$$

Since $k \le n/2$ and $|J_1| \ge |J_2|$, we have that if $k - (h-1) = |J_1|$, then h-1 = 0. In this case, there is only one vertex A in F_{h-1} , and we have defined $\psi(A)$.

If $k - (h + 1) \ge 1$, we have that

$$\{z_1'\} = V(J_1) \setminus \bigcup_{X' \in S_2'} \psi(X'); \tag{7}$$

and for every vertex $w' \in J_1 \setminus \{z_1'\},$

$$\{w'\} = \bigcap_{X' \in S_2', w' \in \widehat{\pi}(X')(1)} \psi(X'). \tag{8}$$

If $h + 1 < |J_2|$, we have that

$$\{z_2'\} = \bigcap_{X' \in S_n'} \psi(X'); \tag{9}$$

and for every vertex $w' \in J_2 \setminus \{z_2'\},\$

$$\{w'\} = \left(\bigcap_{X' \in S'_2, w' \in \widehat{\pi}(X')(1)} \psi(X')\right) \setminus \{z'_2\}. \tag{10}$$

If $h-1 \geq 1$, we have that

$$\{z_2'\} = V(J_2) \setminus \bigcup_{X' \in S_1'} \psi(X');$$
 (11)

and for every vertex $w' \in J_2 \setminus \{z_2'\},$

$$\{w'\} = \bigcap_{X' \in S'_1, w' \in \widehat{\pi}(X')(2)} \psi(X'). \tag{12}$$

Suppose that k=2. Thus, $H=F_1$ in this case. We define ψ on all the vertices of F_2 . If J_2 is an edge, then F_2 consists of a single vertex, for which we have already defined ψ . Suppose that $|J_2| \geq 3$. We use Lemma 6.4 with $F^*=F_2$, $J^*=J_2$ and $T^*=\emptyset$ and $k^*=2$. Note that we have conditions (b) and (c) for F_2 being a definable subgraph of F. Since there a no edges from F_0 to F_2 we also have condition (a). From (9) and (10) we get condition (d). If $|J_2| \neq 4$ then we have condition (1) of Lemma 6.4. Suppose that $|J_2|=4$. If a component of $J_2\setminus\{z_2'\}$ has three vertices, then we have condition (3) of Lemma 6.4. Suppose that no component of $J_2\setminus\{z_2'\}$ has three vertices. Suppose J_2 is not a path. Thus, every automorphism of J_2 leaves z_2' fixed; and we have condition (2) of Lemma 6.4. Suppose that J_2 is a path (x_1', x_2', x_3', x_4') , such that z_2' is equal to x_2' or x_3' . Without loss of generality assume that $z_2'=x_2'$. Note that we have defined $\psi(A)$ on all vertices $A \in F_2$ such that $z_2 \in \varphi(A)$. Let $A \in V(F_2)$ be such that $\psi(A) = \{x_1', x_2'\}$. Let B be the only neighbor of A in F_2 . Note that $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along x_2x_3 . We define $\widehat{\pi}(B)(1) = \emptyset$ and $\widehat{\pi}(B)(2) = \{x_1', x_3'\}$. Let C be the only neighbor

of B in F_2 for which we have not defined ψ ; that is, $\varphi(C)$ does not contain x_2 . $\varphi(C)$ is obtained from $\varphi(B)$ by sliding along x_3x_4 . We define $\widehat{\pi}(C)(1) = \emptyset$ and $\widehat{\pi}(C)(2) = \{x_1', x_4'\}$. Let D be the only vertex of F_2 for which we have not defined ψ . We define $\widehat{\pi}(D)(1) = \emptyset$ and $\widehat{\pi}(D)(2) = \{x_3', x_4'\}$. Thus, we can define ψ on every vertex of F_2 . To define ψ on every vertex of F_0 we proceed in a similar way.

Assume that k > 2. Our assumption that r = 2 and k > 2 implies that J cannot contain 3 disjoint edges. Therefore, J_1 and J_2 are either edges, triangles or stars. Thus, J_1 is a star on at least four vertices. Let u' be the center of J_1 . We proceed by cases on whether J_2 is an edge, a triangle or a star of at least three vertices.

• J_2 is an edge.

Thus, h=1 in this case. We use Lemma 6.4 with $F^*=F_0$, $J^*=J_1$ and $T^*=\emptyset$. Note that we have conditions (a), (b) and (c) for F_0 being a definable subgraph of F. From (5) and (6) we get condition (d). Since every automorphism of J_1 leaves u' fixed we have condition (2) of Lemma 6.4. Thus, we can define ψ on every vertex of F_0 . We use Lemma 6.4 with $F^*=F_2$, $J^*=J_1$ and $T^*=V(J_2)$. Note that we have conditions (a), (b) and (c) for F_0 being a definable subgraph of F. From (9) and (10) we get condition (d). Since every automorphism of J_1 leaves u' fixed we have condition (2) of Lemma 6.4. Thus, we can define ψ on every vertex of F_2 . Since $V(F)=V(F_0)\cup V(F_1)\cup V(F_2)$, we are done in this case.

• J_2 is triangle.

A leaf of J_1 cannot be adjacent to a vertex of J_2 ; otherwise J contains three disjoint edges. Thus every $J_1 - J_2$ edge contains u' as an endpoint, in particular $u' = z'_1$. Note that H is equal to F_1 or F_2 . Suppose that H is equal to F_1 .

We use Lemma 6.4 with $F^* = F_0$, $J^* = J_1$ and $T^* = \emptyset$. Note that we have conditions (b) and (c) for F_0 being a definable subgraph of F. Since all vertices in F_0 are adjacent to vertices in either to F_0 or F_1 , we also have condition (a). From (5) and (6) we get condition (d). Since every automorphism of J_1 leaves u' fixed we have condition (2) of Lemma 6.4. Thus, we can define ψ on every vertex of F_0 .

Let w' and x' be the vertices of J_2 distinct from z_2' . Let $F_{z_2'}$, $F_{w'}$, $F_{x'}$ be subgraphs of F_2 induced by the vertices $A \in F_2$ such that $\varphi(A)$ does not contain z_2' , w' and x', respectively. We use Lemma 6.4 with $F^* = F_{x'}$, $J^* = J_1$ and $T^* = \{z_2', w'\}$. Note that we have conditions (b) and (c) for $F_{x'}$ being a definable subgraph of F. From (7) and (8) we get condition (d). Let $A \in V(F_{x'}) \cap S_2'$. Note that a neighbor $B \notin F_{w'} \cup F_1$ of A is in $F_{x'} \cup F_{z_2'}$. Let $C \in V(F_{w'})$ be such that $\widehat{\pi}(C)(1) = \widehat{\pi}(A)(1)$. We have that B is a common neighbor of A and C if and only if $B \in F_{z_2'}$. In this way we can determine the remaining vertices of $F_{x'}$ and we have condition (a). By condition (2) of Lemma 6.4 we can define ψ on every vertex of $F_{x'}$. By analogous arguments we can define ψ on every vertex of $F_{w'}$. Let $A_1 \in V(F_{w'})$ and $A_2 \in V(F_{x'})$ such that $\widehat{\pi}(A_1)(1) = \widehat{\pi}(A_2)(1)$. Note that all the neighbors of A_1 and A_2 are contained in $F_1 \cup F_2$. Let B be a common neighbor of A_1 and A_2 in F_2 . We define $\widehat{\pi}(B)(1) = \widehat{\pi}(A_1)(1)$ and $\widehat{\pi}(B)(2) = \{w', x'\}$. For every such pair (A_1, A_2) we define $\widehat{\pi}(B)$ accordingly. As these accounts for all the vertices of $F_{z_2'}$, we have defined ψ on very vertex of $F_{z_2'}$. Thus, we have defined ψ on every vertex of F_2 .

Let S be the set of vertices $A \in F_{z'_2}$ such that $u' \in \widehat{\pi}(A)(1)$. For every vertex $A \in S$, let A' be the vertex such that $\varphi(A')$ results from $\varphi(A)$ from sliding a token along uv. Note that A' is the only neighbor of A not in F_2 . We define $\widehat{\pi}(A')(1) := \widehat{\pi}(A)(1) \setminus \{u'\}$ and $\widehat{\pi}(A')(2) := V(J_3)$. If k = 3, then F_3 consists of a single vertex and we have defined ψ on all vertices of F_3 . Suppose that k > 3. We use Lemma 6.4 with $F^* = F_3$, $J^* = J_1$ and $T^* = V(J_2)$. Note that we have conditions (b) and (c) for F_3 being a definable subgraph of F. Since all the edges in F_3 go either to F_2 or to F_3 , we also have condition (a). From (7) and (8) we get condition (d). By condition (2) of Lemma 6.4 we can define ψ on every vertex of F_3 . Thus, we have defined ψ on every vertex of F. If $H = F_2$, then we proceed with similar arguments as above. In this case, we first define ψ on every vertex of F_3 ; afterwards we define ψ on every vertex of $F_0 \cup F_1$.

• J_2 is a star on at least three vertices.

Let v' be the center of J_2 . A leaf of J_1 cannot be adjacent to a leaf of J_2 ; otherwise, J would contain three disjoint edges. Therefore, all $J_1 - J_2$ edges contain u' as an endpoint or all $J_1 - J_2$ edges contain v' as endpoint. Without loss of generality assume that all $J_1 - J_2$ edges contain u' as an endpoint. Thus, $u' = z'_1$. No two leaves x' and y' of J_2 are adjacent simultaneously to u'.

Otherwise, x', v', u', y' are the vertices of a 4-cycle in J; which implies that x' and y' are adjacent, which contradicts the assumption that J_2 is a star.

Suppose that we have defined ψ on the vertices of F_i , for some $i \leq k-1$. We show that we can define ψ on the vertices of F_{i-1} ; the proof that we can define ψ on the vertices of F_{i+1} for $i \geq 1$, follows by similar arguments.

Let

$$S := \{ X \in V(F_i) : u' \notin \widehat{\pi}(X)(1) \text{ and } |\widehat{\pi}(X)(2) \cap N(u')| = 1 \}.$$

Let $X \in S$. Let y' be the only vertex in $\widehat{\pi}(X)(2) \cap N(u')$. Let X' be the vertex of F such that $\varphi(X')$ is obtained from $\varphi(X)$ by sliding a token along y'u'. Thus, $X' \in F_{i-1}$. Note that X' is the only neighbor of X not in F_i . We define $\widehat{\pi}(X')(1) := \widehat{\pi}(X)(1) \cup \{u'\}$ and $\widehat{\pi}(X')(2) := \widehat{\pi}(X)(2) \setminus \{y'\}$. Let

$$S' := \{X' : X \in S\}.$$

Let $A \in S$. We use Lemma 6.4 with $F^* = \mathbf{Move}(A', 1)$, $J^* = J_1$ and $T^* = \widehat{\pi}(A')(2)$. We have conditions (b) and (c) for $\mathbf{Move}(A', 1)$ being a definable subgraph of F. Note that

$$\{u'\} = \bigcap_{X' \in S' \cap \mathbf{Move}(A',1)} \psi(X')$$

and

$$\{x'\} = \left(\bigcap_{\substack{X' \in S' \cap \mathbf{Move}(A',1)' \text{ and} \\ x' \in \widehat{\pi}(X')(1)}} \psi(X')\right) \setminus \{u'\}.$$

Thus, we have condition (d). Let B be a neighbor of A', and C' a neighbor of B such that

- (i) $B \notin F_i$;
- (ii) $C \in S \setminus \{A\}$; and
- (iii) $\widehat{\pi}(C')(2) = \widehat{\pi}(A')(2)$.

Property (i) implies that $B \in F_{i-1}$. Properties (ii) and (iii) imply that $\varphi(B)$ is obtained from $\varphi(A')$ by sliding a token along an edge of G_1 . Thus $B' \in \mathbf{Move}(A', 1)$. By considering all such B and C' we can determine which vertices of F belong to $\mathbf{Move}(A', 1)$; and we have condition (a). By condition (2) of Lemma 6.4 we can define ψ on every vertex of $\mathbf{Move}(A', 1)$.

If i-1=0, then $\mathbf{Move}(A',1)=F_{i-1}$, and we have defined ψ on every vertex of F_{i-1} . Suppose that i>1. Let now B be a vertex of $\mathbf{Move}(A',1)$. We show that we can define ψ on every vertex of $\mathbf{Move}(B,2)$. We use Lemma 6.4 with $F^*=\mathbf{Move}(B,2)$, $J^*=J_2$ and $T^*=\widehat{\pi}(B)(1)$. We have conditions (b) and (c) for $\mathbf{Move}(B,2)$ being a definable subgraph of F.

For every $A'_1 \in S'$, consider every pair of vertices $B_1 \in \mathbf{Move}(A'_1, 1)$ and $C \notin \mathbf{Move}(A', 1) \cup F_i$, such that $\widehat{\pi}(B_1)(1) = \widehat{\pi}(B)(1)$ and C is a neighbor of B_1 . Note that $C \in \mathbf{Move}(B, 2)$. If $|E(J_1, J_2)| = 1$ then we have determined all the vertices of F that belong to $\mathbf{Move}(B, 2)$. Suppose that $|E(J_1, J_2)| = 2$. Let A'_1 , B_1 and C be chosen as above. Let C_1 be a neighbor of C, and let $B_2 \in \mathbf{Move}(A'_1, 1)$ be a neighbor of B_1 , such that B_1, B_2, C and C_1 are in a common induced 4-cycle of F. This implies that $\varphi(C_1)$ is obtained from $\varphi(C)$ by moving a token along an edge of G_2 . In this case we have that $C_2 \in \mathbf{Move}(B, 2)$. By considering all such A'_1, B_1, C, B_2 and C_1 , we determine all the vertices of F that belong to $\mathbf{Move}(B, 2)$. Thus, we have property (a).

We now show condition (d). For every $x' \in J_2$ not adjacent to u', let $S_{x'}$ be the set of all vertices $X \in \mathbf{Move}(B, 2)$, for which we have defined ψ , and such that $x' \in \widehat{\pi}(X)(2)$. Note that

$$\{x'\} = V(J_2) \cap \left(\bigcap_{Y \in S_{x'}} \psi(Y)\right).$$

If $|E(J_1, J_2)| = 1$, then

$$\{v'\} = V(J_2) \setminus \bigcup_{x' \in V(J_2) \setminus \{v'\}} \{x'\};$$

and we have condition (d) in this case.

Suppose that $|E(J_1, J_2)| = 2$, and let w' the neighbor of u' in J_2 distinct from v'. Let $C \in \mathbf{Move}(B, 2)$ such that

$$v, w \notin \varphi(C)$$
.

Note that there exists $A_1 \in S$, such that $C \in \mathbf{Move}(A_1, 1)$. Therefore, we have defined $\psi(C)$. If $u' \notin \psi(C)$, let C' be a neighbor of C in $\mathbf{Move}(A_1, 1)$ such that $u' \in \psi(C)$; otherwise let C' := C. Let $D \notin F_i$ be a neighbor of C, and D' a neighbor of C', such that CD and C'D' are in the same ladder. Note that $D, D' \in F_{i-1}$ and $\varphi(C) \triangle \varphi(D) = \varphi(C') \triangle \varphi(D')$. Let X be neighbor of D' in F_i . Let y' be the only vertex of $\widehat{\pi}(C)(2)$ not in $\widehat{\pi}(X)(2)$. Note that $\varphi(D)$ is obtained from $\varphi(C)$ by sliding a token along y'v'. We define $\widehat{\pi}(D)(1) := \widehat{\pi}(B)(1)$ and $\widehat{\pi}(D)(2) := \widehat{\pi}(C)(2) \setminus \{y'\} \cup \{v'\}$. Let S_1 be the set of all such D for every choice of C as above. We have that

$$\{v'\} = V(J_2) \cap \left(\bigcap_{D \in S_1} \psi(D)\right),$$

and

$$\{w'\} = V(J_2) \setminus \bigcup_{D \in S_1} \psi(D).$$

We have condition (d) in this case. By condition (2) of Lemma 6.4 we can define ψ on every vertex of $\mathbf{Move}(B,2)$. Since these accounts for all the vertices in F_{i-1} we have defined ψ on every vertex of F_{i-1} . Proceeding iteratively in this way we can define ψ on all vertices of F.

This completes the proof for when r=2.

Suppose that r > 2.

Let P be the graph whose vertex set is $\{J_1, \ldots, J_r\}$ and where J_i is adjacent to J_j if and only if $E(J_i, J_j) \neq \emptyset$. Since F is connected, P is connected. Therefore, there exists at least two vertices of P, say J_1 and J_2 , such that $P \setminus (J_1 \cup J_2)$ is connected. Without loss of generality assume that $|J_1| \leq |J_2|$. By Lemma 6.6 we can define ψ on all the vertices of $\mathbf{Split}(k_1, 1)$ and $\mathbf{Split}(k_2, 2)$

Suppose that $n - |J_1| - |J_2| \ge 3$.

for all
$$0 \le s \le \min\{k - 2, |J_1|\}$$
, we can define ψ on all the vertices of $\mathbf{Split}(s, 1)$. (*)

We claim that:

- a) if for some $1 \le s \le k-2$ we have defined ψ on every vertex of $\mathbf{Split}(s,1)$, then we can define ψ on every vertex of $\mathbf{Split}(s-1,1)$; and
- b) if for some $1 \le s \le \min\{k-3, |J_1|-1\}$ we have defined ψ on every vertex of $\mathbf{Split}(s,1)$, then we can define ψ on every vertex of $\mathbf{Split}(s+1,1)$.

In both a) and b), we have that

$$\begin{split} k-s &\leq k-1 \\ &\leq \frac{|J|}{2}-1 \\ &= \frac{|J|-|J_1|-|J_2|}{2} + \frac{|J_1|+|J_2|}{2} - 1 \\ &\leq \frac{|J|-|J_1|-|J_2|}{2} + |J_2|-1 \\ &= (|J|-|J_1|-|J_2|) - \frac{n-|J_1|-|J_2|}{2} + |J_2|-1 \\ &\leq |J|-|J_1|-\frac{5}{2}. \end{split}$$

Since k - s is an integer we have that

$$k - s \le |J| - |J_1| - 3$$

We first prove a). Since $2 \le k - s \le |J| - |J_1| - 3$ there exist $1 \le t' \le n - |J_1| - |J_2| - 2$ and $1 \le t \le |J_2| - 1$, such that t' + t = k - s. We use Lemma 6.5 to define ψ on every vertex of **Split**(t,2). Let $A \in \mathbf{Split}(s,1)$ such that $|\widehat{\pi}(A)(2)| = t$. Since $A \in \mathbf{Split}(s,1)$, we have defined ψ on every vertex of $\mathbf{Move}(A,2)$; thus, we have condition 1 of Lemma 6.5. Since $n - |J_1| - |J_2| \ge 3$ and $1 \le t' \le n - |J_1| - |J_2| - 2$, we have condition 2a. For every vertex $x' \in J \setminus \{J_1\}$, let

$$S_{x'} := \{ B \in \mathbf{Split}(s,1) \cap \mathbf{Split}(t,2) : \widehat{\pi}(B)(1) = \widehat{\pi}(A)(1) \text{ and } x' \in \psi(B) \}.$$

Note that

$$\{x'\} = V(J \setminus J_1) \cap \left(\bigcap_{B \in S_{x'}} \psi(B)\right).$$

Thus, we have condition 2b and we can define ψ on all the vertices of $\mathbf{Split}(t,2)$.

We now use Lemma 6.5 to define ψ on every vertex of $\mathbf{Split}(s-1,1)$. Let $A \in \mathbf{Split}(t,2)$ such that $|\widehat{\pi}(A)(1)| = s-1$. We have defined ψ on every vertex of $\mathbf{Move}(A,1)$; thus, we have condition 1 of Lemma 6.5. Let t'' := k - (s-1) - t = t' + 1. Since $n - |J_1| - |J_2| \ge 3$ and $2 \le t'' \le n - |J_1| - |J_2| - 1$, we have condition 2a. For every vertex $x' \in J \setminus \{J_2\}$, let

$$S_{x'} := \{ B \in \mathbf{Split}(s-1,1) \cap \mathbf{Split}(t,2) : \widehat{\pi}(B)(2) = \widehat{\pi}(A)(2) \text{ and } x' \in \psi(B) \}.$$

Note that

$$\{x'\} = V(J \setminus J_2) \cap \left(\bigcap_{B \in S_{x'}} \psi(B)\right);$$

and we have condition 2b. If s-1=0 then we also have condition 3). Thus, we can define ψ on all the vertices of $\mathbf{Split}(s-1,1)$. This proves a).

We now prove b). Since $3 \le k - s \le |J| - |J_1| - 3$, there exist $2 \le t' \le n - |J_1| - |J_2| - 2$ and $1 \le t \le |J_2| - 1$, such that t' + t = k - s. We use Lemma 6.5 to define ψ on every vertex of **Split**(t,2). Let $A \in \mathbf{Split}(s,1)$ such that $|\widehat{\pi}(A)(2)| = t$. Since $A \in \mathbf{Split}(s,1)$, we have defined ψ on every vertex of $\mathbf{Move}(A,2)$; thus, we have condition 1 of Lemma 6.5. Since $n - |J_1| - |J_2| \ge 3$ and $2 \le t' \le n - |J_1| - |J_2| - 2$, we have condition 2a. For every vertex $x' \in J \setminus \{J_1\}$, let

$$S_{x'} := \{ B \in \mathbf{Split}(s,1) \cap \mathbf{Split}(t,2) : \widehat{\pi}(B)(1) = \widehat{\pi}(A)(1) \text{ and } x' \in \psi(B) \}.$$

Note that

$$\{x'\} = V(J \setminus J_1) \cap \left(\bigcap_{B \in S_{x'}} \psi(B)\right).$$

Thus, we have condition 2b and we can define ψ on all the vertices of **Split**(t,2).

We now use Lemma 6.5 to define ψ on every vertex of $\mathbf{Split}(s+1,1)$. Let $A \in \mathbf{Split}(t,2)$ such that $|\widehat{\pi}(A)(1)| = s+1$. We have defined ψ on every vertex of $\mathbf{Move}(A,1)$; thus, we have condition 1 of Lemma 6.5. Let t'' := k - (s+1) - t = t' - 1. Since $n - |J_1| - |J_2| \ge 3$ and $1 \le t'' \le n - |J_1| - |J_2| - 3$, we have condition 2a. For every vertex $x' \in J \setminus \{J_2\}$, let

$$S_{x'} := \{ B \in \mathbf{Split}(s-1,1) \cap \mathbf{Split}(t,2) : \widehat{\pi}(B)(2) = \widehat{\pi}(A)(2) \text{ and } x' \in \psi(B) \}.$$

Note that

$$\{x'\} = V(J \setminus J_2) \cap \left(\bigcap_{B \in S_{x'}} \psi(B)\right);$$

and we have condition 2b. If $s + 1 = |J_1|$ then we also have condition 3). Thus, we can define ψ on all the vertices of **Split**(s + 1, 1). This proves b). Statement (*) now follows from successive applications of a) and b).

Suppose that $k-2 \ge |J_1|$. We have that

$$F = \bigcup_{s=0}^{|J_1|} \mathbf{Split}(s,1);$$

thus, by (*) we have defined ψ on every vertex of F. Suppose that $k-2<|J_1|$. Since,

$$\mathbf{Split}(2,2) = \bigcup_{s=\max\{0,(k-2)-(|J|-|J_1|-|J_2|)\}}^{k-2} \mathbf{Split}(s,1),$$

we have defined ψ on all the vertices of $\mathbf{Split}(2,2)$. With similar arguments as for the proof of (*) we show that we can define ψ on all the vertices of $\mathbf{Split}(1,2)$. Let $A \in \mathbf{Split}(k-2,1)$ such that $\widehat{\pi}(A)(2) = 1$. We have conditions 1, 2a, 2b and 3 of Lemma 6.5. Thus, we can define ψ on all vertices of $\mathbf{Split}(1,2)$. Let now $A \in \mathbf{Split}(k-2,1)$ such that $\widehat{\pi}(A)(2) = 0$. We have conditions 1, 2a, 2b and 3 of Lemma 6.5. Thus, we can define ψ on all vertices of $\mathbf{Split}(0,2)$. Note that $\mathbf{Split}(k-1,1) \subset \mathbf{Split}(1,2) \cup \mathbf{Split}(0,2)$; thus, we have defined ψ on every vertex of $\mathbf{Split}(k-1,1)$. If $k \leq |J_1|$, then $\mathbf{Split}(k,1) \subset \mathbf{Split}(1,2) \cup \mathbf{Split}(0,2)$; and we have defined ψ on every vertex of $\mathbf{Split}(k,1)$. Since

$$F = \bigcup_{s=0}^{\min\{k,|J_1|\}} \mathbf{Split}(s,1),$$

we have defined ψ on every vertex of F.

Suppose that $n - |J_1| - |J_2| = 2$.

We have that r=3 and that J_3 is an edge. Let $V(J_3)=:\{x',y'\}$. Since we are assuming that $n\geq 7$, we have that $|J_2|\geq 3$. If $E(J_1,J_2)\neq \emptyset$, then $J\setminus J_3$ is connected; in this case we may proceed as above with the roles of J_2 and J_3 interchanged. Assume that $E(J_1,J_2)=\emptyset$.

Suppose that there exists an edge $u'v'_1 \in E(J_1, J_3)$ such that $J_1 \setminus u'$ contains an edge $w'_1w'_2$. Let v'_2 be the neighbor of v'_1 in J_3 , and let $x'_1x'_2$ be an edge of J_2 . Let $A \in V(F)$ such that

$$v_1, w_1, x_1 \in \varphi(A),$$

and

$$u, v_2, w_2, x_2 \notin \varphi(A)$$
.

Note that w_1w_2, x_1x_2 and v_1v_2 is a matching in $G_{\varphi(A)}$. Therefore, we may use A for line 5 of Initialize. Let e_1, e_2 , and e_3 be the edges in F that correspond to w_1w_2, x_1x_2 and v_1v_2 , respectively. Suppose that e_1, e_2 and e_3 are chosen in line 6 of initialize, and that the order in which they are chosen is e_3, e_1, e_2 . Let J' be the graph isomorphic to G that is obtained by following the previous construction with these choices. Let J'_1, J'_2 and J'_3 be its subgraphs such that J'_i corresponds to e_i . Let G'_i be the subgraph of G that corresponds to J_i . Note that v_1, v_2 and u are vertices of G'_3 . Since $E(G_1, G_2) = \emptyset$, we have that G'_1 is a subgraph of G_1 and that G'_2 is a subgraph of G_2 . Therefore, $J' \setminus J'_1$ and $J' \setminus J'_2$ are connected. Since $|J'_3| \geq 3$ we may proceed as in the case when $n - |J_1| - |J_2| \geq 3$. To find such an A we iterate over all vertices of F and over all the orderings of e_1, e_2 and e_3 . We may apply similar arguments with J_2 instead of J_1 . Thus, we may assume that

if J_1 is not an edge, then J_1 is a star whose center u'_1 is the only vertex of J_1 adjacent to a vertex of J_3 ; J_2 is a star whose center u'_2 is the only vertex of J_2 adjacent to a vertex of J_3 . (*)

Suppose that for some

$$0 < s < \min\{|J_1|, k\} \text{ and } t := k - s - 1,$$

we have defined ψ on all the vertices of $\mathbf{Split}(s,1)$ and $\mathbf{Split}(t,2)$. We show that we can define ψ on all the vertex of $\mathbf{Split}(s-1,1)$ and $\mathbf{Split}(s+1,1)$. Let $A \in \mathbf{Split}(t,2)$ such that $\widehat{\pi}(A)(1) = s-1$. Since $\mathbf{Move}(A,1) \subset \mathbf{Split}(t,2)$ we have condition 1 of Lemma 6.5. Since $|J_2| \geq 3$ we have condition 2a. If s-1=0, then we have condition 3. Similarly, let $A \in \mathbf{Split}(t,2)$ such that $\widehat{\pi}(A)(1) = s+1$. Since

 $\mathbf{Move}(A,1) \subset \mathbf{Split}(t,2)$ we have condition 1 of Lemma 6.5. Since $|J_2| \geq 3$ we have condition 2a. If $s+1=|J_1|$, then we have condition 3.

Let $v' \in V(J_2)$. Let

$$S_{v'} := \{ B \in V(\mathbf{Split}(t, 2)) : v' \in \widehat{\pi}(B)(2) \}.$$

Note that

$$\{v'\} := V(J_2) \cap \left(\bigcap_{B \in S_{v'}} \psi(B)\right).$$

• Suppose that J_1 is an edge.

Thus, s = 1.

- Suppose that there exists a vertex $u' \in J_1$ with only one neighbor in J_3 .

Without loss of generality assume that the neighbor of u' in J_3 is x'. Since G is a $(C_4, \text{diamond})$ -free graph and J_3 is an edge, this implies that x' is the only vertex of J_3 adjacent to a vertex of J_1 . Let $B \in V(\mathbf{Split}(1,1))$ such that $u' \in \widehat{\pi}(B)(1)$, and $\widehat{\pi}(B)(3) = \emptyset$. Let B' be the only neighbor of B not in $\mathbf{Split}(1,1)$. We have that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along ux. We define $\widehat{\pi}(B')(1) := \emptyset$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{x'\}$. We have that

$$\{x'\} := \psi(B') \setminus V(J_2),$$

and

$$\{y'\} := V(J \setminus J_1) \setminus (\{x'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5. Since we also have condition 3. we can define ψ on every vertex of $\mathbf{Split}(s-1,1)$.

Let now $B \in V(\mathbf{Split}(1,1))$ such that $u' \notin \widehat{\pi}(B)(1)$, and $\widehat{\pi}(B)(3) = \{x',y'\}$. Let B' be the only neighbor of B not in $\mathbf{Split}(1,1)$. We have that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along ux. We define $\widehat{\pi}(B')(1) := V(J_1)$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{y'\}$. We have that

$$\{y'\} := (\psi(B') \setminus V(J_2)) \cap V(J \setminus J_1),$$

and

$$\{x'\} := V(J \setminus J_1) \setminus (\{y'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of **Split**(s+1,1).

- Suppose that every vertex in J_1 has two neighbors in J_3 .

This implies that there is only one edge between J_3 and J_2 . Otherwise, since G is $(C_4, \text{diamond})$ -free a vertex of J_1 would be adjacent to a vertex of J_2 ; without loss of generality assume that $x'u_2'$ is the only edge between J_3 and J_2 . Let $B \in V(\mathbf{Split}(t,2)) \cap V(\mathbf{Split}(0,1))$ such that $u_2' \notin \widehat{\pi}(B)(2)$. Let B' be the only neighbor of B not in $\mathbf{Split}(t,2)$. Note that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $x'u_2'$. We define $\widehat{\pi}(B')(1) := \emptyset$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2) \cup \{u_2'\}$ and $\widehat{\pi}(B')(3) = \{y'\}$. We have that

$$\{y'\} := \psi(B') \setminus V(J_2),$$

and

$$\{x'\} := V(J \setminus J_1) \setminus (\{y'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of $\mathbf{Split}(s-1,1)$.

Let now $B \in V(\mathbf{Split}(t,2)) \cap V(\mathbf{Split}(2,1))$ such that $u_2' \in \widehat{\pi}(B)(2)$. Let B' be the only neighbor of B not in $\mathbf{Split}(t,2)$. Note that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $u_2'x'$. We define $\widehat{\pi}(B')(1) := V(J_1)$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2) \setminus \{u_2'\}$ and $\widehat{\pi}(B')(3) = \{x'\}$. We have that

$$\{x'\} := V(J \setminus J_1) \cap (\psi(B') \setminus V(J_2)).$$

and

$$\{y'\} := V(J \setminus J_1) \setminus (\{x'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of **Split**(s+1,1).

• Suppose that J_1 is a star on at least three vertices.

If u_1' and u_2' have both two neighbors in J_3 , then u_1' and u_2' are adjacent. This contradicts our assumption $E(J_1, J_2) = \emptyset$. Assume without loss of generality that the only neighbor of u_2' in J_3 is x'. The proof for when u_1' has only one neighbor in J_3 follows by similar arguments. Let $B \in \mathbf{Split}(s, 1)$ such that $u_1' \in \widehat{\pi}(B)(1)$, $u_2' \in \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B)(3) = \emptyset$.

Suppose that u_1' has only one neighbor z' in J_3 . Let B' be the only neighbor of B not in $\mathbf{Split}(s,1)$. Note that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $u_1'z'$. Let w' be the vertex of J_3 distinct from z'. We define $\widehat{\pi}(B')(1) := \widehat{\pi}(B) \setminus \{u_1'\}, \ \widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{z'\}$. We have that

$$\{z'\} := V(J \setminus J_1) \cap (\psi(B') \setminus V(J_2)).$$

and

$$\{w'\} := V(J \setminus J_1) \setminus (\{z'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of $\mathbf{Split}(s-1,1)$. Let now $B \in \mathbf{Split}(s,1)$ such that $u_1' \notin \widehat{\pi}(B)(1)$, $u_2' \notin \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B)(3) = V(J_3)$. Let B' be the only neighbor of B not in $\mathbf{Split}(s,1)$. Note that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $u_1'z'$. We define $\widehat{\pi}(B')(1) := \widehat{\pi}(B) \cup \{u_1'\}$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{w'\}$. We have that

$$\{w'\} := V(J \setminus J_1) \cap (\psi(B') \setminus V(J_2)).$$

and

$$\{z'\} := V(J \setminus J_1) \setminus (\{y'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of **Split**(s+1,1).

Suppose that u_1' has two neighbors in J_3 . Let $C \in \mathbf{Split}(1,1)$ such that $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along $u_2'x'$. Let $B' \notin \mathbf{Split}(S,1)$ be the neighbor of B such that BB' is not in the same ladder as BC. Note that $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $u_1'x'$. We define $\widehat{\pi}(B')(1) := \widehat{\pi}(B) \setminus \{u_1'\}, \widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{x'\}$. We have that

$$\{x'\} := V(J \setminus J_1) \cap (\psi(B') \setminus V(J_2)).$$

and

$$\{y'\} := V(J \setminus J_1) \setminus (\{x'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of $\mathbf{Split}(s-1,1)$. Let now $B \in \mathbf{Split}(s,1)$ such that $u_1' \notin \widehat{\pi}(B)(1)$, $u_2' \notin \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B)(3) = V(J_3)$. Let $C \in \mathbf{Split}(1,1)$ such that $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along $x'u_2'$. Let $B' \notin \mathbf{Split}(1,1)$ be the neighbor of B such that BB' is not in the same ladder as BC. Note that in both cases $\varphi(B')$ is obtained from $\varphi(B)$ by sliding a token along $u_1'x'$. We define $\widehat{\pi}(B')(1) := \widehat{\pi}(B) \cup \{u_1'\}$, $\widehat{\pi}(B')(2) := \widehat{\pi}(B)(2)$ and $\widehat{\pi}(B')(3) = \{y'\}$. We have that

$$\{y'\} := V(J \setminus J_1) \cap (\psi(B') \setminus V(J_2)).$$

and

$$\{x'\} := V(J \setminus J_1) \setminus (\{y'\} \cup V(J_2)).$$

Thus, we have condition 2b of Lemma 6.5 and we can define ψ on every vertex of **Split**(s+1,1).

By similar arguments we can define ψ on every vertex of $\mathbf{Split}(t-1,2)$ and $\mathbf{Split}(t+1,2)$. Thus, starting from $s=k_1$ and $t=k_2$ we can define ψ on all vertices of $\mathbf{Split}(s,1)$ for all

$$0 \le s \le \min\{|J_1|, k\}.$$

Since

$$F = \bigcup_{s=0}^{\min\{|J_1|,k\}} \mathbf{Split}(s,1),$$

we can define ψ on all vertices of F.

7 Disconnected Graphs

To finish the paper we consider the case when G is disconnected. The first result in this direction is that there exist non-isomorphic, disconnected, $(C_4, \text{ diamond})$ -free graphs G and H, and integers $k \neq l$ such that $F_k(G) \simeq F_l(H)$. See Figure 3, for an example. This example was found by Trujillo-Negrete in her Master's Thesis [20].

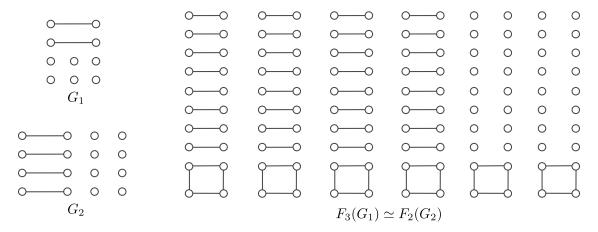


Figure 3: Two non-isomorphic graphs G_1 and G_2 for which $F_3(G_1)$ is isomorphic to $F_2(G_2)$.

On the positive side we have the following.

Theorem 7.1. Let G and H be two $(C_4, diamond)$ -free graphs. If $F_k(G)$ is isomorphic to $F_k(H)$, then G is isomorphic to H.

Proof. We proceed as follows. Suppose that we are given a graph F and an integer k, such that F is isomorphic to the k-token graph of a $(C_4, \text{diamond})$ -free graph. We show that there is a unique G (up to isomorphism) such that $F \simeq F_k(G)$. Since $F_k(G)$ is connected if and only if G is connected [8], we may assume that G is disconnected, as otherwise we are done by Theorem 1.2. Since $F_k(G)$ has $\binom{|G|}{k}$ vertices, we can determine n := |G|. We may assume that $k \leq n/2$. Let G_1, \ldots, G_r be the components of G. Let G be a component of $F_k(G)$. Note that there exist integers k_1, \ldots, k_r , with $0 \leq k_i \leq |G_i|$ and $k = k_1 + \cdots + k_r$, such that G is generated by moving k_i tokens on G_i . Moreover, we have that

$$C \simeq F_{k_1}(G_1) \square \cdots \square F_{k_r}(G_r).$$

Note that since G_i is connected, by Corollary 4.7, we have that if $0 < k_i < |G_i|$, then $F_{k_i}(G_i)$ is a prime graph. Given C, there is a unique Cartesian decomposition (up to the order of the factors) such that

$$C \simeq F_1 \square \cdots \square F_r$$

and every F_i is a non-trivial prime graph [18, 21]. This decomposition can be found in linear time [11]. We compute the Cartesian decompositions of all components of F. Let C^* be the component with the largest number, r^* , of terms; and let $F_1 \square \cdots \square F_{r^*}$ be this decomposition. We proceed by cases depending on the value of r^* .

• $r^* < k$.

Note that G has exactly r^* non trivial components. Let G_1, \ldots, G_{r^*} be these components. By Theorem 1.2 we can reconstruct these components in polynomial time. Finally, the number of isolated vertices of G is given by

$$n - \sum_{i=1}^{r^*} |G_i|.$$

 $\bullet \ r^* = k.$

Suppose that C^* is the only component of F having k factors in its decomposition. This implies that G has exactly k non-trivial components; and we may proceed as in the previous case. Suppose now that there are at least two components of F having k factors in their decomposition. Thus, G has more than k non-trivial components. Let \mathcal{C}_F be the set of components of F with k factors in its decomposition, and let \mathcal{C}_G be the set of non-trivial components of G. Let $q(F) := |\mathcal{C}_F|$ and $q(G) := |\mathcal{C}_G|$. Since $q(F) = \binom{q(G)}{k}$, we can determine the value q(G). Moreover, each $G_i \in \mathcal{C}_G$ is counted in exactly $\binom{q(G)-1}{k-1}$ components of \mathcal{C}_F .

For every $C \in \mathcal{C}_F$, we use Theorem 1.2 to compute a set of graphs H'_1, \ldots, H'_k such that $C \simeq H'_1 \square \cdots \square H'_k$. Let \mathcal{S} be the set of all such graphs. By testing for graph isomorphism we obtain a set of tuples $\mathcal{S}' := \{(G'_1, t_1), \ldots, (G'_s, t_s)\}$, such that: the G'_i are pairwise non-isomorphic; for every $H_i \in \mathcal{S}$ there exists a graph G'_j such that $H_i \simeq G'_j$; and there are exactly t_j graphs in \mathcal{S} isomorphic to G'_i .

Note that each G_i gives way to $\binom{q(G)-1}{k-1}$ graphs in \mathcal{S} . Therefore, for every G_i' there are exactly $t_i/\binom{q(G)-1}{k-1}$ components of \mathcal{C}_G isomorphic to G_i' . Thus we can determine the graphs in \mathcal{C}_G up to isomorphism. Finally, the number of isolated vertices of G is given by

$$n - \sum_{\substack{(G'_i, t_i) \in \mathcal{S}'}} \frac{t_i}{\binom{q(G)-1}{k-1}} |G'|.$$

We point out that in contrast with the connected case we are unable to reconstruct G in polynomial time. The bottleneck of the algorithm implied in the proof of Theorem 7.1 is the Graph Isomorphism Problem.

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