# Cops and Robbers Pebbling in Graphs

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*Dedicated to Oleksandr Stanzhytsky, and others like him who are doing less mathematics than usual at this time.*

**Abstract**

Here we merge the two fields of Cops and Robbers and Graph Pebbling to introduce the new topic of Cops and Robbers Pebbling. Both paradigms can be described by moving tokens (the cops) along the edges of a graph to capture a special token (the robber). In Cops and Robbers, all tokens move freely, whereas, in Graph Pebbling, some of the chasing tokens disappear with movement while the robber is stationary. In Cops and Robbers Pebbling, some of the chasing tokens (cops) disappear with movement, while the robber moves freely. We define the cop pebbling number of a graph to be the minimum number of cops necessary to capture the robber in this context, and present upper and lower bounds and exact values, some involving various domination parameters, for an array of graph classes. We also offer several interesting problems and conjectures.

**Keywords:** cops and robbers, graph pebbling, dominating set

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# 1 Introduction

There are numerous versions of moving tokens in a graph for various purposes. Two popular versions are called *Cops and Robbers* and *Graph Pebbling*. In both cases we have tokens of one type (C) attempting to capture a token of another type (R), and all token movements occur on the edges of a graph. In the former instance, all tokens move freely, whereas in the latter instance, type R tokens are stationary and type C movements come at a cost. In this paper we merge these two subjects to create *Cops and Robbers Pebbling*, wherein type R tokens move freely and type C tokens move at a cost.

We define these three paradigms more specifically in Subsection 1.1 below. The new graph invariant we define to study in this paper is the cop pebbling number of a graph, denoted $\pi^{\mathsf{c}}(G)$; roughly, this equals the minimum number of cops necessary to capture the robber in the cops and robbers pebbling paradigm. In Subsections 1.2–1.4 we present known results about cops and robbers, dominating sets, and optimal pebbling, respectively, that will be used in the sequel. We record in Subsections 2.1–2.3 new theorems on lower bounds, upper bounds, and exact answers for $\pi^{\mathsf{c}}(G)$, respectively, for a range of graph families, including paths, cycles, trees, chordal graphs, high girth graphs, and cop-win graphs, as well as, in some cases, for all graphs. Section 3 contains theorems for Cartesian products of graphs, and we finish in Section 4 with some natural questions left open from this work, including a version of Meyniel's Cops and Robbers Conjecture that may hold in the Cops and Robbers Pebbling world.

## 1.1 Definitions

We use several standard notations in graph theory, including $V(G)$for the set of vertices of a graph $G$ (with $n(G) = |V(G)|$), $E(G)$ for its edge set, $\mathsf{rad}(G)$ for its radius, $\mathsf{diam}(G)$ for its diameter, and $\mathsf{gir}(G)$ for its girth, as well as $\mathsf{deg}(v)$ for the degree of a vertex, and $\mathsf{dist}(u, v)$ for the distance between vertices $u$ and $v$. For a vertex $v$ in a graph $G$, we use the notations $N_d(v) = \{u \mid \mathsf{dist}(u, v) < d\}$ and $N_d[v] = \{u \mid \mathsf{dist}(u, v) \leq d\}$. If $d = 1$ we drop the subscript. We often use $T$ to denote a generic tree, and set $P_n$, $C_n$, and $K_n$ to be the path, cycle, and complete graph on $n$ vertices, respectively. (For convenience, we define $C_2 = P_2$.)

For graphs $G$ and $H$ we define the *Cartesian product* $G \square H$ by having vertices $V(G) \times V(H)$ and edges $(u, v)(w, x)$ with either $uw \in E(G)$ and $v = x$ or $u = w$ and $vx \in E(H)$. The *d-dimensional cube* $Q^d$ is defined by $Q^1 = P_2$ and $Q^d = Q^{d-1} \square Q_1$ for $d > 1$.

### 1.1.1 Graph Pebbling

A *configuration* $C$ of pebbles on a graph $G$ is a function from the vertices of $G$ to the non-negative integers. Its *size* equals $|C| = \sum_{v \in G} C(v)$. For adjacent vertices $u$ and $v$ with $C(u) \geq 2$, a *pebbling step* from $u$ to $v$

removes two pebbles from $u$ and adds one pebble to $v$, while, when $C(u) \geq 1$, a *free step* from $u$ to $v$ removes one pebble from $u$ and adds one pebble to $v$. In the context of moving pebbles, we use the word *move* to mean *move via pebbling steps.*

The *pebbling number* of a graph $G$, denoted $\pi(G)$, is the minimum number $m$ so that, from <u>any</u> configuration of size $m$, one can move a pebble to any specified *target* vertex. The *optimal pebbling number* of a graph $G$, denoted $\pi^*(G)$, is the minimum number $m$ so that, from <u>some</u> configuration of size $m$, one can move a pebble to any specified target vertex.

### 1.1.2 Cops and Robbers

In cops and robbers, the cops are the pebbles, the robber is the target, and the robber is allowed to move. The cops and robbers alternate making moves in *turns.* At each turn, any positive number of cops make one free step, then the robber chooses to make a free step or not. In graph pebbling literature, the activity of moving a pebble to a target is called *solving* or *reaching* the target; here we use the analogous cops and robbers terminology of *capturing* the robber.

The *cop number* $c(G)$ is defined as the minimum number $m$ so that, from <u>some</u> configuration of $m$ cops, it is possible to capture any robber via free steps. If the cops catch the robber on their $t^{\text{th}}$ turn, then we say that the *length* of the game is $t$; if the robber wins then the length is infinite. When $c(G) = m$, the *$m$-capture time* of $G$, denoted $\mathsf{capt}_m(G)$, is defined to be the length of the game on $G$ when both cops and robbers play optimally. That is, it equals the minimum (over all cop strategies) of the maximum (over all robber strategies) of the length of the game on $G$. For graphs of cop number 1, we simply say *capture time* and write $\mathsf{capt}(G)$ without the subscript.

When necessary or convenient to use pronouns, we follow the literature by having female cops and a male robber.

### 1.1.3 Cop Pebbling

The *cop pebbling number* $\pi^{\mathsf{c}}(G)$ is defined as the minimum number $m$ so that, from <u>some</u> configuration of $m$ cops, it is possible to capture any robber via pebbling steps. We call an instance of a graph $G$, configuration $C$, and robber vertex $v$ a *game*, and say that the cops win the game if they can capture the robber; else the robber wins. Note that in standard cops and robbers, the robber must elude capture forever to win the game, since there is no cost to cop movements. However, since we lose a cop with each pebbling step, the cops and robbers pebbling game is finite — the robber wins if not captured within $|C| - 1$ turns.

We may assume that all graphs are simple. Because games on $K_1$ are trivial, we will assume that all graph components have at least two vertices. Additionally, because of the following fact, we will restrict our attention in this paper to connected graphs.

**Fact.** *If $G$ has connected components $G_1, \ldots, G_k$ then $\pi^{\mathsf{c}}(G) = \sum_{i=1}^{k} \pi^{\mathsf{c}}(G_i)$.*

A set $S \subseteq V(G)$ is a *distance-$d$ dominating set* if $\cup_{v \in S} N_d[v] = V(S)$. We denote by $\gamma_d(G)$ the size of the smallest distance-$d$ dominating set.

## 1.2 Cop Results

Here we list the results on cops and robbers that will be used to prove our theorems on cop pebbling. A graph $G$ is *cop-win* if $c(G) = 1$. A vertex $u$ in $G$ is called a *corner* if there is a vertex $v \neq u$ such that $N[u] \subseteq N[v]$. We say that $G$ is *dismantlable* if either $G$ is a single vertex or there is a corner $u$ such that $G - u$ is dismantlable. Note that chordal graphs are dismantlable.

**Result 1.** *[24] A graph is cop-win if and only if it is dismantlable.*

**Result 2.** *[1] For $t \geq 1$ let $G$ be a graph with $\mathsf{gir}(G) \geq 8t - 3$. Then $c(G) > (\delta(G) - 1)^t$.*

The following weaker constant can be achieved for a larger class of graphs.

**Result 3.** *[8] For $t \geq 1$ let $G$ be a graph with $\mathsf{gir}(G) \geq 4t + 1$. Then $c(G) \geq \frac{1}{et}(\delta(G) - 1)^t$.*

**Result 4.** *[5] If $G$ is a chordal graph with radius $r$, then $\mathsf{capt}(G) \leq r$.*

This bound is tight. For example, $P_5$ has both radius 2 and $\mathsf{capt}(G) = 2$.

**Result 5.** *[14] If $G$ is a $d$-regular Cayley graph then $c(G) \leq \lceil \frac{d+1}{2} \rceil$, and $\mathsf{capt}_c(G) \leq |V(G)| \lceil \frac{d+1}{2} \rceil$.*

Result 5 yields the following result as a corollary.

**Result 6.** *[2] For $d \geq 1$, the $d$-dimensional cube $Q^d$ satisfies $c(Q^d) = \lceil \frac{d+1}{2} \rceil$.*

**Result 7.** *[7] With $c = c(Q^d)$ we have $\mathsf{capt}_c(Q^d) = \Theta(d \lg d)$.*

Result 6 for even $d = 2k$ also follows from the following result because $Q^2 = C_4$.

**Result 8.** *[23] If $G = \square_{i=1}^{k} C_{n_i}$ and each $n_i \geq 4$ then $c(G) = k + 1$. If $n_i = m$ for all $i$, then $\mathsf{capt}_2 k(G) \leq k \lfloor \frac{m-1}{2} \rfloor (\lceil \lg k \rceil + 1) + 1 - k$.*

In fact, Result 6 follows as well from the following result because $Q^1 = P_2$ is a tree.

**Result 9.** *[21] If each $T_i$ is a tree on at least two vertices and $G = \square_{i=1}^{d} T_i$ then $c = c(G) = \lceil \frac{d+1}{2} \rceil$ and $\mathsf{capt}_c(G) \leq \lceil \lg n \rceil \sum_{i=1}^{d} \mathsf{rad}(T_i) - \lfloor \frac{d-1}{2} \rfloor + 1$.*

## 1.3 Dominating Set Results

In this section we list results on domination that will be used to prove cop pebbling theorems. The definition of a dominating set immediately yields the following result.

**Result 10.** *If $G$ is a graph with $n$ vertices and maximum degree $\Delta$ then $\gamma(G) \geq \frac{n}{\Delta+1}$.*

**Result 11.** *[9] If $|N_d[v]| \geq c$ for all $v \in G$ then $\gamma_{2d}(G) \leq n/c$.*

**Result 12.** *[6] Almost all cop-win graphs $G$ have $\gamma(G) = 1$.*

**Result 13.** *[15] If $G = P_k \square P_m$ with $16 \leq k \leq m$, then $\gamma(G) \leq \left\lfloor \frac{(k+2)(m+2)}{5} \right\rfloor - 4$.*

Result 10 implies that $\gamma(Q^d) \geq 2^d/(d+1)$. The following result shows that the actual value is not much greater, asymptotically.

**Result 14.** *[16] $\gamma(Q^d) \sim 2^d/d$.*

## 1.4 Optimal Pebbling Results

Finally, we list the optimal pebbling results we use to establish new cop pebbling theorems.

**Result 15.** *[9] For every graph $G$, $\pi^*(G) \leq \lceil 2n/3 \rceil$. Equality holds when $G$ is a path or cycle.*

Fractional pebbling allows for rational values of pebbles. A *fractional pebbling step* from vertex $u$ to one of its neighbors $v$ removes $x$ pebbles from $u$ and adds $x/2$ pebbles to $v$, where $x$ is an rational number such that $0 < x \leq C(u)$. The *optimal fractional pebbling number* of a graph $G$, denoted $\hat{\pi}^*(G)$, is the minimum number $m$ so that, from some configuration of size $m$, one can move, via fractional pebbling moves, a sum of one pebble to any specified target vertex.

**Result 16.** *[18, 22] For every graph $G$ we have $\pi^*(G) \geq \lceil \hat{\pi}^*(G) \rceil$.*

The authors of [18] prove that $\hat{\pi}^*(G)$ can be calculated by a linear program. Furthermore, they use this result to show that there is uniform configuration that witnesses the optimal fractional pebbling number of any vertex-transitive graph; that is, the configuration $C$ defined by $C(v) = \hat{\pi}^*(G)/n(G)$ for all $v$ fractionally solves any specified vertex. From this they prove the following.

**Result 17.** *[18] Let $G$ be a vertex-transitive graph and, for any fixed vertex $v$, define $m = \sum_{u \in V(G)} 2^{-\mathsf{dist}(u,v)}$. Then $\hat{\pi}^*(G) = n(G)/m$.*

**Result 18.** *[9] If $G$ is an $n$-vertex graph with $\mathsf{gir}(G) \geq 2s+1$ and $\delta(G) = k$ then $\pi^*(G) \leq \frac{2^{2s}n}{\sigma_k(s)}$, where $\sigma_k(s) = 1 + k\sum_{i=1}^{s}(k-1)^{i-1}$.*

For a configuration $C$ on a graph $G$, we say that a vertex $v$ is 2-*reachable* if it is possible to move two pebbles to $v$ via pebbling steps. Then $C$ is 2-*solvable* if every vertex of $G$ is 2-reachable.

**Result 19.** *[9] If $C$ is a 2-solvable configuration of pebbles on the path $P_n$ then $|C| \geq n+1$.*

For a subset $W$ of vertices in a graph $G$ we define the graph $G_W$ to have vertices $V(G_W) = V(G) - W \cup \{w\}$ with edges $xy$ whenever $x, y \in V(G) - W$ and $xy \in E(G)$ and $xw$ whenever $x \in V(G) - W$ and $xz \in E(G)$ for some $z \in W$. The process of creating $G_W$ from $G$ is called *collapsing* $W$. If $C$ is a configuration on $G$ then we define the configuration $C_W$ on $G_W$ by $C_W(w) = \sum_{z \in W} C(z)$ and $C_W(x) = C(x)$ otherwise. Note that $|C| = |C_W|$.

**Result 20.** *[9] Let $W$ be a subset of vertices in a graph $G$. If a configuration $C$ on $G$ can reach the configuration $D$ on $G$ then the configuration $C_W$ on $G_W$ can reach the configuration $D_W$ on $G_W$. In particular, we have $\pi^*(G) \geq \pi^*(G_W)$.*

The next three results involve Cartesian products.

**Result 21.** *[17] For all $3 \leq k \leq m$ we have $\pi^*(P_k \square P_m) \leq \frac{2}{7}km + 8 \approx 0.2857km$.*

The above result is conjectured to be best possible, while the best known lower bound is below.

**Result 22.** *[25] For all $k \leq m$ we have $\pi^*(P_k \square P_m) \geq \frac{5092}{28593}km + O(k+m) \approx 0.1781km$.*

Combining Results 16 and 17 yields the lower bound of the following result. The upper bound places $2^k$ pebbles on each vertex of a distance-$k$ dominating set, with $k$ roughly $d/3$.

**Result 23.** *[9, 22] For all $d \geq 1$ we have $(4/3)^d \leq \pi^*(Q^d) \leq (4/3)^{d+O(\lg k)}$.*

# 2 Main Theorems

## 2.1 Lower Bounds

**Theorem 1.** *For any graph $G$, $\pi^{\mathsf{c}}(G) \geq c(G)$, with equality if and only if $G = K_1$.*

*Proof.* Any configuration of cops that can capture the robber via pebbling steps can also capture the robber via free steps.

If $G = K_1$ then $\pi^{\mathsf{c}}(G) = 1 = c(G)$.

If $\pi^{\mathsf{c}}(G) = c(G)$ then capturing the robber requires no steps. That means that a successful pebbling configuration has no vertex without a pebble; i.e. $\pi^{\mathsf{c}}(G) = n(G)$. However, if $n(G) \geq 3$ then $\pi^{\mathsf{c}}(G) < n$ by Corollary 5, below, a contradiction. If $n(G) = 2$ then $G = K_2$ and $\pi^{\mathsf{c}}(K_2) = 2 > 1 = c(K_2)$, a contradiction. Hence $G = K_1$. $\square$

**Theorem 2.** *For any graph $G$, $\pi^{\mathsf{c}}(G) \geq \pi^*(G)$. Equality holds if $G$ is a tree or cycle.*

*Proof.* Any configuration of $k$ cops, where $k < \pi^*(G)$, will contain a vertex $v$ which is unreachable. The robber can then choose to start and stay on $v$ and thus not be captured.

If $G$ is a cycle then Result 15 and Theorem 17 yield the equality.

If $G$ is a tree then place $\pi^*(G)$ cops according to an optimal pebbling configuration $C$. The robber beginning at some vertex $v$ defines a subtree $T$ containing $v$ for which every cop in $T$ is on a leaf of $T$, and any leaf of $T$ with no cop is a leaf of $G$. Thus the robber can never escape $T$. Because $C$ can reach every vertex of $T$, they can capture the robber, regardless of where he moves. Hence $\pi^{\mathsf{c}}(G) \leq \pi^*(G)$, and the equality follows. □

Typically, Theorem 2 gives a sharper lower bound on $\pi^{\mathsf{c}}(G)$ than Theorem 1. However, this may not be true for all graphs.

**Theorem 3.** *If $G$ is a graph with $\delta = \delta(G) \geq 27$, and with $\mathsf{gir}(G) \geq 4t + 2$ and $n(G) \leq \delta^{2t+1}$ for some $t \geq 3$, then $c(G) > \pi^*(G)$.*

*Proof.* Given $G$ as above, set $d = \delta - 1$. Then we have $\mathsf{gir}(G) \geq 4t + 1$, and so $c(G) \geq d^t/et$ by Result 3. Also, with $s = 2t$, we have $\mathsf{gir}(G) > 2s+1$, and so $\pi^*(G) \leq 2^{2s}n/\sigma_\delta(s)$ by Result 18. Note that $2^{2s}n/\sigma_\delta(s) < 4^s n/d^s \leq (4/d)^{2t}(d+1)^{2t+1}$. Thus the result will be proved by showing that $(4/d)^{2t}(d+1)^{2t+1} \leq d^t/et$.

Since $\delta \geq 27$ we have $d \geq 26$. Then it is easy to calculate that $f(d,t) = [4(1+1/d)]^{2t}(d+1)/etd^t < 1$ when $d = 26$ and $t = 3$, and to observe that $f(d,t)$ decreases in $d$ and $t$. From this it follows that $(4/d)^{2t}(d+1)^{2t+1} \leq d^t/et$. □

At issue here is that it is not known if there exists a graph that satisfies the hypothesis of Theorem 3. Indeed, Biggs [3] defines a sequence of $k$-regular graphs $\{G_i\}$ with increasing $n(G_i)$ to have *large girth* if $\mathsf{gir}(G_i) \geq \alpha \log_{k-1}(n(G_i))$ for some constant $\alpha$. It is known that $\alpha \leq 2$, and the greatest known constant is a construction of [20] that yields $\alpha = 4/3$. However, a graph satisfying the hypothesis of Theorem 3 necessarilty has $\alpha = 2$.

## 2.2 Upper Bounds

**Theorem 4.** *Let $G$ be a graph with dominating set $S$. Suppose that $S' \subseteq S$ is a dominating set of $V(G)-S$. Then $\pi^{\mathsf{c}}(G) \leq |S| + |S'|$. In particular, $\pi^{\mathsf{c}}(G) \leq 2\gamma(G)$.*

*Proof.* Place two cops on each vertex of $S'$ and one cop on each vertex of $S - S'$. In order to not be immediately captured, the robber must start in $V(G) - S$, but then is captured in one step by some pair of cops from $S'$. The second statement follows from choosing $S' = S$ to be a minimum dominating set of $G$. $\square$

To illustrate the improvement of $|S| + |S'|$ compared to $2\gamma(G)$, consider the following example.

**Example 1.** *For positive integers $m \geq 2k \geq 2$, let $Y = \{y_1, \dots, y_m\}$ and let $Q = \{Q_1, \dots, Q_k\}$ be a partition of $Y$ with each part size $|Q_i| \geq 2$. Define a bipartite graph $G$ with vertices $Y$, $Z = \{z_1, \dots, z_k\}$, and $x$ as follows. For each $1 \leq j \leq k$ set $z_j \sim y_i$ if and only if $y_i \in Q_j$. Also set $x \sim y_i$ for every $1 \leq i \leq m$. Then $\gamma(G) = k + 1$. Indeed, since the neighborhoods of each $z_j$ are pairwise disjoint, at least $k$ vertices in $Y \cup Z$ are required to dominate $Z$, one from each $N[z_j]$. Suppose that $S$ is a dominating set of size $k$. By the above, $|S \cap N[z_j]| = 1$ for all $j$. But to dominate $x$, some $y_i$ must be in $S$. Let $y_i \in N(z_j)$; then $y_i$ does not dominate any other $y_{i'} \in N(z_j)$. Hence $\gamma(G) \geq k+1$. It is easy to see that $S = Z \cup \{x\}$ is a dominating set, so that $\gamma(G) = k+1$. With this choice of $S$ we have $S' = \{x\}$, so that $\pi^{\mathsf{c}}(G) \leq k+2$, much better than $2\gamma(G) = 2k+2$.*

An obvious corollary of Theorem 4 (recorded as Corollary 15, below) is that any graph $G$ with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$. A more interesting corollary is the following.

**Corollary 5.** *Every graph $G$ satisfies $\pi^{\mathsf{c}}(G) \leq n - \Delta(G) + 1$. In particular, if $n(G) \leq 2$ then $\pi^{\mathsf{c}}(G) = n$, if $n(G) \geq 3$ then $\pi^{\mathsf{c}}(G) \leq n - 1$, and if $n(G) \geq 6$ then $\pi^{\mathsf{c}}(G) \leq n - 2$.*

*Proof.* Let $v$ be a vertex with $\mathsf{deg}(v) = \Delta(G)$ and set $S = V(G) - N[v]$, with $S' = \{v\}$. Then apply Theorem 4. Next, it is easy to see that $\pi^{\mathsf{c}}(K_n) = n$ for $n \leq 2$. Then, a graph with at least three vertices has a vertex of degree at least two, so that $n - \Delta(G) + 1 \leq n - 1$. Finally, if $\Delta(G) \geq 3$ then $n - \Delta(G) + 1 \leq n - 2$, while if $\Delta(G) \leq 2$ then $G$ is a path or cycle, for which Theorem 17 yields $\pi^{\mathsf{c}}(G) = \lceil \frac{2n}{3} \rceil$, which is at most $n - 2$ when $n \geq 6$. $\square$

All three conditional bounds in Corollary 5 are tight: for example, $\pi^*(P_2) = 2$, $\pi^*(P_5) = 4$, and $\pi^*(P_7) = 5$. Furthermore, its more general bound of $n - \Delta(G) + 1$ is tight for a graph with a dominating vertex (see Corollary 15).

A set $A$ of edges of a graph $G$ is called an *induced $k$-star packing* of $G$ if every component of the subgraph $G[A]$ induced by $A$ is a star with at most $k$ edges and is an induced subgraph of $G$. Kelmans [19] showed that there is a polynomial algorithm for finding a $k$-star packing that covers the maximum number of vertices in a graph. Of course, this problem is $\mathsf{NP}$-hard if $k$ is not fixed. *Perfect* star packings cover all vertices in a graph, and have been studied in [27]. Suppose that $X = \{X_1, \dots, X_m\}$ is a packing of stars in $G$, with corresponding centers $U = \{u_1, \dots, u_m\}$. If $X$ covers all but the vertices $W$, then Theorem 4 implies that $\pi^{\mathsf{c}}(G) \leq |W| + 2m = n - \sum_i \mathsf{deg}(u_i) + m$; this follows from setting $S = W \cup U$ and $S' = U$. In fact, the resulting placement of cops is a *roman dominating set* of $G$, defined in [12] as a $\{0, 1, 2\}$-labeling of $V(G)$ so that every vertex labeled 0 is adjacent to some vertex labeled 2. They define the *roman domination number* $\gamma_R(G)$ to be the minimum sum of labels of a roman dominating set. Hence we obtain the following bound.

**Theorem 6.** *Every graph $G$ satisfies $\pi^{\mathsf{c}}(G) \leq \gamma_R(G)$.*

**Theorem 7.** *Let $H$ be an induced subgraph of a graph $G$. Then, for any $s$, if $\pi^{\mathsf{c}}(H) \leq n(H) - s$ then $\pi^{\mathsf{c}}(G) \leq n(G) - s$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(H) \leq n(H) - s$. Then there is a configuration $C_H$ of $n(H) - s$ cops on $H$ that captures any robber on $H$. Define the configuration $C_G$ of $n(G) - s$ cops on $G$ by placing one cop on each vertex of $G - H$ and $C_H(v)$ cops on each vertex $v \in H$. Then $C_H$ captures any robber on G. □

**Corollary 8.** *For all $s \geq 2$ there is an $N = N(s)$ such that every graph $G$ with $n = n(G) \geq N$ has $\pi^{\mathsf{c}}(G) \leq n - s$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(G) \geq n - s + 1$. Then Corollary 5 implies that $\Delta(G) \leq s$. Consider if $\mathsf{diam}(G) \geq 3s$. Then there exists an induced path $P$ of length $3s$ in $G$. By Theorem 17 we have $\pi^{\mathsf{c}}(P_{3s}) = 2s \leq 2s + 1 = n(P) - s$. By Theorem 7, we must have that $\pi^{\mathsf{c}}(G) \leq n - s$, contradicting our assumption that $\pi^{\mathsf{c}}(G) \geq n - s + 1$. Thus, we conclude that $\mathsf{diam}(G) < 3s$. Since there are finitely many (at most $\Delta(G)^{\mathsf{diam}(G)}$) such graphs, there must be some $N$ such that $\pi^{\mathsf{c}}(G) \leq n - s$ for all $s \geq N$. □

One might be interested in measuring the gap between the size of a graph and its cop pebbling number. For this we define the *cop deficiency* of a graph $G$ to be $\text{Ï}^{\mathsf{c}}(G) := n(G) - \pi^{\mathsf{c}}(G)$. Then Theorem 7 and Corollary 8 can be restated as follows.

**Theorem 9.** *Let $H$ be an induced subgraph of a graph $G$. Then $\text{Ï}^{\mathsf{c}}(G) \geq \text{Ï}^{\mathsf{c}}(H)$.*

**Corollary 10.** *For all $s \geq 2$ there is an $N = N(s)$ such that every graph $G$ with $n = n(G) \geq N$ has $\text{Ï}^{\mathsf{c}}(G) \geq s$.*

**Theorem 11.** *If $G$ is a cop-win graph with $\mathsf{capt}(G) = t$, then $\pi^{\mathsf{c}}(G) \leq 2^t$. More generally, if $c(G) = k$ and $\mathsf{capt}_k(G) = t$ then $\pi^{\mathsf{c}}(G) \leq k2^t$.*

*Proof.* If $G$ is a cop-win graph with $\mathsf{capt}(G) = t$, then there is some vertex $v$ at which the cop begins and the robber can be caught with free steps in at most $t$ moves. If $2^t$ cops are placed on $v$, the cops can use the same capture strategy, and there will be sufficiently many cops for up to $t$ pebbling steps. Similarly, by placing $2^t$ on each of $c(G)$ cops, there will be sufficiently many cops for up to $t$ rounds of pebbling steps. □

For example, let $T$ be a complete $k$-ary tree of depth $t$. Then $\mathsf{capt}(T) = t$ by Result 4, and so $\pi^{\mathsf{c}}(T) \leq 2^t$. Theorem 11 is tight for some graphs, as witnessed by any graph $G$ with a dominating vertex (see Corollary 15, below). It is also tight for any complete $k$-ary tree of depth two, when $k \geq 3$ (see Corollary 19, below).

**Corollary 12.** *If $G$ is a chordal graph with radius $r$, then $\pi^{\mathsf{c}}(G) \leq 2^r$.*

*Proof.* Follows from Result 4 and Theorem 11. □

**Theorem 13.** *If $T$ is an $n$-vertex tree, then $\pi^{\mathsf{c}}(T) \leq \lceil \frac{2n}{3} \rceil$.*

*Proof.* Consider a maximum length path $P$ in $T$. Let $z$ be an endpoint of $P$ (necessarily a leaf), let $y$ be the neighbor of $z$, and let $x$ be the other neighbor of $y$ on $P$.

*Base case:* For $n = 3$, the only tree is $P_3$. Place two cops on the central vertex, and the robber will be caught on the cops' first move.

*Inductive Step:* Assume that for trees with $n < k$ vertices, $\pi^{\mathsf{c}}(T) \leq \lceil \frac{2n}{3} \rceil$. If $d(y) > 2$, form a new tree $T' = T - \{z\} - \{y\} - (\{N[y] - \{x\}\})\}$. By our inductive hypothesis, we can distribute the cops in such a way that the robber is caught if the robber starts on $T'$. By placing two cops on $y$, we can also ensure that the robber is caught on the first move if the robber starts on $T - T'$.

On the other hand, if $d(x) = d(y) = 2$, form a new tree $T' = T - \{x, y, z\}$. By our inductive hypothesis, we can distribute the cops in such a way that the robber is caught if the robber starts on $T'$. By placing two cops on $y$, we can also ensure that the robber is caught if the robber starts on $T - T'$.

If $d(y) = 2$ and $d(x) > 2$, and $x$ has a leaf neighbor $u$, form a new tree $T' = T - \{u, y, z\}$. By our inductive hypothesis, we can distribute the cops in some distribution $D'$ so that the robber is caught if the robber starts on $T'$. By placing two cops on $y$, we can also ensure that the robber is caught if the robber starts on vertices $y$ or $z$. To capture a robber on $u$, one cop can reach $x$ from $D'$, and another cop can reach $x$ from $y$. We then can reach $x$ from $u$.

Finally, suppose $d(y) = 2$ and $d(x) > 2$, and $x$ has no leaf neighbor $u$. Denote the neighborhood of $x$ which is not on $P$ as $N[x \cap P^c] = N_2[x] \cap T[V(T) \setminus V(P)]$, and let $u \in N[x] \cap T[V(T) \setminus V(P)]$. Since $P$ has maximum length, $N[u] - \{x\}$ consists only of leaves. Let $v \in N[u] - \{x\}$, and let $T' = T - \{v, y, z\}$. By our inductive hypothesis, we can distribute the cops in some distribution $D'$ that the robber is caught if the robber starts on $T'$. If two cops can reach $x$ in $T'$, we can add 2 more cops to $x$ to catch the robber on the vertices $\{v, y, z\}$. If two cops can reach $u$ in $T'$, then $v$ and $x$ are reachable, so we can add 2 more cops to $y$ to catch the robber on the vertices $\{x, z\}$. Last, if two cops cannot reach $x$ or $u$, then no sequences of cop moves in $T'$ will use the edge $uv$ (otherwise, we would be able to get two cops on at least one of the two vertices). Thus, we can simultaneously get one cop on $x$ and one cop on $u$. By adding two cops onto $y$, the cops can reach the vertices $\{v, y, z\}$. □

Note that, on a tree, cops move greedily toward the robber, so if a cop $p$ can reach a vertex $v$ then the robber cannot ever occupy $v$, as the robber has no access to $v$ except through $p$. Hence if $G$ is a tree then $\pi^{\mathsf{c}}(G) = \pi^*(G)$. We note that Theorems 11 and 13 can each be stronger then each other, as the following two examples show. Define the *spider* $S(k, d)$ to be the tree having a unique vertex $x$ of degree greater than 2, all $k$ of whose leaves have distance $d$ from $x$.

**Example 2.** *For integers $k$ and $d$, the spider $S = S(k, d)$ has $c(S) = 1$ and $\mathsf{capt}(S) = d$, with $n = kd + 1$. Thus Theorem 11 yields $\pi^{\mathsf{c}}(S) \le 2^d$, while Theorem 13 yields $\pi^{\mathsf{c}}(S) \le \lceil (2kd + 2)/3 \rceil$. Hence one bound is stronger than the other depending on how $k$ compares, roughly, to $3 \cdot 2^{d-1}/d$.*

**Example 3.** *For integers $k, t \ge 1$, let $T$ be the complete $k$-ary tree of depth $t$. Then $n(T) = \sum_{i=0}^{t} k^i = (k^{t+1} - 1)/(k - 1)$. Thus Theorem 11 is stronger than Theorem 13 for $k \ge 3$ and for $k = 2$ with $t \ge 2$, while Theorem 13 is stronger than Theorem 11 when $k = 1$ and $t \ge 5$ (because $\mathsf{capt}(P_t) = \lceil t/2 \rceil$).*

**Theorem 14.** *For any positive integer $d$, if $G$ is a graph with $\mathsf{gir}(G) \ge 4d - 1$, then $\pi^{\mathsf{c}}(G) \le 2^d \gamma_d(G)$.*

*Proof.* Let $S = \{v_1, v_2, ...\}$ be a minimum $d$-distance dominating set of $G$, and place $2^d$ cops on each $v_i$. Suppose the robber starts at vertex $v$. Since $\mathsf{gir}(G) \ge 4d-1$, we know that $T = N_d[u]$ is a tree for all $u$. We write $T_i = N_d[v_i]$ and, for each $v \in T$, denote the unique $vu$-path in $T$ by $P_v$.

Let $J$ be such that $T \cap T_j \ne \emptyset$ if and only if $j \in J$, and set $Q_j = T \cap T_j$. Note that $\mathsf{gir}(G) \ge 4d - 1$ implies that, for each $j \in J$, there is some $v \in T$ such that $Q_j \subseteq P_v$. Moreover, by the definition of $S$, we have $\cup_{j \in J} Q_j = T$. In addition, $\mathsf{gir}(G) \ge 4d - 1$ implies that, for each $j \in J$, the shortest $v_i v$-path $P_i^*$ is unique.

For each $j \in J$, each cop at $v_j$ adopts the strategy to move at each turn toward $v$ along $P_i^*$ until reaching $T$, at which time then moving toward the robber along the unique path in $T$. This strategy ensures the property that, at any point in the game, if some cop is on vertex $x$ while the robber is on vertex $z$, then the robber can never move to a vertex $y$ for which the unique $yz$-path in $T$ contains $x$ — which includes $x$ itself. It also implies that the game will last at most $d$ turns. Hence, if we suppose that the robber wins the game, then the game lasted exactly $d$ turns and the robber now sits on some vertex $z$. However, by the definition of $S$, some cop reached $z$ within $d$ turns, which implies by the property just mentioned that the robber cannot move to $z$, a contradiction. Hence the cops win the game, capturing the robber. □

An obvious corollary (recorded as Corollary 15, below) is that any graph $G$ with a dominating vertex has $\pi^{\mathsf{c}}(G) = 2$.

We remark that Theorem 14 applies to trees. $P_5$ is an example for which this bound is tight. In the case of the spider $S(k, 2)$, this bound is significantly better than Corollary 5 when $k$ is large. The case $d = 1$

yields the same upper bound of $2\gamma(T)$ from Theorem 4, which is better than the bound of Theorem 13 if and only if $\gamma(T) < \lceil (n-1)/3 \rceil$. Since $\gamma(T)$ can be as high as $n/2$, both theorems are relevant. The following example shows that Theorem 14 can be stronger than Theorem 13 for any $d$.

**Example 4.** *For $1 \le i \le 3$, define the tree $T_i$ to be the complete binary tree of depth $d-1$, rooted at vertex $v_i$, and define the tree $T$ to be the union of the three $T_i$ with an additional root vertex adjacent to each $v_i$. Then $\gamma_d(T) = d$, and $n = 3(2^d - 1) + 1$, so that the bound from Theorem 14 is stronger than the bound from Theorem 13.*

Theorem 14 can be stronger than other prior bounds as well, as shown by the following example.

**Example 5.** *For integers $k$ and $d$, define the theta graph $\Theta(k, d)$ as the union of $k$ internally disjoint $xy$-paths, each of length $d$. Then $\Theta = \Theta(k, 2d)$ has $n = k(d-1) + 2$, $c(\Theta) = 2$, $\mathsf{capt}_2(\Theta) = d$, $\mathsf{gir}(\Theta) = 4d$, $\gamma(\Theta) = k\lceil (2d-3)/3 \rceil$, and $\gamma_d = 2$. Thus Theorem 4 yields an upper bound of roughly $4kd/3$, while Theorems 11 and 14 both yield the upper bound of $2^{d+1}$, which is better or worse than Theorem 4 when $k$ is bigger or less than, roughly, $3 \cdot 2^{d-1}/d$.*

The following example illustrates the need for stronger bounds than given by Theorem 14.

**Example 6.** *Consider the $(3, 7)$-cage McGee graph $M$, defined by $V = \{v_i \mid i \in \mathbb{Z}_{24}\}$, with $v_i \sim v_{i+1}$ for all $i$, $v_i \sim v_{i+12}$ for all $i \equiv 0 \pmod 3$, and $v_i \sim v_{i+7}$ for all $i \equiv 1 \pmod 3$. We have $\gamma_2(M) \le 4$ (e.g. $\{v_0, v_6, v_9, v_{15}\}$), and so $\pi^*(M) \le \pi^{\mathsf{c}}(M) \le 16$ by Theorem 14. However, this bound is not tight, as $\pi^{\mathsf{c}}(M) \le 12$: the vertex set $\{v_i \mid i \equiv 0 \pmod 3\}$ induces a matching of size 4 — for each edge, place 2 cops on one of its vertices and 1 cop on the other. Incidentally, this yields $\pi^*(M) \le 12$; the best known lower bound on $\pi^*$ comes from Result 17: $\pi^*(M) \ge \lceil \hat{\pi}^*(M) \rceil = \lceil 64/7 \rceil = 10$. Hence we are left with a gap in the bounds for $M$: $10 \le \pi^*(M) \le \pi^{\mathsf{c}}(M) \le 12$.*

## 2.3 Exact Results

The following is a corollary of Theorem 4, as well as of Theorem 14.

**Corollary 15.** *If $G$ is a graph with a dominating vertex then $\pi^{\mathsf{c}}(G) = 2$.*

The following is a corollary of Results 12 and Theorem 7.

**Corollary 16.** *Almost all cop-win graphs $G$ have $\pi^{\mathsf{c}}(G) = 2$.*

**Theorem 17.** *For all $n \ge 1$ we have $\pi^{\mathsf{c}}(P_n) = \pi^{\mathsf{c}}(C_n) = \lceil \frac{2n}{3} \rceil$.*

*Proof.* We have from Theorem 2 that $\pi^{\mathsf{c}}(P_n) \geq \pi^*(P_n) = \lceil \frac{2n}{3} \rceil$ and $\pi^{\mathsf{c}}(C_n) \geq \pi^*(C_n) = \lceil \frac{2n}{3} \rceil$.

We have from Theorem 13 that $\pi^{\mathsf{c}}(P_n) \leq \lceil \frac{2n}{3} \rceil$. For $C_n$, partition $C_n$ into $\lfloor \frac{n}{3} \rfloor$ copies of $P_3$ and, possibly, an extra $P_1$ or $P_2$. Place two cops on the center vertex of each $P_3$, and one cop on each vertex of the remaining one or two vertices. The robber can only choose to start on one of the copies of $P_3$, where he is next to a pair of cops, and so will be captured on the first move. Thus $\pi^{\mathsf{c}}(C_n) \leq \lceil \frac{2n}{3} \rceil$. □

**Theorem 18.** *If $T$ is a tree with $\mathsf{rad}(T) = 2$ and $\mathsf{diam}(T) = 4$ then $\pi^{\mathsf{c}}(T) = 4$.*

*Proof.* The upper bound follows from Result 4 and Theorem 11. The lower bound follows from Theorem 17 since $T$ contains $P_5$.) □

**Corollary 19.** *If $T$ is a complete $k$-ary tree of depth 2 with $k \geq 3$, then $\pi^{\mathsf{c}}(T) = 4$.*

## 3 Cartesian Products

For ladders ($P_2 \square P_m$) we have the following theorem.

**Theorem 20.** *For all $m \geq 1$ we have $\pi^{\mathsf{c}}(P_2 \square P_m) = m + 1$.*

*Proof.* Let $G = P_2 \square P_m$ and for each $i \in [m]$ define the vertex subset $E_i = [2] \times \{i\}$. Suppose that $C$ can catch any robber on $G$. Then $C$ must be able to move two pebbles to any $E_i$. Indeed, assume that a cop catches the robber on $E_i$; without loss of generality, on vertex $(0, i)$. If the robber didn't move, it is because each of its neighbors contained a cop. If the robber did move, say, from $(1, i)$, then it was because a cop could have moved onto the robber. In each of those cases we see that two cops could be moved onto $E_i$. If the robber moved instead from, say $(0, i-1)$ (or symmetrically $(1, i+1)$), then it was because cops could move into both $(0, i-1)$ and $(1, i-1)$ (or $(0, i+1)$ and $(1, i+1)$), and so two cops could be moved onto $E_{i-1}$ (respectively $E_{i+1}$). Now, by collapsing every $E_i$ we obtain the graph $G' = P_m$, with corresponding collapsed configuration $C'$. The above argument then shows by Result 20 that $C'$ is 2-solvable. Consequently, Result 19 proves that $|C| = |C'| \geq m + 1$.

For the upper bound, we define $[k] = \{0, 1, \ldots, k-1\}$, write $m$ uniquely as $m = 4r + 2s + t$, with $s, t \in [2]$, and let $V(G) = [2] \times [m]$. Next define the sets $S = \{(0, 1)\}$ when $m$ is even and $S = \{(0, 1), (m-1, s)\}$ when $m$ is odd, and $T = \{(4i + 2j + 1, j) \mid 0 \leq i \leq r, j \in [2], 4i + 2j + 1 \in [m]\}$ — alternately, $T = \{(x, 0) \mid x \equiv 1 \pmod{m}\} \cup \{(x, 1) \mid x \equiv 3 \pmod{m}\}$. Then $S \cup T$ is a dominating set. Now place two cops on each vertex of $T$ and one cop on each vertex of $S$. It is simple to check that any robber can be captured in one step and that the number of cops in each case equals $m + 1$. □

More general grids have cop pebbling numbers linear in their number of vertices, but there is a gap in the bounds for its coefficient.

**Theorem 21.** *For all* $16 \le k \le m$ *we have* $\frac{5092}{28593}km + O(k+m) \le \pi^{\mathsf{c}}(P_k \square P_m) \le 2\left\lfloor \frac{(k+2)(m+2)}{5} \right\rfloor - 8.$

*Proof.* Result 22 and Theorem 2 produce the lower bound (which actually holds for all $1 \le k \le m$), while Result 13 and Theorem 4 produce the upper bound. □

For cubes, upper and lower bounds on their cop pebbling numbers currently exhibit an exponential gap. We can use Results 6 and 7 with Theorem 11 to obtain the upper bound $\pi^{\mathsf{c}}(Q^d) \le \lceil \frac{d+1}{2} \rceil d^{\Theta(d)}$. However, by adding cops we can reduce the capture time and therefore also the upper bound.

**Theorem 22.** $\left(\frac{4}{3}\right)^d \le \pi^{\mathsf{c}}(Q^d) \le \frac{2^{d+1}}{d+1} + o(d)$.

*Proof.* The lower bound follows from Result 23 and Theorem 2. The upper bound follows from Result 14 and Theorem 4. □

**Theorem 23.** *For every graph* $G$ *we have* $\pi^{\mathsf{c}}(G \square K_t) \le t\pi^{\mathsf{c}}(G)$.

*Proof.* Let $C$ be a configuration of $\pi^{\mathsf{c}}(G)$ cops on $G$ that can capture any robber. Define the configuration $C'$ on $G \square K_t$ by $C'(u,v) = C(u)$ for all $u \in V(G)$ and $v \in V(K_t)$; then $|C'| = t|C|$. Let $C'_v$ be the restriction of $C'$ to the vertices $V_v = \{(u,v) \mid u \in V(G)\}$. Then each $C'_v$ is a copy of $C$ on $V_v$. Now imagine, for any robber on some vertex $(u,v)$, placing a copy of the robber on each vertex $(u',v)$ and maintaining that property with every robber movement. Then the cops on each $V_v$ will move in unison to catch their copy of the robber in $V_v$, one of which is the real robber. □

When $G = P_m$ and $t = 2$, Theorems 17 and 23 yields $\pi^{\mathsf{c}}(P_m \square K_2) \le 2\lceil \frac{2m}{3} \rceil$. However, it is not difficult to see that $\pi^{\mathsf{c}}(P_m \square K_2) \le m+1$. Indeed, label the vertices $\{v_{i,j} \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_2\}$ and define the configuration $C$ by $C(v_{0,0}) = 1$, $C(V_{i,1}) = 2$ for all $i \equiv 1 \pmod 4$, and $C(v_{i,0}) = 2$ for all $i \equiv 3 \pmod 4$. If $m$ is odd then also define $C(v_{i,m-1}) = 1$ for $i = (m+1)/2 \mod 2$. Then $|C| = m+1$ and, since the set of vertices with pebbles on them is a dominating set, $C$ can catch any robber.

A famous conjecture of Graham [10] postulates that every pair of graphs $G$ and $H$ satisfy $\pi(G \square H) \le \pi(G)\pi(H)$. This relationship was shown by Shiue to hold for optimal pebbling.

**Theorem 24.** *[28] Every pair of graphs* $G$ *and* $H$ *satisfy* $\pi^*(G \square H) \le \pi^*(G)\pi^*(H)$.

One might ask whether or not the analogous relationship holds between $\pi^{\mathsf{c}}(G \square H)$ and $\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$. Theorem 23 shows that this is true for $H = K_2$. However, the inequality is false in general, as the following theorem shows. For any graph $G$ define $G^1 = G$ and $G^d = G \square G^{d-1}$ for $d > 1$.

**Theorem 25.** *There exist graphs $G$ and $H$ such that $\pi^{\mathsf{c}}(G\Box H) > \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$.*

*Proof.* Suppose that $\pi^{\mathsf{c}}(G\Box H) \le \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G$ and $H$. For fixed $k \ge 2$, let $d \ge 25k^2$, $v \in V(C_k^d)$, and $m = \sum_{u\in V(C_k^d)} 2^{-\mathsf{dist}(u,v)}$. Then $\sqrt{d} > \ln d$, so that $d/\ln d > \sqrt{d} \ge 5k > \frac{2}{\ln(3/2)}k$, which implies that $d^{2k} < (3/2)^d$. Also $d \ge \sqrt{k/8}+2$, so that $\sqrt{k/8} \ge d-2$. Thus

$$\begin{aligned}\left(\frac{2}{3}\right)^d \sum_{u\in V(C_k^d)} 2^{-\mathsf{dist}(u,v)} \;&\le\; \left(\frac{2}{3}\right)^d \sum_{i=0}^{kd/2} \binom{i+k-1}{k-1} 2^{-i} \;\le\; \left(\frac{2}{3}\right)^d \sum_{i=0}^{kd/2} \binom{i+k-1}{k-1} \;\le\; \left(\frac{2}{3}\right)^d \binom{kd/2+k}{k}\\ &\le\; \left(\frac{2}{3}\right)^d (kd/2+k)^k/k! \;\le\; \left(\frac{2}{3}\right)^d (d+2)^k \sqrt{k/2}^k 2^{-k} \;\le\; \left(\frac{2}{3}\right)^d (d+2)^k (d-2)^k\\ &\le\; \left(\frac{2}{3}\right)^d d^{2k} \;<\; 1.\end{aligned}$$

Therefore we would have

$$\pi^{\mathsf{c}}(P_k^d) \;\le\; \pi^{\mathsf{c}}(P_k)^d \;\le\; \left(\frac{2}{3}k\right)^d \;=\; \left(\frac{2}{3}\right)^d n(P_k^d) \;<\; n(C_k^d)/m \;=\; \hat{\pi}^*(C_k^d) \;\le\; \hat{\pi}^*(P_k^d) \;\le\; \pi^*(P_k^d),$$

by Fact 16 and Theorem 17. This, however, contradicts Theorem 2. □

# 4 Open Questions

Theorem 3 shows one potential way to find a graph with larger cop number than optimal pebbling number. But can such a graph be found by a different method?

**Question 26.** *Is there a graph $G$ that satisfies $c(G) > \pi^*(G)$?*

**Question 27.** *Can the bounds $.1781 \le \pi^{\mathsf{c}}(P_k\Box P_m)/km \le .4$ in Theorem 21 be improved?*

The argument in the proof of Theorem 25 shows that, for any constant $a < 3/2$, there exists a large enough $d = d(a)$ so that $P_k^d$ is a counterexample to the statement that $\pi^{\mathsf{c}}(G\Box H) \le a\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$. This begs two questions.

**Question 28.** *Is there an infinite family of graphs $\mathcal{G}$ for which $\pi^{\mathsf{c}}(G\Box H) \le \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G, H \in \mathcal{G}$?*

**Question 29.** *Is there some constant $a \ge 3/2$ such that $\pi^{\mathsf{c}}(G\Box H) \le a\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G$ and $H$?*

In addition to chordal graphs and cartesian products discussed above, it would be interesting to study other graph classes. It was proved in [11] that $c(G) \le 2$ for outerplanar $G$, and in [1] that $c(G) \le 3$ for planar $G$. Additionally, it was shown in [26] that if $G$ is a planar graph on $n$ vertices then $\mathsf{capt}_3(G) \le 2n$.

**Problem 30.** *Are there constant upper bounds on $\pi^{\mathsf{c}}(G)$ when $G$ is planar or outerplanar? If $k = \pi^{\mathsf{c}}(G)$ then is $\mathsf{capt}_k(G)$ linear?*

Finally, Meyniel [13] conjectured in 1985 that every graph $G$ on $n$ vertices satisfies $c(G) = O(\sqrt{n})$. Some evidence in support of this conjecture is found in [4], where it is proved for $G \in \mathcal{G}_{n,p}$ that when $0 < \epsilon < 1$ and $p > 2(1+\epsilon)\log(n)/n$ we have $c(G) < \frac{10^3}{\epsilon^3} n^{\frac{1}{2}\log(n)}$ almost surely. (In fact, they also show that when $p \gg 1/n$ we have $c(G) > \frac{1}{(pn)^2} n^{\frac{1}{2}\left(\frac{\log\log(pn)-9}{\log\log(pn)}\right)}$ almost surely.) Along these lines, we make the following conjecture.

**Conjecture 31.** *Every graph $G$ on $n$ vertices satisfies $\pi^{\mathsf{c}}(G) = 2n/3 + o(n)$.*

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