Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Quantum gravity phenomenology

Bachelor Thesis

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Study Programme:	Physics
Field of Study:	Physics
Department:	Department of Theoretical Physics
Supervisor:	Mgr. Samuel Kováčik, PhD.

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Aim: The goal of the work is to compute the threshold anomaly for GRB with realistic formulation of the dispersion law and verify the validity of various approximations of that expression.

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Abstrakt

Cieľom tejto práce je skúmať dôsledky kvantovej štruktúry priestoru a previazať ich s pozorovateľnými parametrami. Jej cieľom je vypočítať prahovú anomáliu pre GRB s realistickou formuláciou disperzného zákona a overiť presnosť rôznych aproximácií tohto vzťahu. Práca sa zameriava na disperzný vzťah, ktorý vzniká zavedením nekomutatívnej štruktúry na kvantovom priestore. Tento vzťah je porovnávaný s disperznými vzťahmi z rôznych teórií a poukazuje na potenciálny dolný limit pre škálovací parameter.

Kľúčové slová: GRB221009A, vákuový rozptyl, kvantový priestor

Abstract

The thesis aims to investigate the consequences of the quantum space structure and establish their connection to observable parameters. Its objective is to calculate the threshold anomaly for GRB using a realistic formulation of the dispersion law and to verify the accuracy of various approximations of that formulation. The thesis focuses on a dispersion relation resulting from the introduction of a noncommutative structure on the quantum space. This relation is compared with dispersion relations from different theories and a potential lower limit on the scaling parameter is suggested.

Keywords: GRB221009A, vacuum dispersion, quantum space

Preface

The discovery of a photon with an energy level of approximately 18 TeV on 09.10.2022 posed a challenge to the principles of special relativity. While explanations such as those offered by string theories and doubly special relativity exist for this phenomenon, this work explores the potential of noncommutative quantum space (NCQS) theory as a solution to this problem of quantum gravity. We demonstrate the ability to remodel the dispersion relation to match other theories and examine the lower bound of the scaling parameter in NCQS. This thesis explores the possibilities of NCQS as a potential solution to this quantum gravity problem.

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Foreword

In the 17th century, Isaac Newton introduced the law of universal gravitation in his work "Philosophiæ Naturalis Principia Mathematica", which was derived from empirical observations. As time went on, anomalies in the world of physics appeared, challenging scientists to find solutions. It wasn't until the 20th century, when Albert Einstein presented his theory of relativity, that many of these problems were solved, providing physicists with precise predictions for large-scale phenomena like planets, stars, and black holes. In 1864, Maxwell's equations completed the theory of classical electromagnetism, explaining the relationship between electric fields, magnetic fields, electric current, and electric charge. Despite its enormous success, classical electromagnetism couldn't account for discrete lines in atomic spectra or the distribution of blackbody radiation across different wavelengths. Max Planck later solved this problem by introducing the idea that only discrete energy values can be exchanged between subatomic particles, naming this "pack" of the energy "quantum". In the same century, quantum field theory emerged from the work of generations of theoretical physicists. The weak and electromagnetic forces have been successfully united in electroweak theory, while the strong force is described by a different quantum field theory called quantum chromodynamics. Scientists have come very close to discovering the theory of everything, with only the need to quantise gravity remaining. However, despite considerable effort, it remains a significant challenge.

Over the past decade, various approaches have been put forward that enable us to investigate certain characteristic phenomena of quantum gravity instead of directly testing the fundamental theory itself [3]. This is the first time we can conduct experiments where quantum mechanics and gravity may coexist, despite our inability to merge them at a theoretical level.

One of the approaches centres around the Brownian motion theory. It posits that quantum spacetime might be discrete on the Planck scale, analogous to how matter becomes discrete when observed at the atomic scale. Consequently, the effects of this discreteness could manifest as random fluctuations in the propagation of light or elementary particles, much like in Brownian motion. These effects could register as noise in interferometers that are highly sensitive to measure gravitational radiation. However, presently, there are no indications of noise in the detectors that cannot be explained by ordinary causes, suggesting the absence of such effects at this level.

An alternative method is based on the concept of large extra dimensions. In some models, there is a possibility that quantum gravitational effects could be significantly stronger than usual due to a change in the gravitational interaction on the smallest scales. This change takes place in scenarios with large additional spatial dimensions, whose existence is predicted by string theory. Consequently, quantum gravity could potentially become detectable in Earth-based collider experiments such as the LHC. If this prediction were accurate, we would anticipate the detection of gravitons and black holes at the LHC. However, no such findings were observed in LHC experiments, unfortunately.

Last but not least, we come to the testing of the symmetry of spacetime, which is a topic discussed in this thesis. One plausible hypothesis is that the principle of relativity breaks down at the scale of Planck energy, resulting in a preferred state of motion and rest. Researchers can search for evidence of this by looking for a variation in the speed of light proportional to its energy. This effect can be detected by observing light from astrophysical sources such as gamma-ray bursts (GRB) that have travelled great distances. Although the effect is small, the arrival time of a photon can be offset by a few seconds over billions of light years. The Fermi telescope, launched in June 2008, has detected several bursts where the higher energy photons arrive more than 10 seconds after the onset of the burst in the low energy range. The cause of the delay in high energy photons is still uncertain, whether it's due to emission or propagation. However, there is good news the HERMES (High Energy Rapid Modular Ensemble of Satellite) experiment is in preparation. This experiment is expected to have enough sensitivity to distinguish between emission and propagation [1].

In this thesis, we are also discussing some violations of the Lorentz invariance, but rather than variation in the speed of light, we focus on the energy thresholds of the photons from GRB that can be observed.

Chapter 1

Introduction to noncommutative quantum space

The primary objective of this chapter is to offer the reader an overview of noncommutative quantum space. The main feature of the noncommutative space is its fuzziness when observed at a microscopic level, making it impossible to distinguish between two closely spaced points in space.

1.1 Planck units

Special relativity and quantum mechanics both rely on fundamental constants – the speed of light c for special relativity and the reduced Planck constant $\hbar$ for quantum mechanics. However, using only these two constants, it is impossible to construct units of length or mass. Fortunately, in 1899, Max Planck proposed a solution by adding Newton's gravitational constant κ, which allowed the construction of constants with dimensions of mass, length and time. These constants are known as Planck mass, Planck length, and Planck time and are respectively defined as

$$\begin{aligned} m_{planck} =& \sqrt{\frac{\hbar c}{\kappa}} \approx 2 \cdot 10^{-8}\text{kg}, \\ l_{planck} =& \sqrt{\frac{\hbar \kappa}{c^3}} \approx 10^{-35}\text{m}, \\ t_{planck} =& \sqrt{\frac{\hbar \kappa}{c^5}} \approx 5 \cdot 10^{-44}\text{s}. \end{aligned} \tag{1.1}$$

These constants are particularly significant as they serve as markers for the scale at which the quantum effects of gravitational interaction are predicted to become relevant, as we will explore later.

1.2 Minimal length scale

This section aims to explore the minimal length scale that can be observed in certain thought experiments. As we will see, this will be crucial for defining noncommutative space. We encourage a curious reader to see also [2] for more details about this section.

1.2.1 High energy photon

Let us consider this thought experiment: Suppose we wish to distinguish between two points in space that are separated by a distance of d. To achieve this, we require a photon with a wavelength $\lambda_\gamma \approx d$. However, what would happen if we gradually decreased the distance d? We know that the energy of a photon is determined by

$$E_\gamma = \frac{hc}{\lambda_\gamma}, \tag{1.2}$$

where c is the speed of light and h is the Planck constant. By referring to 1.2, it becomes evident that decreasing the distance d between the two points and consequently decreasing the wavelength λ_γ of the photon results in an increase in the energy of the photon E_γ.

As per general relativity, we understand that a high energy density can curve spacetime to the extent that it may lead to the formation of a black hole, whose Schwarzschild radius is given by

$$R_s = \frac{2\kappa E}{c^4}, \tag{1.3}$$

where κ is the gravitational constant and E is the total energy. The Schwarzschild radius is a parameter in the Schwarzschild solution of the Einstein field equations, and it defines the event horizon of Schwarzschild black holes. These black holes are characterised as static and non-rotating, existing in a vacuum. The event horizon can be defined as a boundary beyond which any events that occur will not have an impact on an observer.

Now, if we substitute energy from 1.2 to 1.3 we get

$$R_s = \frac{2\kappa h}{\lambda_\gamma c^3}. \tag{1.4}$$

It is known that we can form a wave packet that is localised within a volume proportional to λ_γ^3 using photons with a wavelength of $\lambda'_\gamma \geq \lambda_\gamma$. According to Equation 1.4, reducing the wavelength will increase the Schwarzschild radius. Consequently, there exists a critical value of $\lambda_{\gamma crit}$ at which the Schwarzschild radius will be equal to the wavelength of the photon. This implies that the photon will be concealed under the event horizon and will form a black hole. This implies that there is a certain distance limit beyond which it becomes impossible to distinguish between two points. Surprisingly, this phenomenon happens approximately for $\lambda_{\gamma crit} \approx l_{planck}$, where l_{planck} is Planck length.

1.2.2 The Heisenberg microscope with Newtonian gravity

Werner Heisenberg's thought experiment, the Heisenberg microscope, has been instrumental in shaping certain concepts in quantum mechanics. Specifically, the Heisenberg microscope uses classical optics to demonstrate the uncertainty principle.

Consider the scattering of a photon with frequency ω moving along the x direction with a particle whose position on the x axis we intend to measure. Let the cone of light rays emanating from the microscope lens and converging on the particle, form an angle ϵ with it. As per classical optics, the wavelength of the photon sets a limit to the possible resolution Δx

$$\Delta x \gtrsim \frac{\lambda_\gamma}{\sin\epsilon} = \frac{1}{2\pi\omega\sin\epsilon}. \tag{1.5}$$

It is worth noting that we will henceforth utilise units where $\hbar = c = 1$.

When a photon is used to measure the position of a particle, it transfers momentum to the particle, causing uncertainty in the momentum along the x-direction Δp_x, as the direction of the photon after Compton recoil cannot be determined more accurately than ϵ. The uncertainty is given by

$$\Delta p_x \gtrsim \frac{2\pi\sin\epsilon}{\lambda_\gamma} = \omega\sin\epsilon. \tag{1.6}$$

The first-order form of Heisenberg's uncertainty principle can be obtained by combining equations 1.5 and 1.6

$$\Delta x \Delta p_x \gtrsim \frac{1}{2\pi}. \tag{1.7}$$

As we know from Heisenberg's uncertainty principle, it is meaningless to even consider the position and momentum of the particle at the same time. Consequently, instead of speaking about the photon scattering on a particle at one point, we should speak of a photon interacting strongly with a particle in some region of size R.

Incorporating gravity into this thought experiment, we need to consider the time interval τ between the interaction and measurement, which should be at least the order of the distance R travelled by the photon for the interaction to occur. This implies $\tau \gtrsim R$. If we look at 1.1, we see that for units where $\hbar = c = 1$, we have the same dimension for time and distance. As the photon carries energy, it exerts a small gravitational pull on the particle being measured, resulting in a gravitational acceleration of at least

$$a \approx \frac{\kappa\omega}{R^2}. \tag{1.8}$$

Let us assume that the particle is moving at non-relativistic speeds. The acceleration of the particle due to the gravitational force of the photon lasts for a duration of time that the photon spends in the region of strong interaction. During this time, the particle acquires a velocity of

$$v \approx aR = \frac{\kappa\omega}{R}. \tag{1.9}$$

Therefore, in time R, the acquired velocity enables the particle to cover a distance of

$$L \approx vR = \kappa\omega. \tag{1.10}$$

Due to the unknown direction of the photon within an angle of ϵ, the direction of the acceleration and the resulting velocity of the particle becomes uncertain as well. As a result, the projection of the particle's position onto the x-axis also has an uncertainty of

$$\Delta x \gtrsim \kappa\omega \sin\epsilon. \tag{1.11}$$

If we combine 1.5 and 1.12 we get

$$\Delta x \gtrsim \sqrt{\kappa} = l_{planck}. \tag{1.12}$$

Once more, we have derived the Planck length as the uncertainty in position.

1.3 Noncommutative quantum space

Let us say we have a commutator in basic quantum mechanics. If the commutator is not a 0, the Heisenberg uncertainty principle applies. It says that the more accurately we measure one element of the commutator, the less accurately we measure the second element of the commutator. One of the most famous commutators in physics is a momentum-position commutator $[\hat{x}, \hat{p_x}] = i\hbar$.

Usually, the space that we consider is commutative, which means $[\hat{x}^i, \hat{x}^j] = 0$, where $\hat{x}^i$ is a position operator. This space is certainly not fuzzy because the commutator is equal 0. From now on, let us denote noncommutative quantities with capital letters. How can we construct a noncommutative space if we want it to be fuzzy? We can simply define it as

$$\left[\hat{X}^i, \hat{X}^j\right] = 2i\lambda\epsilon^{ijk}\hat{X}^k, \tag{1.13}$$

where λ is a constant of noncommutativity, i is an imaginary unit and ϵ^{ijk} is the Levi-Civita symbol. If we have a closer look at the constant of noncommutativity λ, we can assume that it will be somehow related to Planck length. We see that it has a dimension of length and if we take $\lambda \to 0$ we will have a commutative case. There is one more beautiful property that we have inserted. This relation is rotationally invariant.

Because it is a non-zero commutator, a Heisenberg uncertainty principle applies to it, which means that if we, for example, want to measure position in x more precisely, we measure position in y less precisely. This means that we can never measure both x and y at the same time, which means, that it is fuzzy.

One of the significant results of noncommutative space, after considerable effort, is the dispersion relation that emerges from it. The dispersion relation is given by

$$\frac{1}{2}\hat{V}^2 = \hat{H}\left(1 - \frac{\lambda^2}{2}\hat{H}\right), \tag{1.14}$$

where $\hat{H}$ is a Hamiltonian and $\hat{V}$ is a velocity operator. Not surprisingly, if we send $\lambda \to 0$, we get the kinetic energy (without mass). If the reader is interested in understanding the origin of this relation, they may refer to [5]. The equation 1.14 reveals a relationship between energy and velocity. However, as explained in Chapter 2, it is preferable to express the dispersion relation in terms of momentum rather than velocity. To do so, we need to do a Legendre transformation.

1.3.1 The Legendre transformation of dispersion relation

First, we will go from operators of physical quantities to the corresponding physical quantities. To do so, we will take eigenvalues of the operators and will stop writing hat on physical quantities ($\hat{A} \to A$). After doing so, the equation 1.14 will take form

$$\frac{1}{2}V^2 = H\left(1 - \frac{\lambda^2}{2}H\right) = H\left(1 - aH\right). \tag{1.15}$$

As the next step, we will express the equation in the form of $H(V)$

$$H^2 - \frac{1}{a}H + \frac{V^2}{2a} = 0 \implies H(V) = \frac{1}{2a} \pm \frac{1}{2a}\sqrt{1 - 2aV^2}. \tag{1.16}$$

Now we will express $p(V)$ and $V(p)$

$$p(V) = \frac{\partial H}{\partial V} = \mp\frac{V}{\sqrt{1 - 2aV^2}} \implies V(p) = \pm\frac{p}{\sqrt{1 + 2ap^2}}. \tag{1.17}$$

Now, let us make the Legendre transformation

$$\tilde{H}(p) = pV - H(V) = \pm\frac{p^2}{\sqrt{1 + 2ap^2}} - \frac{1}{2a} \mp \frac{1}{2a}\sqrt{1 - 2aV^2}. \tag{1.18}$$

We have some flexibility in choosing the signs, so we will select the ones that result in an equation resembling our initial equation 1.15. With additional simplification and substitution ($\tilde{H} \to H$), we obtain

$$\frac{1}{2}p^2 = H\left(1 + aH\right) = H\left(1 + \frac{\lambda^2}{2}H\right). \tag{1.19}$$

If we once again send $\lambda \to 0$, we obtain

$$\frac{1}{2}p^2 = H, \tag{1.20}$$

which is the classical relation between energy and momentum (without mass).

Chapter 2

Dispersion relation

The focus of this chapter is on the dispersion relation, energy thresholds, and properties of Lorentz invariance violation (LIV), as they are closely interconnected, as shown by [7]. The dispersion relation that we get from special relativity for particles with mass is

$$E^2(p) = p^2 + m^2, \tag{2.1}$$

where m is the mass of the particle. For massless particles (in this thesis, we will only work with photons), it is

$$\omega^2(k) = k^2, \tag{2.2}$$

where ω is the angular frequency and k is the wave number.

2.1 Photon annihilation process $\gamma\gamma \to e^+e^-$

In special relativity, the production of an electron-positron pair through the annihilation of two photons is possible only above a certain lower threshold. This threshold refers to the minimum energy of the second photon (in case the energy of one photon is fixed) required to initiate the process. Ultra-high energy (UHE) photons are unable to travel long distances through space due to attenuation caused by cosmic background light, such as the cosmic microwave background (CMB) and extragalactic background light (EBL).

Let us examine the process of photon annihilation, as depicted in 2.1. The angles between the incoming photons and the outgoing electron/positron are α and β, respectively. Luckily, since we are only interested in thresholds, we can choose the photon with 4-momentum p_2 as the fixed one and assume that its energy ε, as well as its momentum magnitude $|\vec{p_2}|$, are much smaller than those of the other photon with 4-momentum p_1.

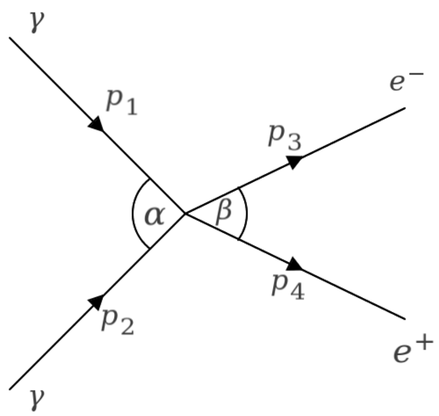

Figure 2.1: The diagram of photon annihilation into an electron-positron pair.

From special relativity, the lower threshold of the annihilation process occurs when $\alpha = \pi$ and $\beta = 0$ and electron/positron carries half of the energy and momentum. It is good to say that the upper threshold does not exist. Since we already know the configuration, it is easy to calculate the threshold

$$\begin{aligned} p_1 &= (E, 0, 0, E)\,, \\ p_2 &= (\varepsilon, 0, 0, -\varepsilon)\,, \\ p_3 = p_4 &= \left(\frac{E+\varepsilon}{2}, 0, 0, \frac{E-\varepsilon}{2}\right). \end{aligned} \tag{2.3}$$

We know that

$$m^2 \equiv p_3^2 = \left(\frac{E+\varepsilon}{2}\right)^2 - \left(\frac{E-\varepsilon}{2}\right)^2, \tag{2.4}$$

where m is the mass of electron/positron, and we use the metric tensor $(+,-,-,-)$. From 2.4, we obtain the lower threshold condition for photon annihilation

$$E \geq E_{th} = \frac{m^2}{\varepsilon}. \tag{2.5}$$

If the energy of a photon exceeds E_{th}, photon attenuation by low-energy photons will occur.

The average energy of a CMB photon is approximately 6.35×10^{-4} eV, which gives rise to a threshold energy of $E_{th} = 411$ TeV. Therefore, the CMB is not a viable source for our purposes. It is worth noting that the highest energy photon ever observed was emitted by GRB221009A and had an energy of around 18 TeV. Conversely, the EBL photons will play a critical role in all the effects that we will discuss in the next chapter. The energy of EBL photons that cannot be ignored lies in the range of 10^{-3} eV to 1 eV, which implies a threshold energy in the range of 261 GeV to 261 TeV. We can find a better analysis of EBL photons in [6].

2.2 Lorentz invariance violation (LIV)

Symmetries are crucial in physics, and among them, Lorentz symmetry and Lorentz invariance hold a significant position. In modern physics, Lorentz invariance is considered a fundamental building block. It has been verified by nearly all experiments with high accuracy. Nevertheless, some quantum gravity theories, like certain string theories, loop quantum gravity, and doubly special relativity, predict certain LIV properties. These properties result in the modified dispersion relation, which can be described, regardless of the specific theory, by a model-independent form

$$E^2(p) = p^2 + m^2 - \eta p^n, \tag{2.6}$$

where m denotes the mass of the particle, p represents the magnitude of the momentum vector $\vec{p}$, η is a parameter dependent on the particle species, and n takes values of 3, 4, and so on (although we will mainly anticipate that $n = 3$). The value of parameter η is greatly suppressed and can be positive, negative, or zero. It is expected to be suppressed by the Planck scale.

Modified dispersion relation 2.6 for massless particles will be

$$\omega^2(k) = k^2 - \xi k^n, \tag{2.7}$$

where ω is the angular frequency, k is the wave number, and ξ is the LIV parameter.

Once again, we consider the same configuration as in 2.1 under special relativity. The primary characteristic of the modified dispersion relation is that it can increase or decrease the threshold, and in certain cases, it can give rise to a region where both upper and lower thresholds coexist. We need to modify the expression for p_1 introduced in the special relativity case to $p_1 = (\omega(k), 0, 0, k)$, where $\omega(k)$ is defined in equation 2.7. We leave p_2 unchanged. This modification is possible because ε is so small that the contribution from LIV can be considered negligible. After same process as in 2.3 and 2.4 we get

$$\begin{aligned} m^2 &= \left(\frac{\omega+\varepsilon}{2}\right)^2 - \left(\frac{k-\varepsilon}{2}\right)^2 \\ &= \frac{\omega^2 - k^2 + 2\epsilon(\omega + k)}{4} \\ &= \frac{4\varepsilon k - \xi k^{n-1}(\varepsilon - k)}{4} + O(\xi^2) \\ &= \frac{4\varepsilon k - \xi k^n}{4} + O(\xi^2) + O(\xi\varepsilon). \end{aligned} \tag{2.8}$$

Once again, we assumed that ε is tiny. After we solve 2.8 for ξ we get

$$\xi = \frac{4\varepsilon k - 4m^2}{k^n}, \tag{2.9}$$

where we assumed that $k > 0$. In the following discussion, we will focus on the simplified case where $n = 3$ in the equation 2.9. This case has been well studied in phenomenological studies and allows us to easily investigate the threshold behaviour by analysing the properties of 2.9. The properties of 2.9 are as follows:

- There exists only one zero point at $k_0 = \frac{m^2}{\varepsilon}$, which coincides with E_{th} from 2.5.
- A global maximum occurs at the critical point $k_c = \frac{3m^2}{2\varepsilon}$ with corresponding $\xi_c = \frac{16\varepsilon^3}{27m^4}$.
- As $k \to 0^+$, ξ approaches negative infinity, whereas as $k \to \infty$, ξ approaches zero.

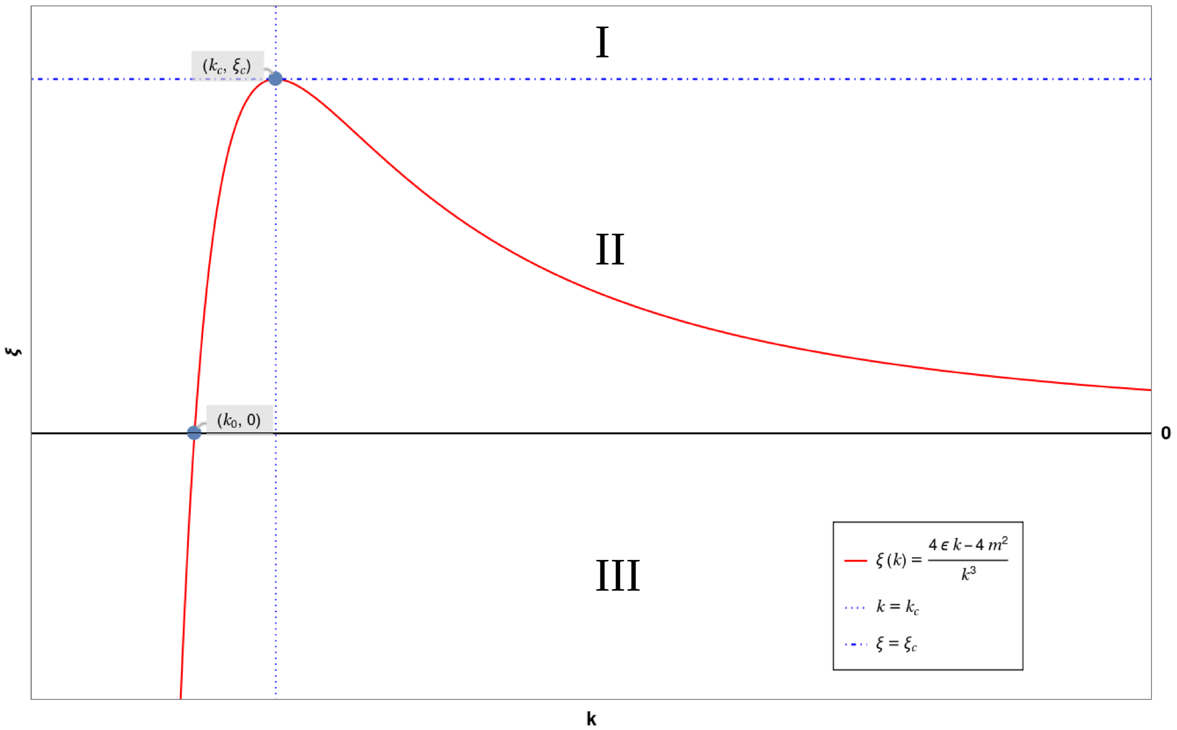

Figure 2.2: Graphical representation of the $\xi(k)$. We can divide it into three regions. In I, we can see all photons; in II, there is an interval of photons that we do not see; and in III, there is a lower threshold after which we can not see any photon.

These properties can be observed in Figure 2.2. The parameter space of 2.9 can be divided into three regions based on these properties.

Region I corresponds to the interval $\xi > \xi_c$. If our LIV parameter falls within this region, no annihilation with low-energy photons will occur, and we will be able to detect all photons.

Region II corresponds to the interval $0 < \xi < \xi_c$. In this region, we obtain two distinct solutions for 2.9, denoted as $k_<$ and $k_>$, with $k_<$ representing the lower threshold and $k_>$ representing the upper threshold. If the photon's wave number falls into the interval $k_< < k < k_>$, then annihilation will occur, and we will not be able to observe photons with energy corresponding to k.

Region III corresponds to the interval $\xi < 0$. In this region, we obtain only one solution corresponding to the lower threshold.

It should be noted that the annihilation process is a probabilistic phenomenon that depends on the distance travelled. For instance, if we had a high-energy photon source up close, we would be able to observe some photons. However, since all the sources of high-energy photons we have are located far away, some photons in between thresholds are so improbable that they cannot be detected [6].

Chapter 3

Results of the work

In the upcoming chapter of this thesis, we will present the findings of our work. We will investigate the dispersion relation to higher orders and evaluate its significance. Additionally, we will closely examine the potential values of the scaling parameter. Our primary reference for numerical values will be the gamma-ray burst (GRB) detected on 9 October 2022, commonly referred to as GRB221009A. The Large High Altitude Air Shower Observatory (LHAASO) recorded several photons from this burst with an energy of approximately 18 TeV. For numerical computations, we used WOLFRAM MATHEMATICA 13.2.

3.1 Dispersion relation in different theories

As mentioned in the previous chapter, several theories lead to a dispersion relation

$$\omega^2(k) = k^2 - \xi k^3. \tag{3.1}$$

This relation is just the first order of ξ. It is only natural that the whole relation will be

$$\omega^2(k) = k^2 - a_1\xi^1 k^3 - a_2\xi^2 k^4 - a_3\xi^3 k^5 - ... = k^2 - \sum_{1}^{\infty} a_i \xi^i k^{i+2}, \tag{3.2}$$

where a_i is a dimensionless factor and ξ is our scaling parameter that should be closely related to Planck length.

To assess the level of negligibility of the higher-order terms, we must examine the values of the constants a_i in those terms for various theories.

3.1.1 Relativistic dispersion relation in noncommutative quantum space

We know that 1.15 was constructed without adding any relativistic effect. How can one construct something relativistic out of non-relativistic physics?

The answer is elementary, my dear Watson. We want our new relativistic dispersion relation to satisfy two conditions. Firstly, we need to have an equation in which after we send $\lambda \to 0$, we get a relativistic formula for momentum

$$E = pc = p. \tag{3.3}$$

Secondly, we can require our constructed space to influence slow particles the same as fast particles. The way to meet all of our assumptions is very simple. We can make substitution in 1.19 where $\frac{p^2}{2} \to p$ and $\lambda^2 \to \xi$, after which we get

$$p = H\left(1 + \frac{\xi}{2}H\right). \tag{3.4}$$

The structure of the equations is left untouched, so our constructed space influences slow particles in the same way as fast particles, and if we send $\lambda \to 0$, we get relativistic formula 3.3.

Now let us take 1.19, solve it for H, square it and for better visualisation of what we have done, let us make Taylor series of it

$$H^2 = \frac{\left(\sqrt{p^2\lambda^2+1}-1\right)^2}{\lambda^2} = \frac{p^4}{4} - \frac{p^6\lambda^2}{8} + \frac{5p^8\lambda^4}{64} - \frac{7p^{10}\lambda^6}{128} + O\left(\lambda^8\right). \tag{3.5}$$

After our substitution, we get

$$H^2 = \frac{p^4}{4} - \frac{p^6\lambda^2}{8} + \frac{5p^8\lambda^4}{64} + O\left(\lambda^6\right) \to H^2 = p^2 - p^3\xi - \frac{5}{4}p^4\xi^2 + O(\xi^3). \tag{3.6}$$

It is remarkable that we obtained this relation without any intentional effort but rather through making reasonable assumptions. Furthermore, we can observe that a_2 is not an excessively large number but rather a constant of the order of one.

3.1.2 Amelino-Camelia Doubly Special Relativity

As we have observed, the Planck length l_{planck} holds a crucial significance in quantum gravity theories, acting as a threshold for quantum effects. Hence, it is expected that l_{planck} remains invariant in all inertial frames of reference. However, this poses a challenge as it contradicts the length contraction predicted by Special Relativity. To address this conflict, Amelino-Camelia Doubly Special Relativity proposes a solution by modifying Einstein's Special Relativity postulates as

1. The laws of physics involve a fundamental velocity scale c and fundamental length scale l_{planck}.

2. Each inertial observer can establish the value of l_{planck} (same value for all inertial observers) by determining the dispersion relation for photons, which takes the form $E^2 - c^2p^2 + f(E, p, l_{planck}) = 0$, where f is the same for all inertial observers and in particular all inertial observers agree on the leading dependence of f: $f(E, p, l_{planck}) \simeq Ecp^2 l_{planck}$.

For more information about this theory, we encourage the reader to check [4]. Let us have a closer look at the dispersion relation in the second postulate. By using $\hbar = c = 1$ units and substituting $l_{planck} = \xi$, we can rewrite it to

$$\omega^2 + \xi k^2 \omega - k^2 = 0 \implies \omega^2 = \frac{-k^2\xi + k\sqrt{4 + k^2\xi^2}}{4}. \tag{3.7}$$

After the Taylor series, we get

$$\omega^2 = k^2 - k^3\xi + \frac{k^4\xi^2}{2} + O(x^3). \tag{3.8}$$

Once again, we see that a_2 is not an enormous number but rather a reasonably large constant.

3.2 Higher-order terms in dispersion relation

From leading-order correction, we get graph 2.2. It is apparent that even with different theories, the relations produced are quite similar, and they only differ in higher corrections. Hence, it is useful to investigate the significance of higher-order terms.

As we have already seen in the previous section, constants in higher-order terms are not big, so let us have a closer look at the behaviour and relevance of these terms. For simplicity, let us look at

$$\omega^2(k) = k^2 - a_1\xi^1 k^3 - a_2\xi^2 k^4. \tag{3.9}$$

Having obtained the new dispersion relation 3.9, we will proceed in the same manner as we did for 2.8. After some simplification, we arrive at

$$m^2 = \frac{4\varepsilon k - \varepsilon a_1 k^2 \xi - a_1 k^3 \xi - k^4 a_2 \xi^2}{4} + O(\xi^3) + O(\varepsilon\xi^2). \tag{3.10}$$

We want to know how negligible the higher order is, so we will substitute for our parameters. For m, we use the mass of the electron, and for the energy of the low energy photon, we substitute $\varepsilon = 0.1$ eV.

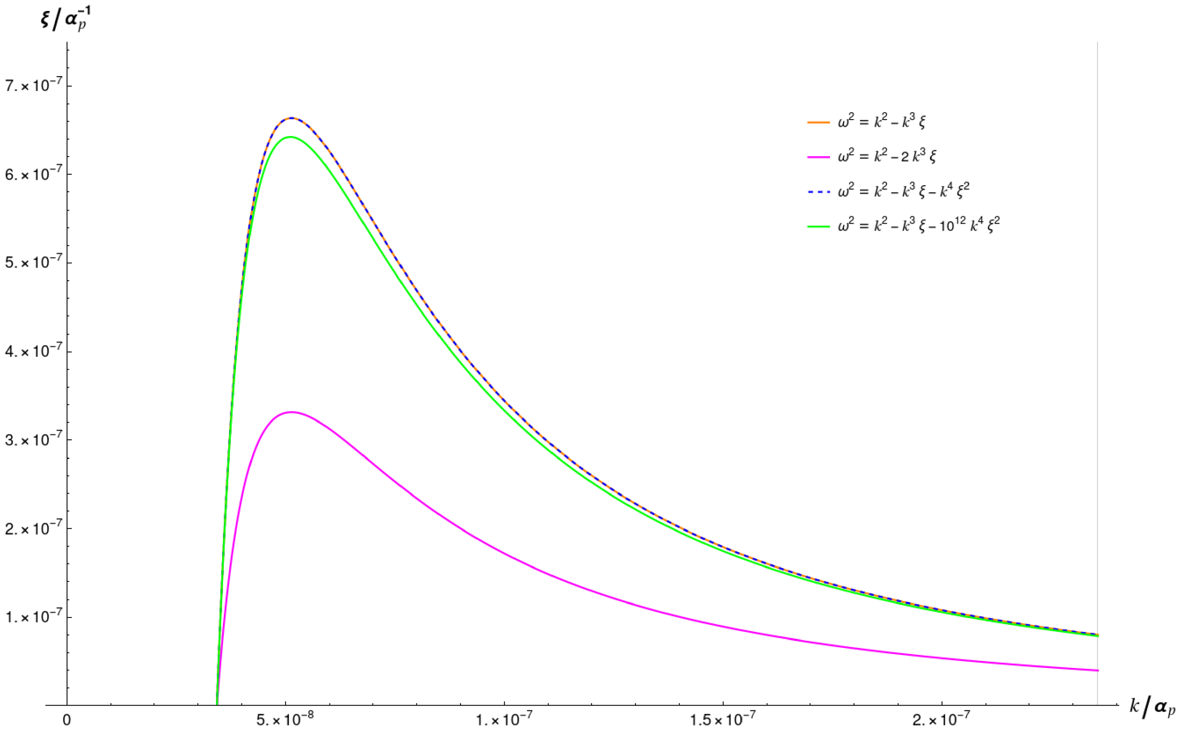

Figure 3.1: Comparison of different disperse relations. We can see, that terms with k^3 are the significant ones, while terms with k^4 are relevant only for huge a_2.

Firstly, it is important to consider the units we are using. If we were to use electron-volts as the unit for k, the resulting value would be huge, on the order of TeV, while our scaling parameter ξ would be tiny, on the order of $10^{-26}eV^{-1}$. Through a trial and error process, we have determined that the most suitable units to use for our photon, with energy $\varepsilon = 0.1$ eV, are $\frac{E_{planck}}{10000} \equiv \alpha_p$, where E_{planck} is defined as $E_{planck} = \sqrt{\frac{\hbar c^5}{\kappa}}$, in order to ensure that all values are of a reasonable order of magnitude.

The information provided by Figure 3.1 regarding dispersion relations is extensive. The orange and magenta lines are particularly noteworthy. By observing these lines, we can see that even a slight alteration of the constant a_1, which appears with k^3 in 3.9, results in a significant difference in the height of the graph. This is an intriguing observation since it implies that we could easily differentiate between two theories that vary in this term. As a result, it would be simpler for us to verify them experimentally.

Next, let us examine the orange, blue, and green lines closely in Figure 3.1. Upon inspection, we notice that the orange and blue lines overlap almost entirely. The distinction between them only becomes noticeable when the value of a_2 reaches the order of 10^{12}, which is an enormous value. To facilitate better visualisation, it would be more effective to display some values in a table. Before that, we will mark the solution of the orange dispersion relation as ξ_1, the solution of the blue dispersion relation as ξ_2 and the solution of the green dispersion relation as ξ_3.

The actual values in Table 3.1 themselves are not as significant as their order.

Table 3.1: Table of solution ξ for particular values of k.

$k/\left(10^{-8}\alpha_p\right)$	$\xi_1/\left(10^{-7}\alpha_p^{-1}\right)$	$\xi_2/\left(10^{-7}\alpha_p^{-1}\right)$	$\xi_3/\left(10^{-7}\alpha_p^{-1}\right)$
4	4.76240775064666	4.76240775064642	4.67498578238452
8	4.68567739935731	4.68567739935706	4.52208348686017
12	2.60031224589149	2.60031224589138	2.52387302531643
16	1.60830378255128	1.60830378255123	1.56891964110724
20	1.08523562821995	1.08523562821992	1.06265108178746

Table 3.2: Table of differences in results ξ for specific values of k. We can confirm our observation from Figure 3.1.

$k/\left(10^{-8}\alpha_p\right)$	$\lvert\xi_1-\xi_2\rvert/\left(10^{-20}\alpha_p^{-1}\right)$	$\lvert\xi_1-\xi_3\rvert/\left(10^{-9}\alpha_p^{-1}\right)$
4	2.4	8.742197
8	2.5	16.35939
12	1.0	7.643922
16	0.5	3.938414
20	0.3	2.258455

In order to better observe the differences between the lines, we will subtract their respective values. As shown in Table 3.2, the discrepancies between ξ_1 and ξ_2 are in the range of 10^{-20}. If we convert these values back to electronvolts, we get a negligible value of approximately 10^{-39}. However, the difference between ξ_1 and ξ_3 is only two orders of magnitude lower, which makes it easy for us to distinguish between the two lines. It should be noted that terms involving k^5 and higher would be even smaller than the difference between ξ_1 and ξ_2. Therefore, there is no need to analyse them in detail.

The insignificance of higher-order terms has both positive and negative aspects. On the positive side, we can simply use the first order of the dispersion relation without much consideration. However, on the negative side, we lose the ability to differentiate between two theories due to the unimportance of higher order terms.

Another notable observation is that the value of a_2 in Equation 3.8 is negative, while we have only been considering positive values of a_i so far. It is important to investigate the behaviour of the dispersion relation when a_2 is negative.

It can be observed from Figure 3.2 that the sign of a_i affects the height of the graph of the dispersion relation. Negative values of a_i result in an increase of magnitude, while positive values lead to a decrease.

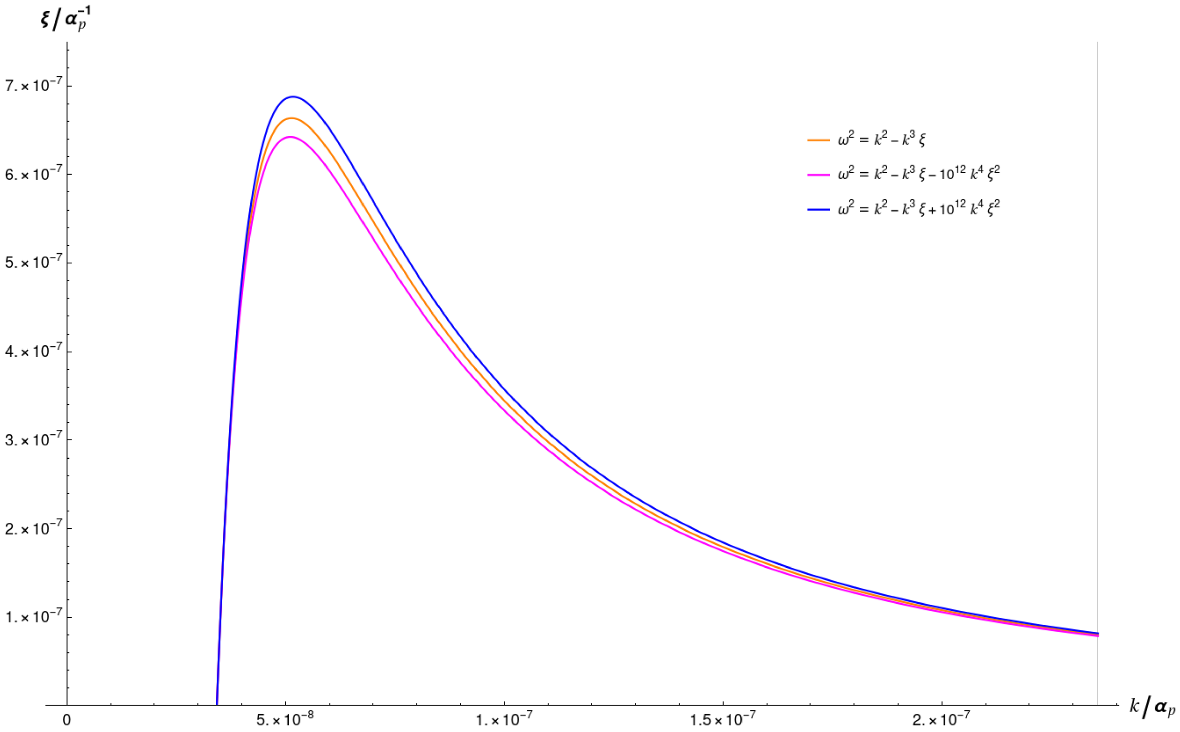

Figure 3.2: Comparison of signs in terms with k^4. We can see that positive a_2 is pushing magnitude lower, while negative a_2 is pushing magnitude higher.

3.3 Lower-bound of the scaling parameter

In Chapter 2, we have discussed three regions of the solution for 2.9. We have learned that if the energy of photons falls into the area under the graph, we will not be able to see these photons because the annihilation process will occur. This fact gives us an opportunity to find the lower bound of the scaling parameter.

Let us concentrate on the LHAASO 18 TeV photon. We know the energy of this photon, so we can substitute it in 2.9

$$\xi_{lb}(\varepsilon) = \frac{4}{k^2_{\text{LHAASO}}}\varepsilon - \frac{4m^2}{k^3_{\text{LHAASO}}} = a\varepsilon - b. \tag{3.11}$$

The energy ε of the EBL photon is the only free parameter for us. If we fix the energy $\varepsilon = \varepsilon_0$ we can say, that scaling parameter will be $\xi(\varepsilon_0) \geq \xi_{lb}(\varepsilon_0)$.

We know that relevant EBL photons have their energy in the range 10^{-3} eV $-$ 1 eV. If we insert this values into 3.11, we get

$$\xi_{lb} \in (-1.274 \cdot 10^{-8}\ \alpha_p^{-1}, 9.295 \cdot 10^{-7}\ \alpha_p^{-1}). \tag{3.12}$$

Units α_p have been defined in the previous section. To find a general lower bound, we have to take the biggest number in the interval 3.12, after which we get

$$\xi \geq 9.295 \cdot 10^{-7}\ \alpha_p^{-1}, \tag{3.13}$$

which corresponds to energy $\varepsilon = 1$ eV. For better visualisation, we shall make graphs of this solution.

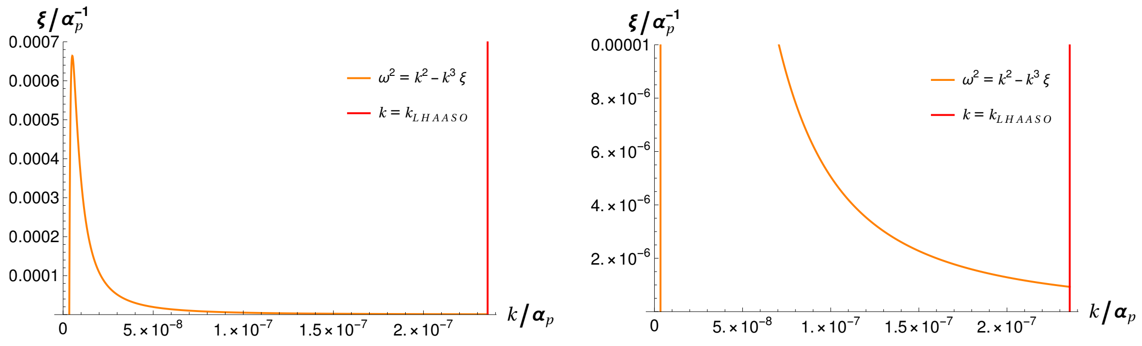

Figure 3.3: The graphs show the solution for ξ if $\varepsilon = 1$ eV. We can see that if ξ was less than ξ in the interception, there would be a huge interval of photons that we would not be able to see.

As we can see in the figure 3.3, using $\xi_{lb1} = 9.295 \cdot 10^{-7}\ \alpha_p^{-1}$ as a lower-bound is very optimistic because it excludes a big interval of photons. The most pessimistic lower bound is

$$\xi_c(\varepsilon = 1\text{ eV}) = 6.640 \cdot 10^{-4}\ \alpha_p^{-1} = \xi_{lb2}. \tag{3.14}$$

For better visualisation, the value of the Planck length in α_p^{-1} is, by definition

$$l_{planck} = \frac{1}{E_{planck}} = 10^{-4}\ \alpha_p^{-1}. \tag{3.15}$$

Surprisingly, our lower bounds are around the same order as Planck length. Although we cannot simply take it as our ξ because it could be, for example, ten times the Planck length or a hundred times the Planck length.

We have calculated two potential lower bounds for the scaling parameter: ξ_{lb1} and ξ_{lb2}. If the value of the scaling parameter falls between these two bounds, it would be highly significant. This would mean that if we observe any interval of photons that did not reach us, we could experimentally test the validity of the theory.

Conclusion

It remains to conclude the results of our work. In this thesis, we have continued the study of threshold anomaly for GRB.

We have analysed the dispersion relation given by NCQS and have applied the Legendre transformation to transform it into the form of $H = H(p)$. After the transformation, we obtained a relation that resulted in the classical physics relation between energy and momentum. We have made some clever assumptions to make it relativistic. As a result, we have obtained a dispersion relation that agrees with other theories, such as Doubly Special Relativity.

Through our comparison of artificially made dispersion relations, we have sought to better understand the significance of higher-order terms, as many theories differ in them. Our findings indicate that anything higher than the first-order is negligible, which is a double-edged sword. The bright side is that we can simplify all equations by using only the first order terms. On the other hand, this prevents us from experimentally distinguishing between two theories.

In our research, we have examined the possible values for the scaling parameter obtained from the observed measurements of GRB221009A. Through our analysis, we have identified two distinct lower bounds for this parameter. While the higher lower bound is not particularly noteworthy, as it is still possible to observe photons with any energy, the second lower bound holds significant potential for experimental measurements. Based on our theory, there will be an interval of photons that cannot be detected, providing us with an excellent opportunity for further investigation and experimentation.

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